Exponential stability in non-conservative dynamical systems

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2011-10-21

Outline of the talk

- **Normal form theory**
  - Non-conservative, nearly integrable systems
  - Close to (non-)resonant initial conditions
- **Stability estimates**
  - Proofs (sketched)
  - Linear stability vs. exponential stability
  - Theorem
- **Possible applications**
  - Stability estimates in Celestial Mechanics
  - Normal form solutions as reference solutions

References

- **A. Celletti**, C. Lhotka, Stability for exponential times in nearly-Hamiltonian systems: the non resonant case, *Preprint* 2011.
- A. Celletti, C. Lhotka, Stability of nearly-Hamiltonian systems with resonant frequency, *Preprint* 2011.
- A. Celletti, C. Lhotka, A comparison of numerical integration methods in nearly-Hamiltonian systems, *Preprint* 2011.
- J. Pöshel, Nekhoroshev's estimates for quasi–convex Hamiltonian systems, *Math. Z.* 213, 187–216, 1993.
- N.N. Nekhoroshev. Exponential estimates of the stability time of near-integrable Hamiltonian systems, *Russ.Math.Surv.*, 32:1–65, 1977.

Normal form theory

Consider the system of ODEs:

$$\dot{x} = \omega(y) + \varepsilon\, h_{10,y}(y,\, x,\, t) + \mu\, f_{01}(y,\, x,\, t)$$
$$\dot{y} = -\varepsilon\, h_{10,x}(y,\, x,\, t) + \mu(g_{01}(y,\, x,\, t) - \eta(y,\, x,\, t)) \qquad (1)$$

With the notation: $h_{10} \quad f_{01} \quad g_{01} \quad \eta$ **(analytic functions):**

$$y \in A \subset \mathbb{R}^n \qquad (x,\, t) \in \mathbb{T}^{n+1} \qquad \varepsilon \in \mathbb{R}_+ \qquad \mu \in \mathbb{R}_+$$

We are looking for a transformation:

$$(X,\, Y) = \Xi_d{}^{(N)} \circ \Xi_c{}^{(N)}(x,\, y)$$

To write (1) in normal form coordinates (in the best case):

$$\underbrace{\dot{X} = \Omega(Y) + O_{N+1}}_{\text{normalized frequency}} \qquad \underbrace{\dot{Y} = O_{N+1}}_{\text{higher order terms}}$$

The change of coordinates

The explicit form of the transformation (normalization order N):

Original to intermediate variables:

$$\tilde{x} = x + \sum_{j=1}^{N} \psi_{j0,y}(\tilde{y}, x, t)\, \varepsilon^{j}$$

$$y = \tilde{y} + \sum_{j=1}^{N} \psi_{j0,x}(\tilde{y}, x, t)\, \varepsilon^{j}$$

Intermediate to final (normalized) variables:

$$X = \tilde{x} + \sum_{m=0}^{N} \sum_{j=1}^{m} \alpha_{m-j,j}(\tilde{y}, \tilde{x}, t)\, \varepsilon^{m-j}\, \mu^{j}$$

$$Y = \tilde{y} + \sum_{m=0}^{N} \sum_{j=1}^{m} \beta_{m-j,j}(\tilde{y}, \tilde{x}, t)\, \varepsilon^{m-j}\, \mu^{j}$$

The normal form equations

The normal form equation for the conservative contributions:

$$\omega(Y)\,\psi_{\mathrm{j}0,x}(Y,\ X,\ t) + \psi_{\mathrm{j}0,t}(Y,\ X,\ t) + \tilde{L}_j(Y,\ X,\ t) = 0$$

The pair of dissipative normal form equations:

$$\omega_y(Y)\,\alpha_{m-j,j,x}(Y,\ X,\ t) + \alpha_{m-j,t}(Y,\ X,\ t) + \omega_y(Y)\,\beta_{m-j,j}(Y,\ X,\ t) + \tilde{f}_{m-j,j}(Y,\ X,\ t) = 0$$

$$\omega_y(Y)\,\beta_{m-j,j,x}(Y,\ X,\ t) + \beta_{m-j,t}(Y,\ X,\ y) + \tilde{g}_{m-j,j}(Y,\ X,\ t) = 0$$

The notation:

$$L_j^{(\leq K)}(Y,\ X,\ t) = \bar{L}_j(Y) + \tilde{L}_j(Y,\ X,\ t) + \tilde{L}_j^{(r)}(Y,\ X,\ t)$$

$$f_{m-j.j}^{(\leq K)}(Y,\ X,\ t) = \bar{f}_{m-j}(Y) + \tilde{f}_{m-j.j}(Y,\ X,\ t) + \tilde{f}_{m-j.j}^{(r)}(Y,\ X,\ t)$$

$$g_{m-j.j}^{(\leq K)}(Y,\ X,\ t) = \bar{g}_{m-j}(Y) + \tilde{g}_{m-j.j}(Y,\ X,\ t) + \tilde{g}_{m-j.j}^{(r)}(Y,\ X,\ t)$$

The generic solutions

Poisson series of analytic functions with two parameters:

$$\sum_{(k,m)\in U\subset\mathbb{Z}^{n+1}} a_{km}\,\varepsilon^{b_{km}}\,\mu^{c_{km}}\,\frac{p_{km}(Y)}{q_{km}(Y)}\,e^{-i\,(X\cdot k+m\,t)}$$

The generic form of the solutions:

$$\sum_{(k,m)\in U\subset\mathbb{Z}^{n+1}} d_{km}\,\varepsilon^{b_{km}}\,\mu^{c_{km}}\,\frac{r_{km}(Y)}{(\omega(Y)\cdot k+m)}\,e^{-i\,(X\cdot k+m\,t)}$$

$a_{km},\, d_{k,m}\in\mathbb{C}^n$ $\quad b_{km},\, c_{km}\in\mathbb{Z}_+$ $\quad p_{km},\, q_{km},\, r_{km}$ **(polynomial)**

The non-resonance condition:

$$|\,\omega(y)\cdot k+m\,| > 0$$

$$K\in\mathbb{Z}_+ \qquad |\,k\,|+|\,m\,| \leq K$$

$$y\in D\subset A \qquad (k,\, m)\in\mathbb{Z}^{n+1}$$

The normal form 1/4

The equation for Y:

$$\omega_y(Y)\,\beta_{m-j,j,x}(Y,\,X,\,t) + \beta_{m-j,t}(Y,\,X,\,y) + \left(\bar{g}_{m-j}(Y) + \tilde{g}_{m-j.j}(Y,\,X,\,t) + \tilde{g}_{m-j.j}{}^{(r)}(Y,\,X,\,t) - \eta_{m-j,j}\right) = 0$$

Depending on the generic form of eta…

$$\eta = \sum_{m=0}^{N}\sum_{j=0}^{m}\eta_{m-j,j}$$

$$\eta = \eta(Y,\,X,\,t) = \eta(y,\,x,\,t)$$
$$\eta = \eta(Y) = \eta(y)$$
$$\boxed{\eta = \eta_* = const.}$$

…we get different kinds of normal forms:

$$\dot{X} = \Omega^{(N)}(Y;\,\varepsilon,\,\mu) + F_\varepsilon{}^{(N)}(Y,\,X,\,t) + F_\mu{}^{(N)}(Y,\,X,\,t) + O_{N+1}(\varepsilon,\,\mu)$$
$$\dot{Y} = G_\varepsilon{}^{(N)}(Y,\,X,\,t) + G_\mu{}^{(N)}(Y,\,X,\,t) + O_{N+1}(\varepsilon,\,\mu)$$

The normal form 2/4

The equation for Y:

$$\omega_y(Y)\,\beta_{m-j,j,x}(Y,\,X,\,t)+\beta_{m-j,t}(Y,\,X,\,y)+$$
$$\left(\bar{g}_{m-j}(Y)+\tilde{g}_{m-j.j}(Y,\,X,\,t)+\tilde{g}_{m-j.j}{}^{(r)}(Y,\,X,\,t)-\eta_{m-j,j}\right)=0$$

Depending on the generic form of eta…

$$\eta=\sum_{m=0}^{N}\sum_{j=0}^{m}\eta_{m-j,j}$$

$$\boxed{\eta=\eta(Y,\,X,\,t)=\eta(y,\,x,\,t)}$$
$$\eta=\eta(Y)=\eta(y)$$
$$\eta=\eta_*=const.$$

…we get different kinds of normal forms:

$$\dot{X}=\Omega^{(N)}(Y;\varepsilon,\,\mu)+F_\varepsilon{}^{(N)}(Y,\,X,\,t)+F_\mu{}^{(N)}(Y,\,X,\,t)+O_{N+1}(\varepsilon,\,\mu)$$
$$\dot{Y}=G_\varepsilon{}^{(N)}(Y,\,X,\,t)+O_{N+1}(\varepsilon,\,\mu)$$

The normal form 3/4

The equation for Y:

$$\omega_y(Y)\,\beta_{m-j,j,x}(Y,\,X,\,t) + \beta_{m-j,t}(Y,\,X,\,y) +$$
$$\left(\bar{g}_{m-j}(Y) + \tilde{g}_{m-j.j}(Y,\,X,\,t) + \tilde{g}_{m-j.j}{}^{(r)}(Y,\,X,\,t) - \eta_{m-j,j}\right) = 0$$

Depending on the generic form of eta…

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…we get different kinds of normal forms:

$$\dot{X} = \Omega^{(N)}(Y;\,\varepsilon,\,\mu) + O_{N+1}(\varepsilon,\,\mu)$$
$$\dot{Y} = G_\mu{}^{(N)}(Y) + O_{N+1}(\varepsilon,\,\mu)$$

The normal form 4/4

The equation for Y:

$$\omega_y(Y)\,\beta_{m-j,j,x}(Y,\,X,\,t) + \beta_{m-j,t}(Y,\,X,\,y) + \left(\bar{g}_{m-j}(Y) + \tilde{g}_{m-j.j}(Y,\,X,\,t) + \tilde{g}_{m-j.j}{}^{(r)}(Y,\,X,\,t) - \eta_{m-j,j}\right) = 0$$

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…we get different kinds of normal forms:

$$\dot{X} = \Omega^{(N)}(Y;\,\varepsilon,\,\mu) + O_{N+1}(\varepsilon,\,\mu)$$
$$\dot{Y} = O_{N+1}(\varepsilon,\,\mu)$$

The Hamiltonian in normal form

The Hamiltonian in normal form coordinates (mu=0) :

$$H_r^{(N)}(Y,\,X,\,t) = h_\varepsilon^{(N)}(Y;\,\varepsilon) + T + h_r^{(N)}(Y,\,X,\,t;\,\varepsilon) + O_{N+1}(\varepsilon)$$

$$F_\varepsilon^{(N)}(Y,\,X,\,t) = \frac{\partial H_r^{(N)}}{\partial Y} \qquad\qquad G_\varepsilon^{(N)}(Y,\,X,\,t) = -\frac{\partial H_r^{(N)}}{\partial X}$$

The total time derivative (mu>0) :

$$\frac{d\,H_r^{(N)}}{d\,t} = \frac{\partial H_r^{(N)}}{\partial Y}\,\dot{Y} + \frac{\partial H_r^{(N)}}{\partial X}\,\dot{X} + \frac{\partial H_r^{(N)}}{\partial T}\,\dot{T} + \frac{\partial H_r^{(N)}}{\partial t}\,\dot{t}$$

Including the dissipative normal form equations:

$$\frac{d\,H_r^{(N)}}{d\,t} = \phi_N(Y,\,X,\,t) + O_{N+1}$$

Stability estimates

The normal form equation for the action Y:

$$\dot{Y} = G_\varepsilon(Y, X, t) + \underline{F_N{}^{(>K,y)}(Y, X, t) + F_{N+1}(Y, X, t)}$$

We want to bound the time developement in the actions:

$$\| y(t) - y(0) \| \leq \| y(t) - Y(t) \| + \| Y(t) - Y(0) \| + \| Y(0) - y(0) \|$$

The variation due to the transformation:

$$\| y(0) - Y(0) \| \leq r_1$$

$$\| y(t) - Y(t) \| \leq r_1$$

The variation of the actions due to the normal form dynamics:

$$\| Y(t) - Y(0) \| \leq \int_0^t \| G_\varepsilon \| + \| F^{(>K,y)} \| + \| F_{N+1} \| \, d\,s$$

Upperbounding norms

Simple expressions to bound the norms ($\pi_0 = max(\varepsilon,\ \mu)$):

$$|| F^{(>K,y)} || \ \leq \pi_0 \, C^{(1)} \, e^{-K\,\tau_0} \qquad\qquad || F_{N+1} || \ \leq C^{(2)} \, {\pi_0}^{N+1}$$

The choice of N makes the remainder exponential small:

$${\pi_0}^{N} = e^{-K\,\tau_0}$$

$$N \mid log\, \pi_0 \mid = -K\,\tau_0$$

$$N = \frac{K\,\tau_0}{\mid log\ \pi_0 \mid}$$

We get the bound on the remainder and "ultraviolet part":

$$|| F^{(>K,y)} || + || F_{N+1} || \ \leq C_Y \, {\pi_0}^{N+1} \ \leq \ e^{-K\,\tau_0}$$

Exponential stability

We bound the integral $(C_Y = C^{(1)} + C^{(2)})$ **:**

$$\| Y(t) - Y(0) \| \le C_Y \, {\pi_0}^{N+1} \cdot t + \int_0^t \| G_\varepsilon \| \, d\, s$$

In the non-resonant case (remaining integral is zero)…

$$C_Y \, {\pi_0}^{N+1} \cdot t \le r_2 \qquad \textbf{for} \qquad t \le \frac{r_2}{C_Y} \, {\pi_0}^{-(N+1)} \le \frac{r_2}{C_Y} \, e^{K \tau_0}$$

$$\textbf{with} \qquad \rho_0 = 2\, r_1 + r_2 \qquad \textbf{and} \qquad C_t = \frac{r_2}{C_Y \, \pi_0}$$

…we get the local stability result:

$$\| y(t) - y(0) \| \le \rho_0 \quad \textbf{for} \qquad t \le C_t \, e^{K \tau_0}$$

Resonant initial conditions

The resonant construction gives:

$$G_\varepsilon^{(N)}(Y, X, t) = -\varepsilon \mathrm{p}_X(Y, X, t; \varepsilon) \qquad F_\mu^{(N)} = \mu\, s(Y, X, t; \varepsilon, \mu)$$

The stability proof relies on the conservation of the energy:

$$\frac{d\, H_r^{(N)}}{d\, t} = \varepsilon\, \mu\, p_X\, s + \ldots$$

In general we only get linear stability (C_1, C_2, C_3, C_4 constant):

$$\frac{m}{2} \,||\, \Delta \mathrm{Y} \,||^2 \leq 2\, \delta \,||\, \Delta \mathrm{Y} \,|| + C_1\, \pi_0{}^N\, t + \underline{C_2\, \varepsilon\mu\, t} + \varepsilon\, C_3 + C_4\, \pi_0{}^N$$

(linear stability time): $\qquad ||\, p_X \,||\; ||\, s \,|| \leq C_2$

(exponential stability time): $\qquad ||\, p_X \,||\; ||\, s \,|| = 0 \Rightarrow p_X = 0 \;\; or\; s = 0$

Theorem-1

Consider the vector field (1) defined on $A \times \mathbb{T}^{n+1}$, and let $D \subset A$ be such that for any $y \in D$ the frequency $\omega = \omega(y)$ satisfies a suitable non-resonant condition.

Assume there exists $\varepsilon_0,\ \mu_0$ such that for $\varepsilon < \varepsilon_0,\ \mu < \mu_0$ the Normal Form Lemma holds.

Then, there exist positive parameters $\rho_0,\ \tau_0$, such that for every solution at time $t > 0$ with initial position $(y(0),\ x(0)) \in D \times \mathbb{T}^n$ one has:

$$|| \, y(t) - y(0) \, || \le \rho_0 \quad \text{for} \quad t < C_t \, e^{K \tau_0}$$

for some positive constant C_t .

Theorem-2

Consider the vector field (1) defined on $A \times \mathbb{T}^{n+1}$, satisfying a quasi-convexity condition. Let $y_0 \in A$ be non resonant outside the main resonance.

Assume there exists $\varepsilon_0, \ \mu_0$ such that for $\varepsilon < \varepsilon_0, \ \mu < \mu_0$ the Resonant Normal Form Lemma holds ($\tau_0, \pi_0 \quad K \in \mathbb{Z}_+$ const.).

i) for $p_X = 0$ or $s = 0$ there exist $\rho_1 > 0, \ C_0 > 0$ s.t.:

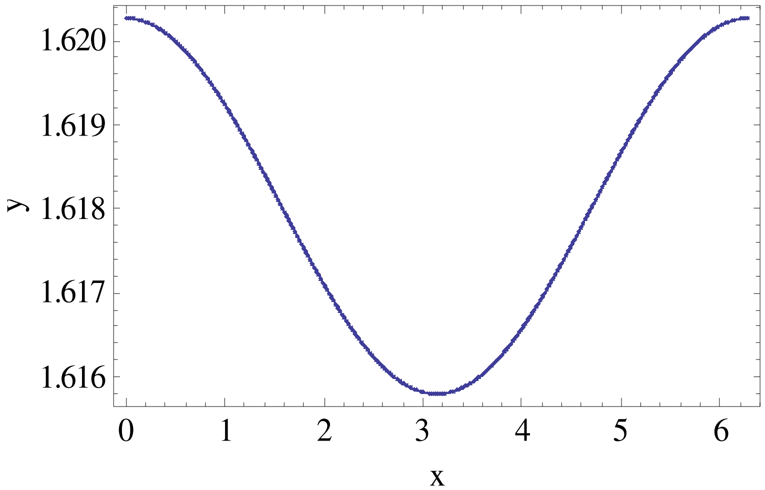
$$\| y(t) - y(0) \| \le 2\, C_p\, \pi_0 + \rho_1 \text{ for } t \le C_0\, e^{K\, \tau_0}$$

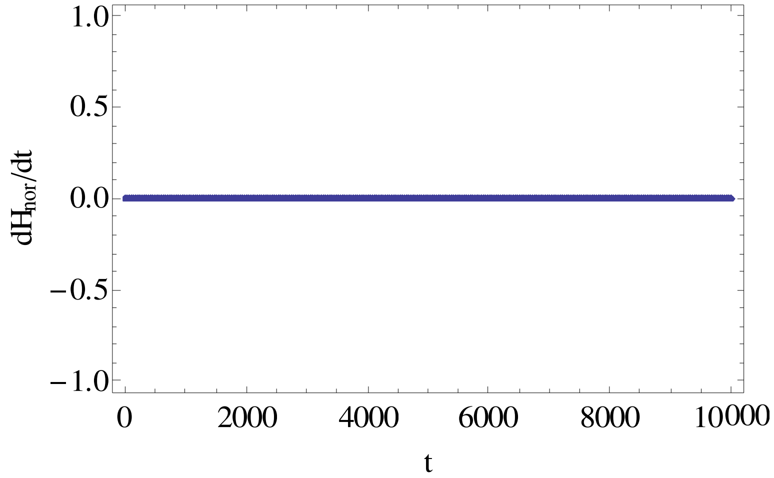
ii) for $p_X \neq 0$ and $s \neq 0$ there exist $\rho_2 > 0, \ C_0' > 0, \ C_0'' > 0$ with:

$$\| y(t) - y(0) \| \le 2\, C_p\, \pi_0 + \rho_2 \quad t \le \min\left(C_0'\, e^{K\, \tau_0}, \ \frac{C_0''}{\varepsilon\, \mu} \right)$$

The non resonant case

Phase portrait & variation of the energy, non resonant case:





$$y(0) = \frac{1}{2}\left(\sqrt{5} + 1\right)$$

$$\varepsilon = 7 \cdot 10^{-4}$$

$$\mu = 7 \cdot 10^{-4}$$

$$\dot{x} = y - \mu\,(\sin(x - t) + \sin(x))$$

$$\dot{y} = -\varepsilon\,(\sin(x - t) + \sin(x)) - \mu(y - \eta)$$

1:1 resonance with linear drift

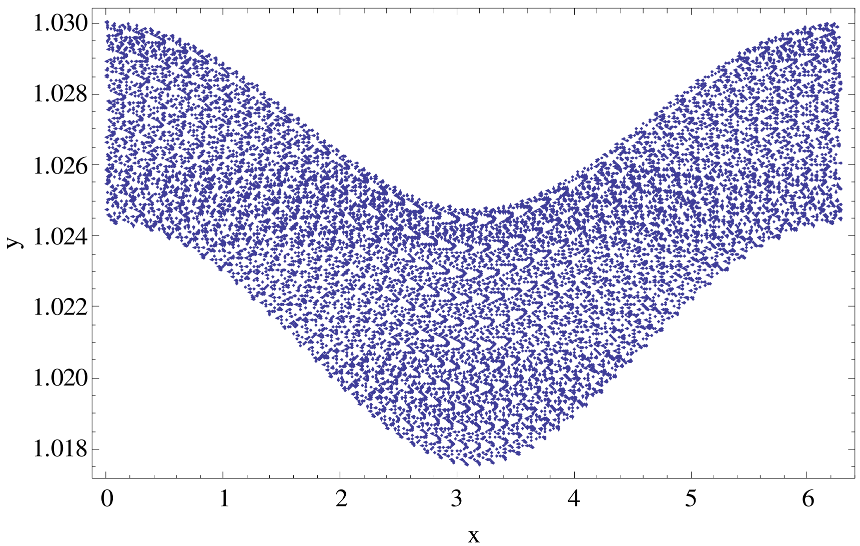
The actions are conserved only on linear times:

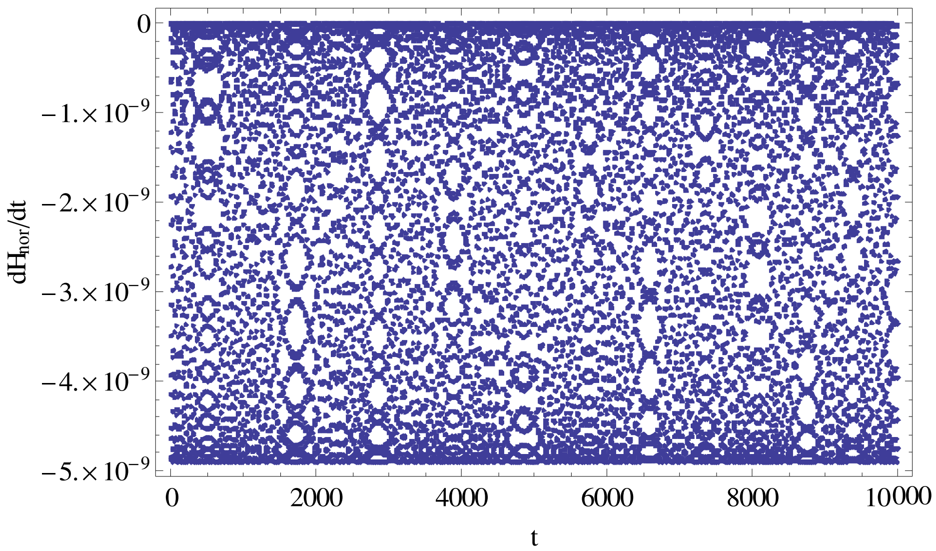
$$\dot{x} = y - \mu\,(\sin(x-t) + \sin(x))$$
$$\dot{y} = -\varepsilon\,(\sin(x-t) + \sin(x)) - \mu(y-\eta)$$

$y(0) = 1.03$

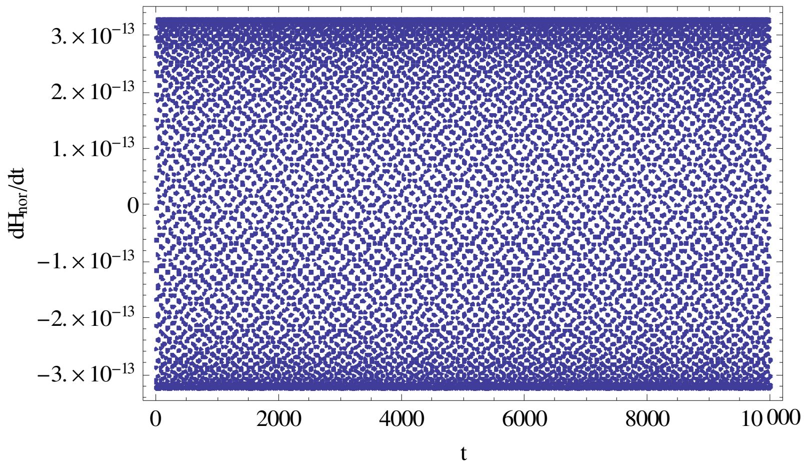
$\varepsilon = 7 \cdot 10^{-4}$

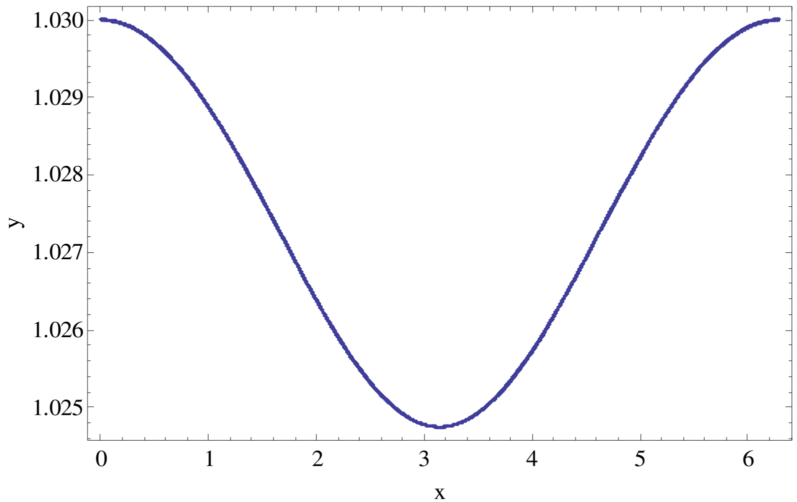
$\mu = 7 \cdot 10^{-4}$





1:1 resonance with slow drift





$$y(0) = 1.03$$

$$\varepsilon = 7 \cdot 10^{-4}$$

$$\mu = 7 \cdot 10^{-4}$$

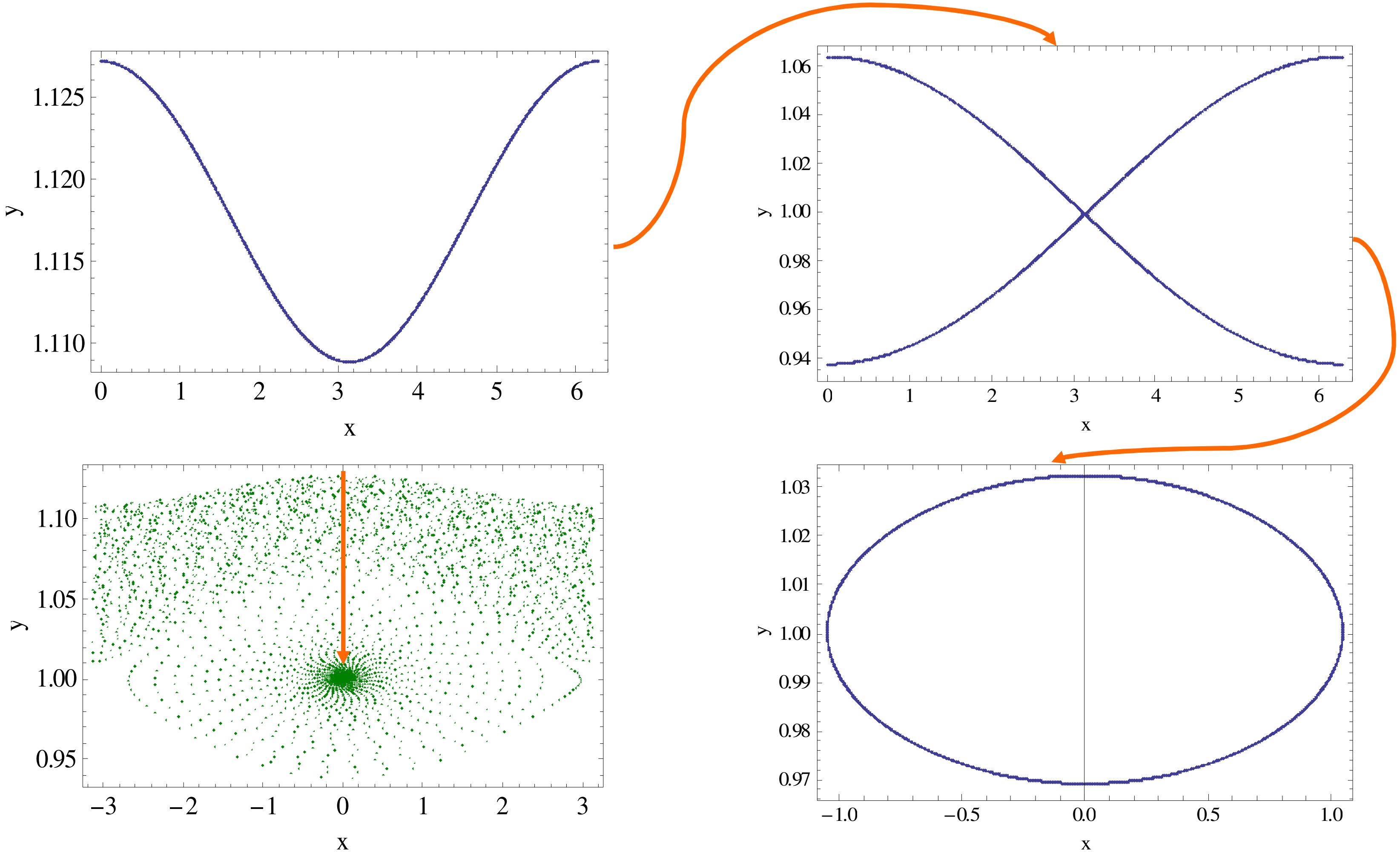
$$\dot{x} = y - \mu\,(\sin(x - t) + \sin(x))$$

$$\dot{y} = -\varepsilon\,(\sin(x - 6\,t) + \sin(x)) - \mu(y - \eta)$$

The actions are conserved for long times.

Resonance capture

Fast capture into resonance vs. capture into resonance on exponentially long time scales:



Possible Applications

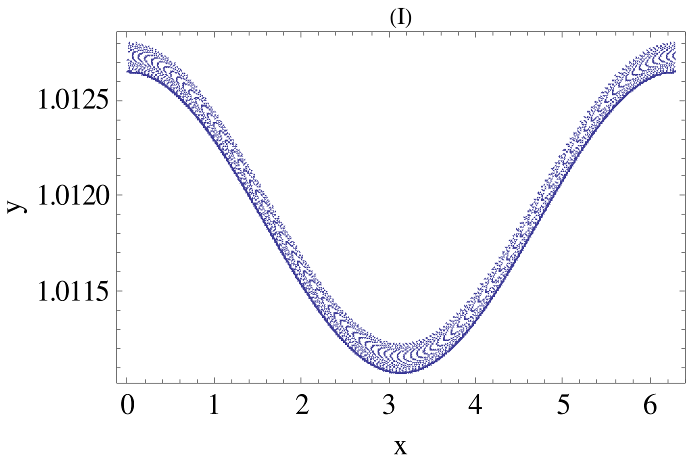
- Application of the Theorem to:
  - The asteroidal problem with dissipation
  - Rotational dynamics
  - Motion of artificial satellites

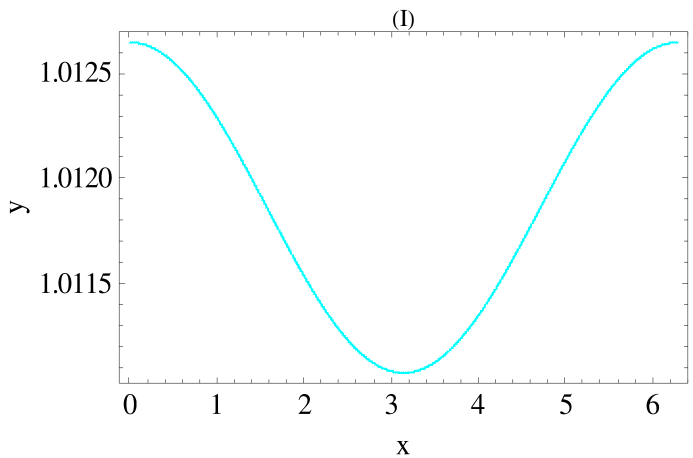
- Reference solution to dynamical systems
  - Qualitative investigation of the dynamics
  - Comparison of numerical integrations schemes

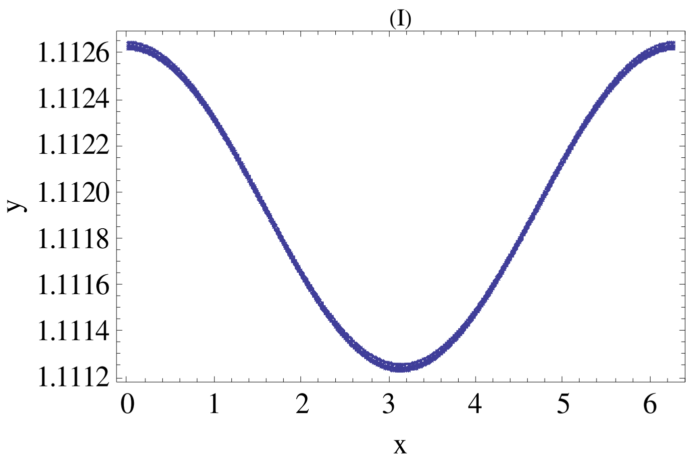
Numerical integrators

- Runge Kutta methods
  - Fehlberg, 4(3) to 9(8)
- Extrapolation methods
  - Adams, Burlish Stoer
- Implicit methods
  - Implicit Euler, implicit Runge-Kutta methods
- Symplectic methods
  - Leap frog, symplectic partitioned Runge-Kutta methods
- Integration by series approximation
  - Taylor methods, Lie-integration

Which integration is correct?



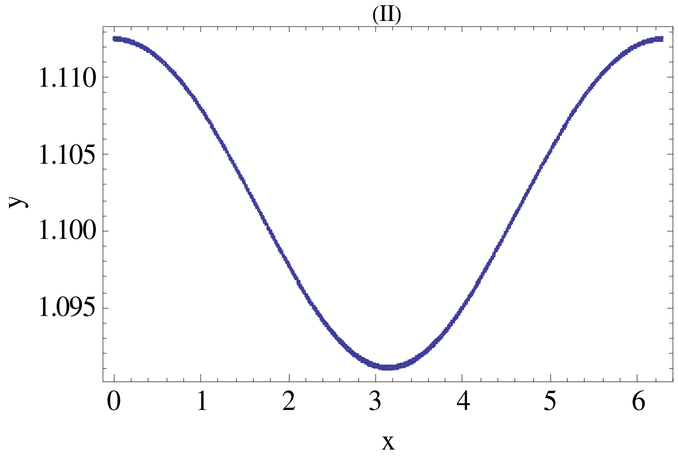


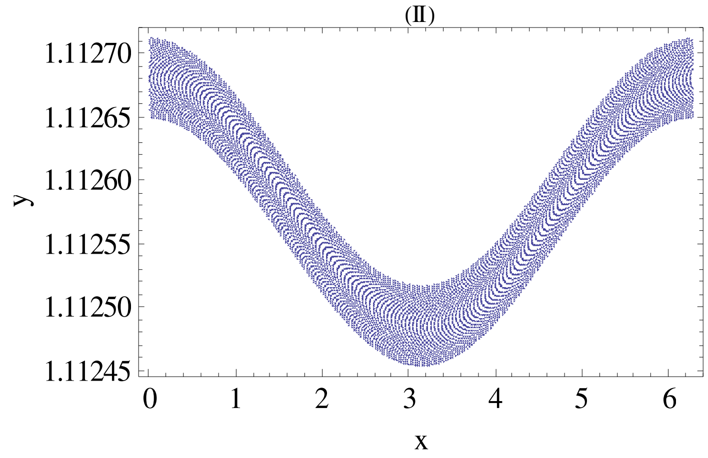


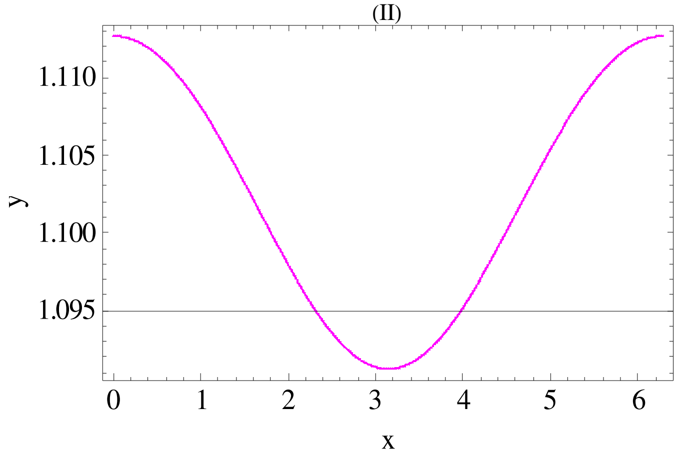
$$\dot{x} = y - \mu\,(\sin(x - t) + \sin(x))$$
$$\dot{y} = -\varepsilon\,(\sin(x - 6\,t) + \sin(x)) - \mu(y - \eta)$$

3 integrators, one dynamical system

See the difference !



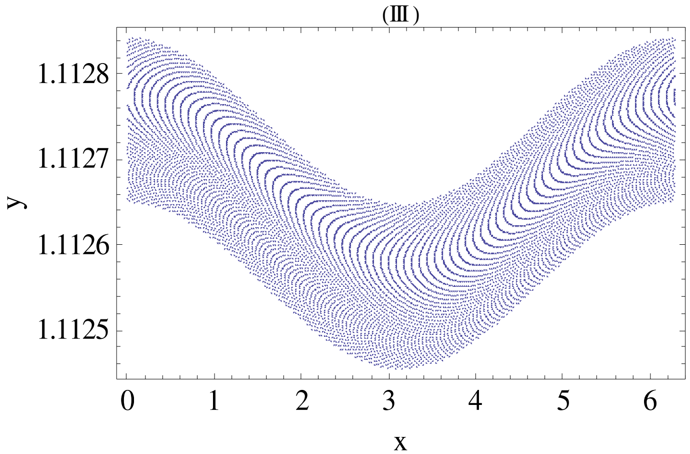


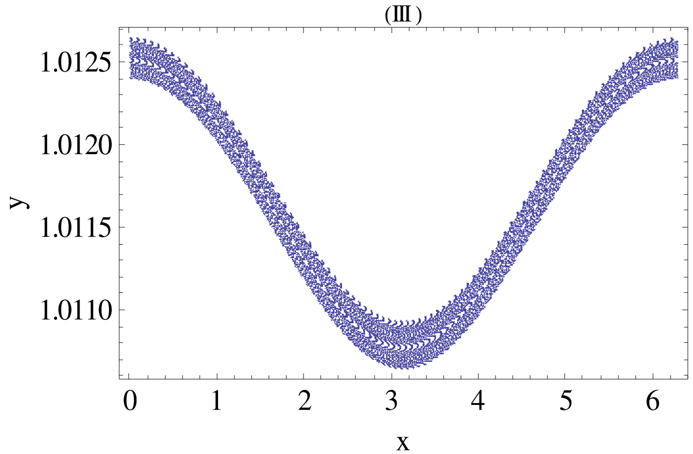


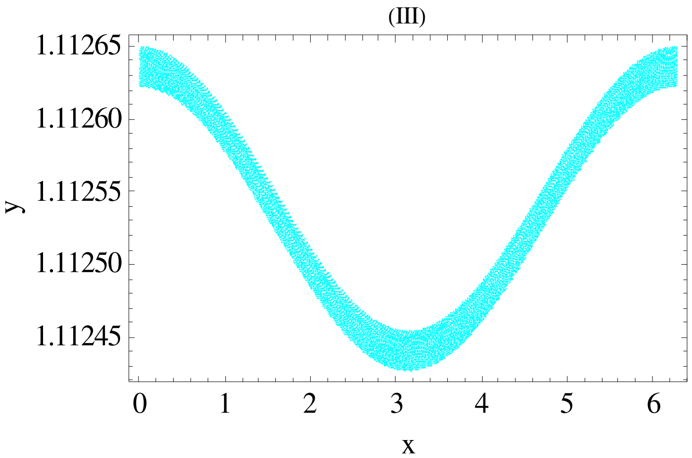
$$\dot{x} = y - \mu\,(\sin(6\,t))$$
$$\dot{y} = -\varepsilon\,(\sin(x - t) + \sin(x)) - \mu(y - \eta)$$

It can become difficult to decide if the drift is fast or slow

How much is the drift?



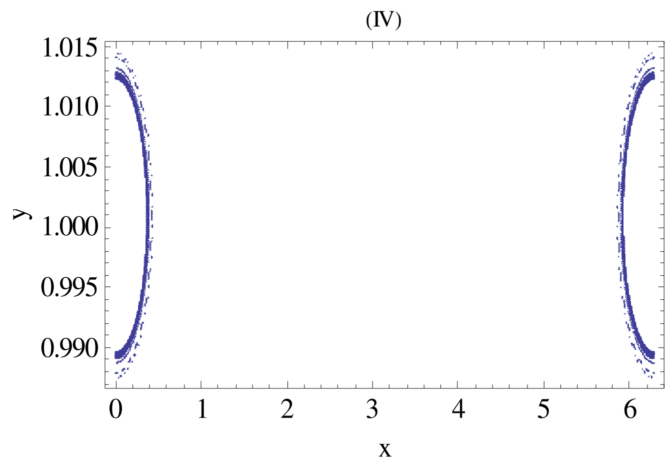


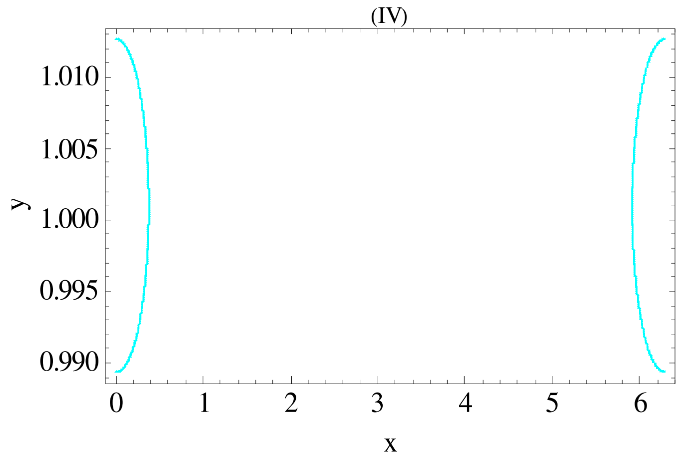


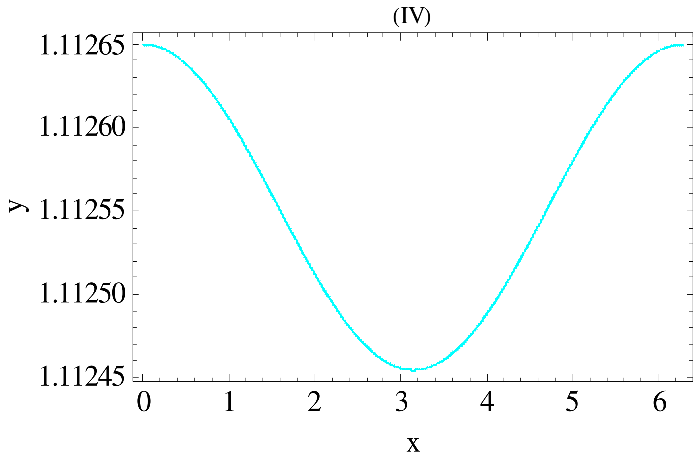
$$\dot{x} = y - \mu\,(\sin(x-t) + \sin(x))$$
$$\dot{y} = -\varepsilon\,(\sin(x-t) + \sin(x)) - \mu(y-\eta)$$

Here there is a drift. Or not?

The final example



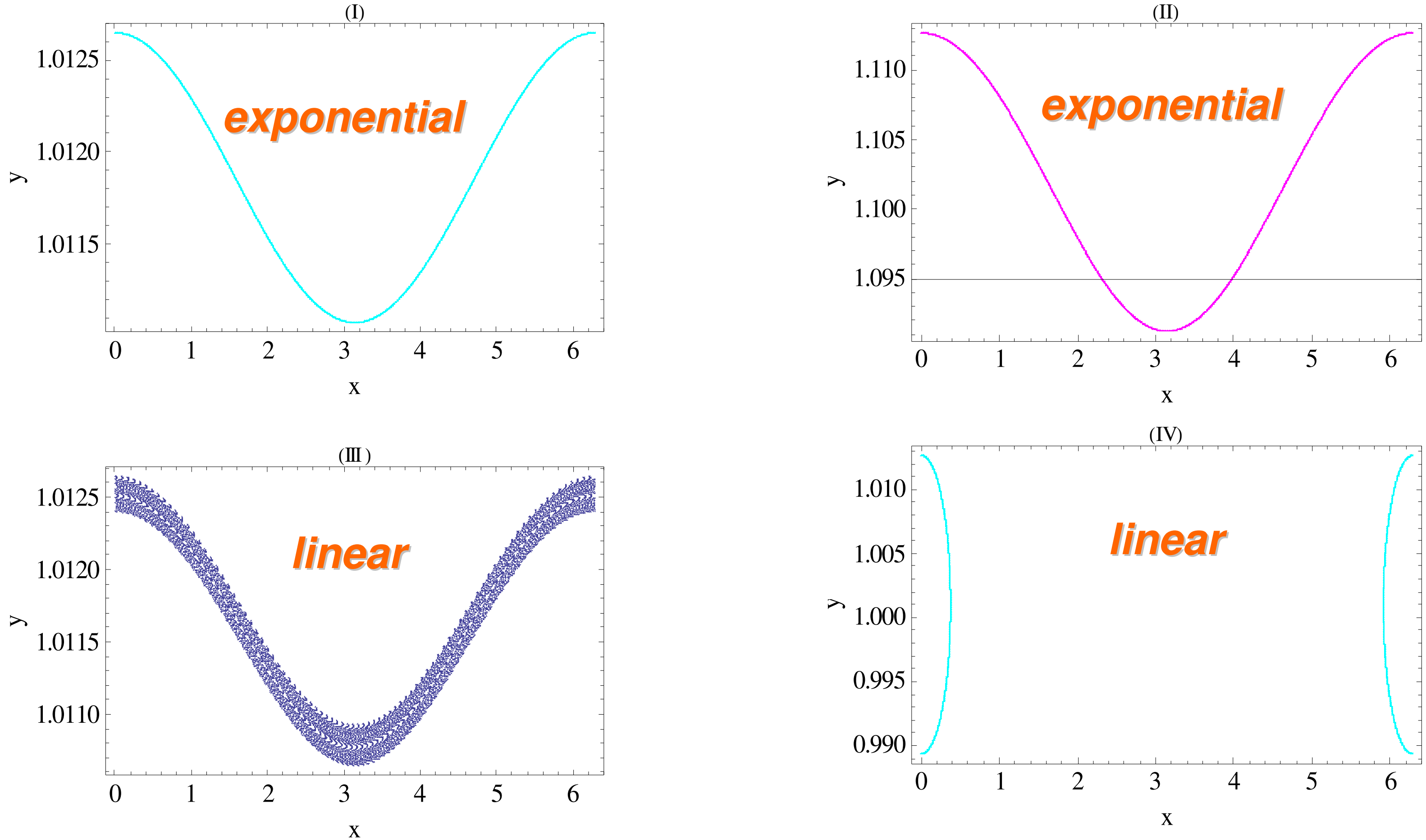




$$\dot{x} = y - \mu\,(\sin(x))$$
$$\dot{y} = -\varepsilon\,(\sin(x - t) + \sin(x)) - \mu(y - \eta)$$

Similar systems of ODEs BUT their normal forms are different

We need a reference solution



Only a reference (normal) form solution helps us to decide if the integration is correct.

Numerical integrator "contest"

The normal form solution allows to choose the best integration method and to adjust its parameters depending on the specific dynamical problem of interest.

Method	Steps	Cost	Error
1	{28849,0}	144247	4.54015*10^-14
2	{28849,0}	115398	8.38051*10^-14
3	{331927,0}	663856	5.46935*10^-12
4	{7033661,0}	7033663	4.80328*10^-11
5	{1667,311}	7914	2.89844*10^-9
6	{572,0}	7438	2.96213*10^-9
7	{413,0}	11178	2.89626*10^-8
8	{413,0}	11178	2.89626*10^-8
9	{413,0}	11178	2.89626*10^-8
10	{4846,0}	13206	7.72971*10^-8

1: implicit, 2: explicit Runge-Kutta (4), 3: midpoint (2), 4: Euler (fixed step)

5: Runge-Kutta (4), 6: Runge-Kutta (8) (variable stepsize)

7-9: extrapolation (order 2/4/8) (variable step)

10: Adams (multistep method)

Thank you for your attention!