The Smaller (SALI) and the Generalized (GALI) Alignment Index Methods of Chaos Detection: Theory and Applications

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Outline

- Smaller ALignment Index – SALI
  - ✓ Definition
  - ✓ Behavior for chaotic and regular motion
  - ✓ Applications

- Generalized ALignment Index – GALI
  - ✓ Definition - Relation to SALI
  - ✓ Behavior for chaotic and regular motion
  - ✓ Applications
  - ✓ Global dynamics
  - ✓ Motion on low-dimensional tori

- Conclusions

Definition of Smaller Alignment Index (SALI)

Consider the n-dimensional phase space of a conservative dynamical system (symplectic map or Hamiltonian flow).

An orbit in that space with initial condition :

$$P(0)=(x_1(0), x_2(0),\ldots,x_n(0))$$

and a deviation vector

$$v(0)=(dx_1(0), dx_2(0),\ldots, dx_n(0))$$

The evolution in time (in maps the time is discrete and is equal to the number N of the iterations) of a deviation vector is defined by:

- **the variational equations (for Hamiltonian flows) and**
- **the equations of the tangent map (for mappings)**

Definition of SALI

We follow the evolution in time of two different initial deviation vectors ($v_1(0)$, $v_2(0)$), and define SALI (Skokos, 2001, J. Phys. A, 34, 10029) as:

$$\mathrm{SALI}(t) = \min\left\{\left\|\hat{v}_1(t) + \hat{v}_2(t)\right\|, \left\|\hat{v}_1(t) - \hat{v}_2(t)\right\|\right\}$$

where

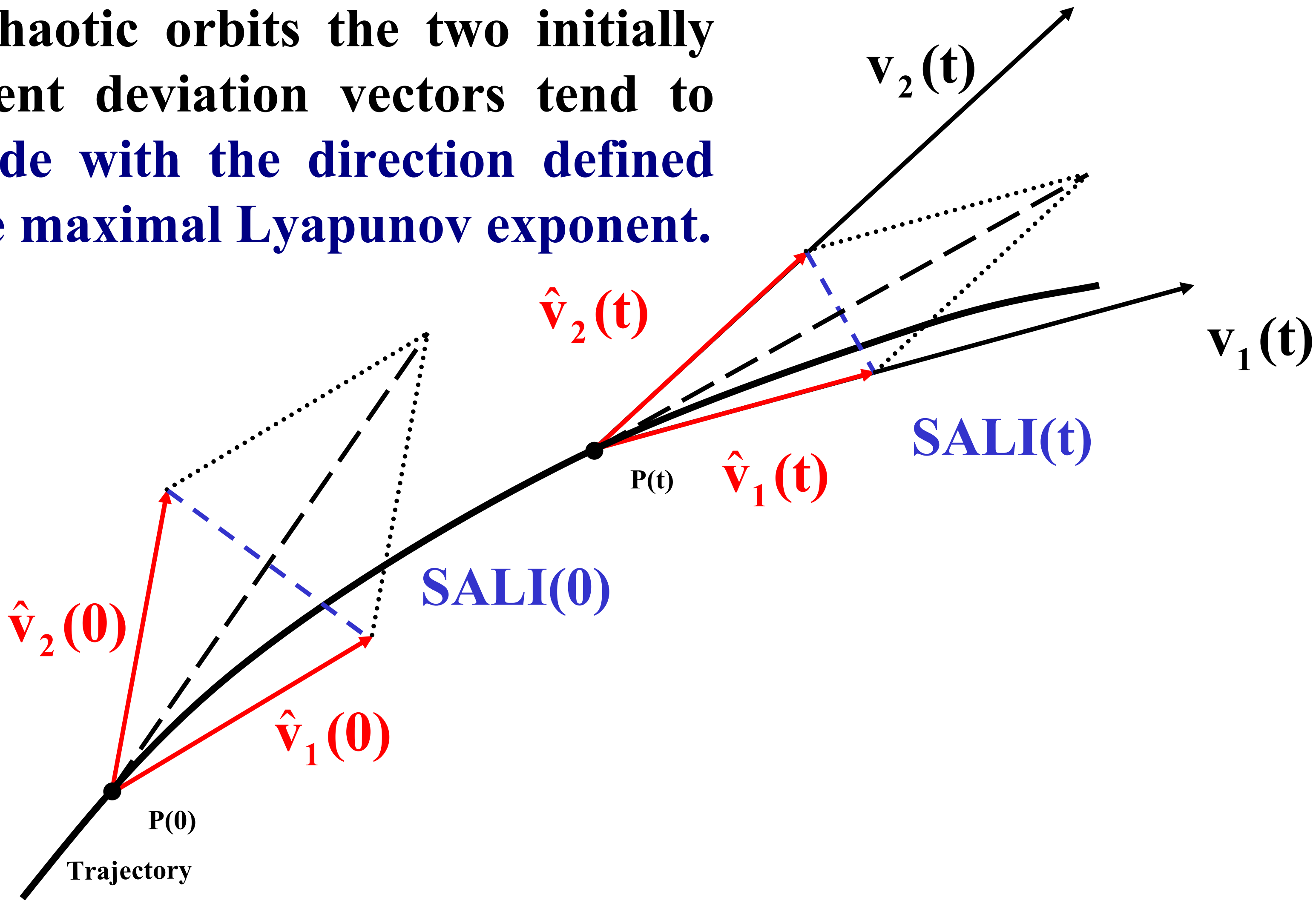
$$\hat{v}_1(t) = \frac{v_1(t)}{\left\|v_1(t)\right\|}$$

When the two vectors become collinear

$$\mathrm{SALI}(t) \rightarrow 0$$

Behavior of SALI for chaotic motion

For chaotic orbits the two initially different deviation vectors tend to coincide with the direction defined by the maximal Lyapunov exponent.

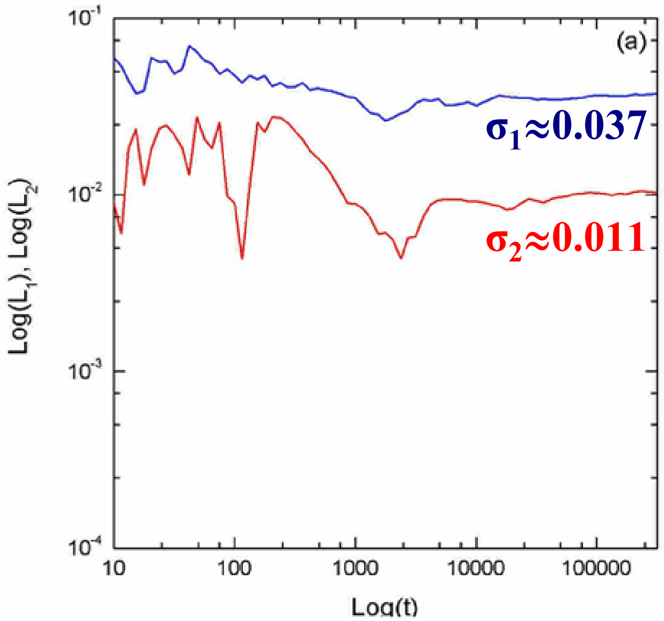


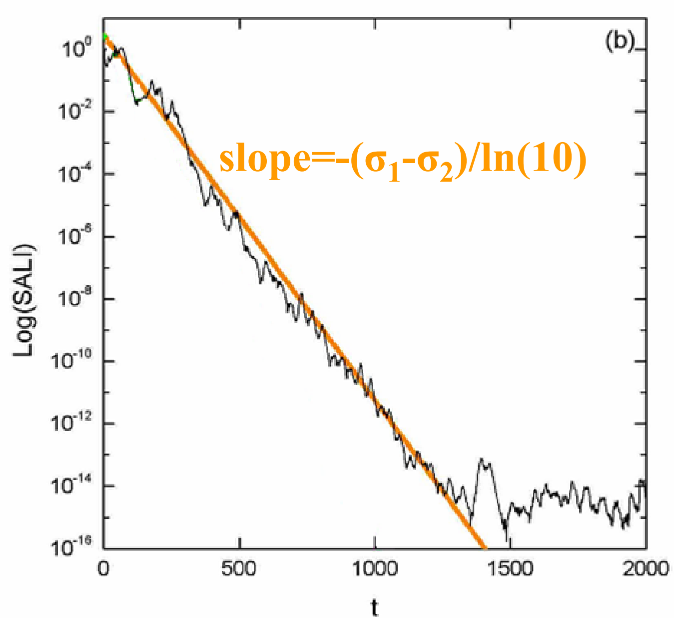
Behavior of SALI for chaotic motion

We test the validity of the approximation $\boxed{SALI \propto e^{-(\sigma 1 - \sigma 2)t}}$ (Skokos et al., 2004, J. Phys. A, 37, 6269) for a chaotic orbit of the 3D Hamiltonian

$$H = \sum_{i=1}^{3} \frac{\omega_i}{2}(q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3$$

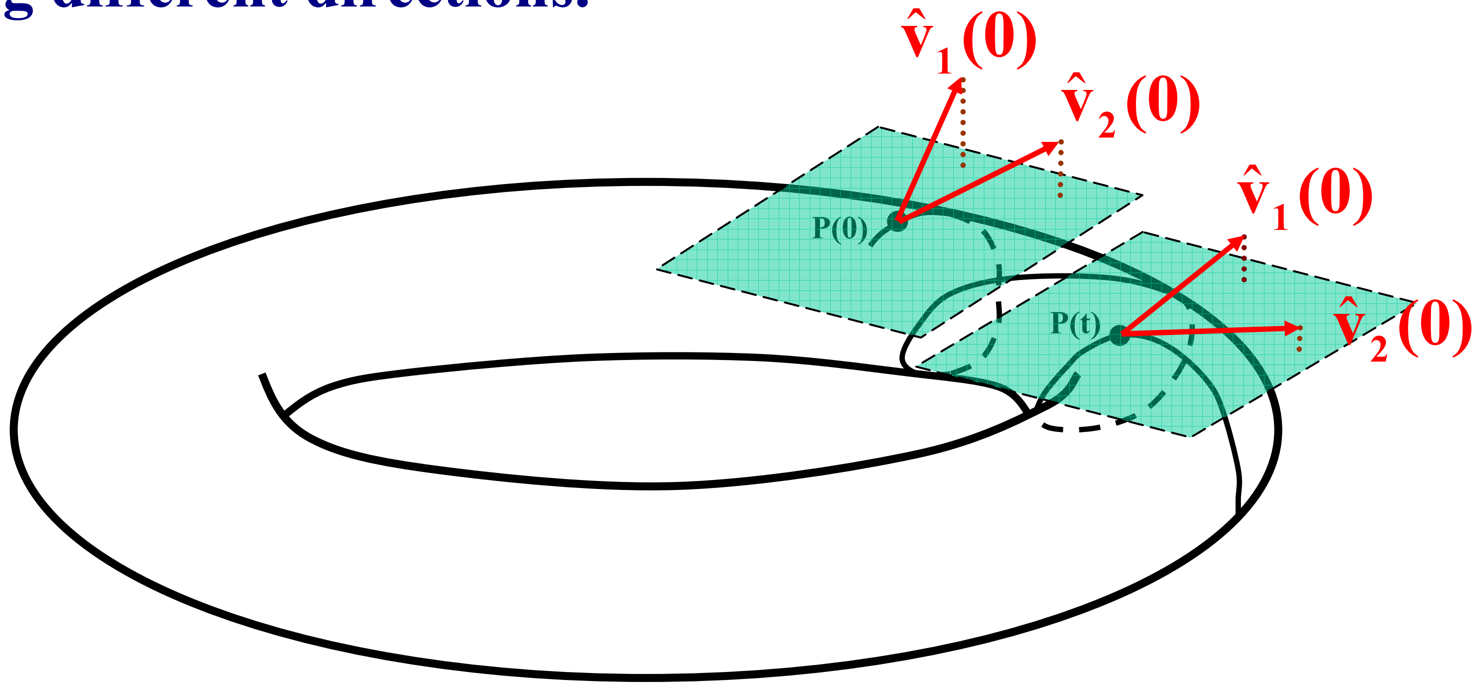
with ω_1=1, ω_2=1.4142, ω_3=1.7321, H=0.09





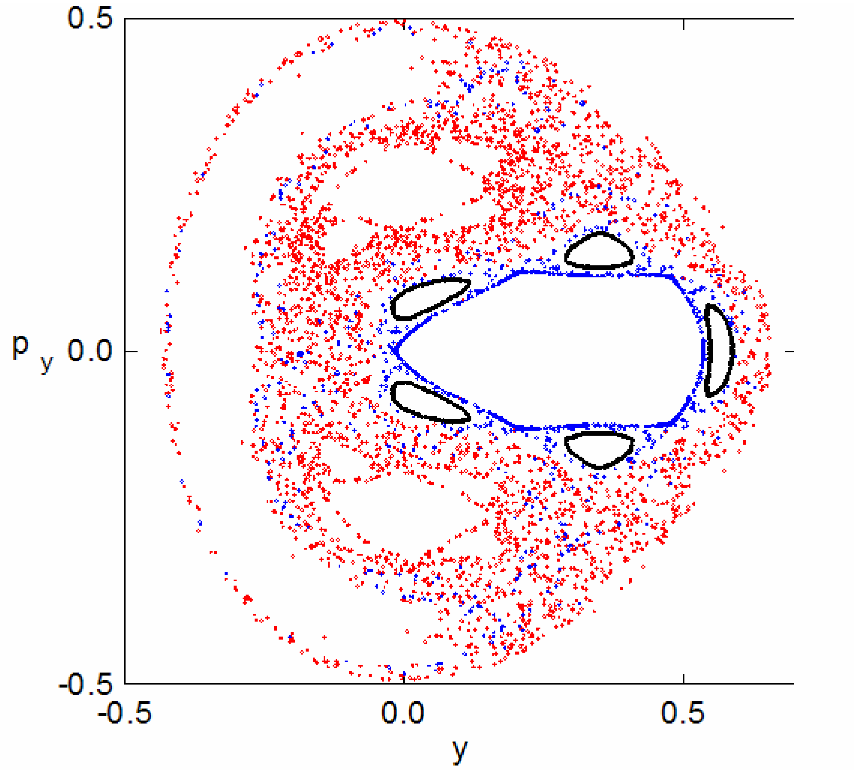
Behavior of SALI for regular motion

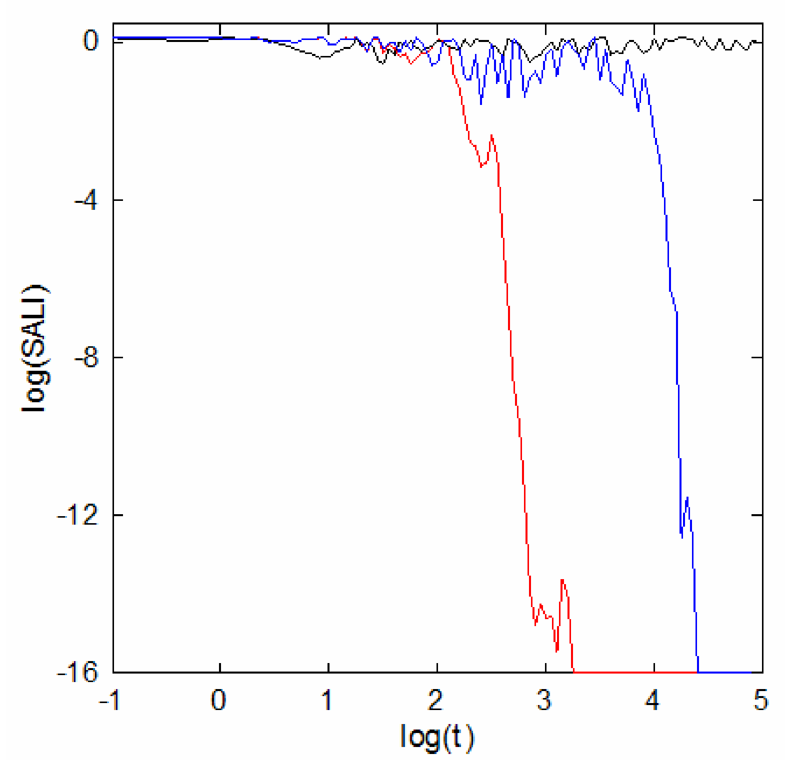
Regular motion occurs on a torus and two different initial deviation vectors become tangent to the torus, generally having different directions.



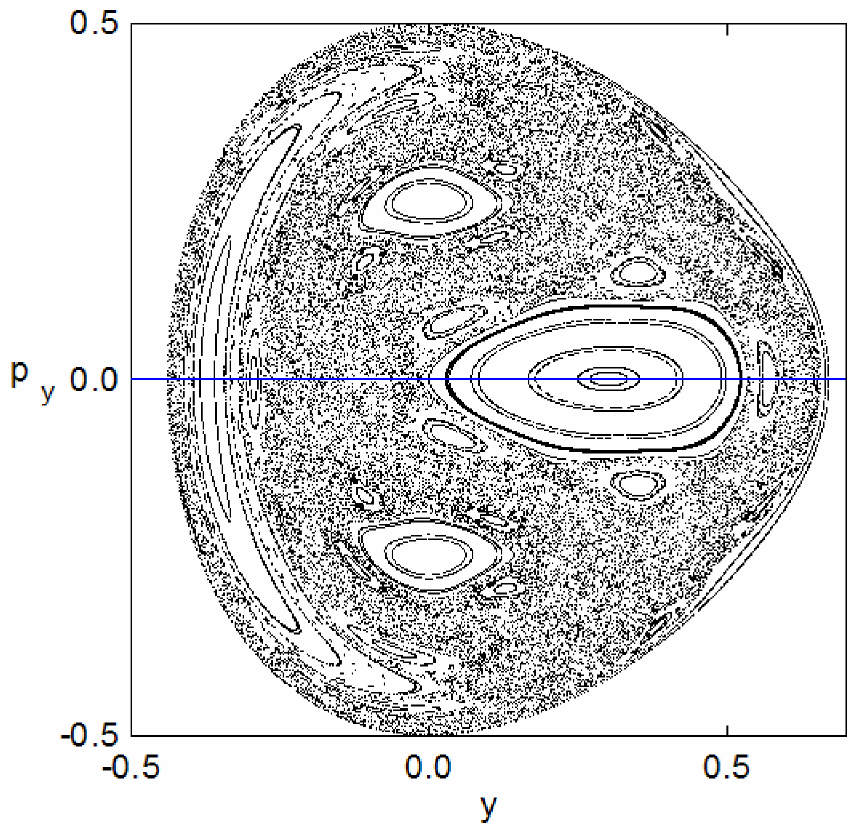
Applications – Hénon-Heiles system

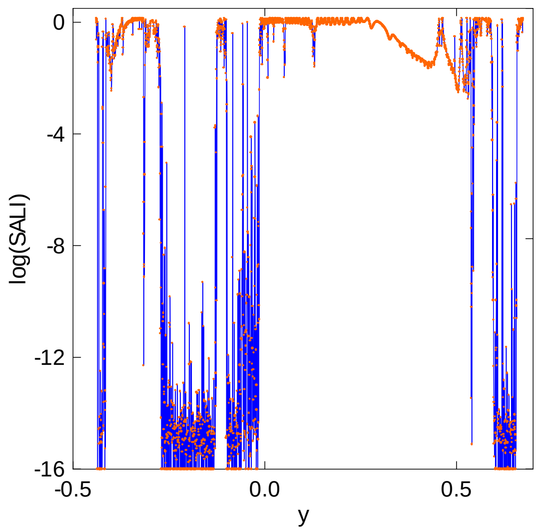
For E=1/8 we consider the orbits with initial conditions:
Ordered orbit, $x=0$, $y=0.55$, $p_x=0.2417$, $p_y=0$
Chaotic orbit, $x=0$, $y=-0.016$, $p_x=0.49974$, $p_y=0$
Chaotic orbit, $x=0$, $y=-0.01344$, $p_x=0.49982$, $p_y=0$



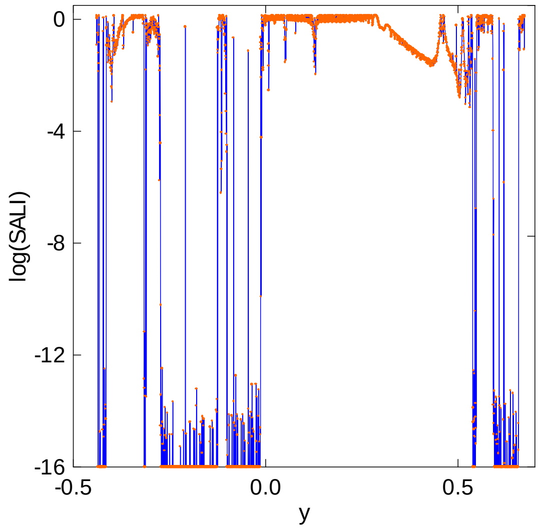


Applications – Hénon-Heiles system



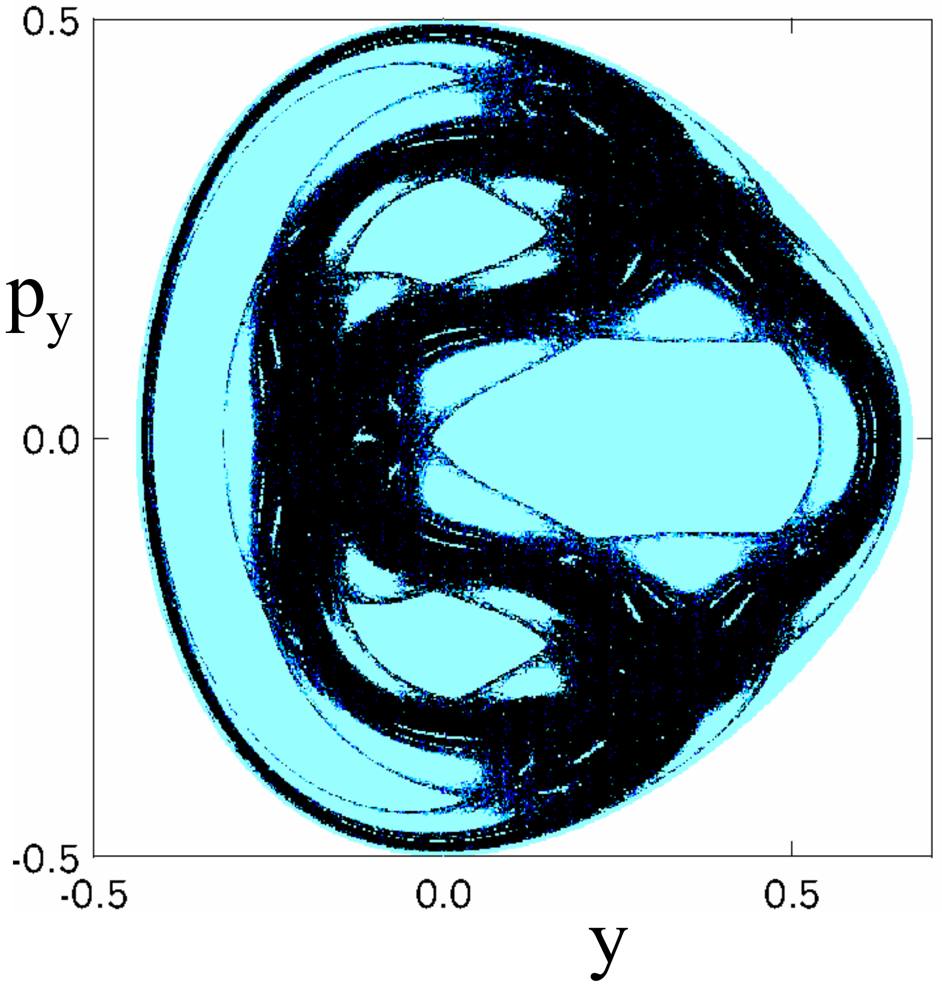


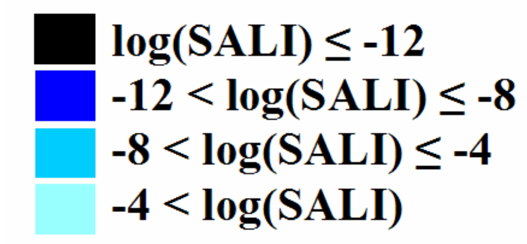
t=1000



t=4000

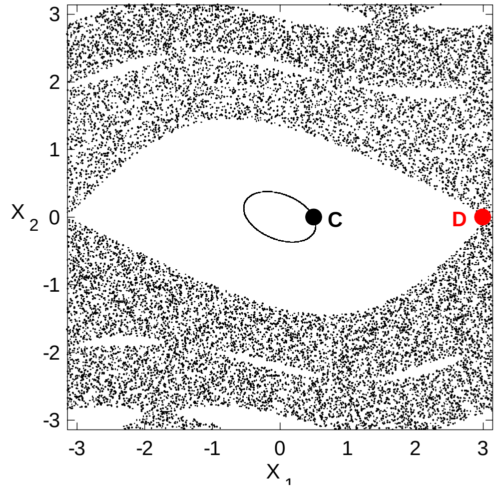
Applications – Hénon-Heiles system





Applications – 4D map

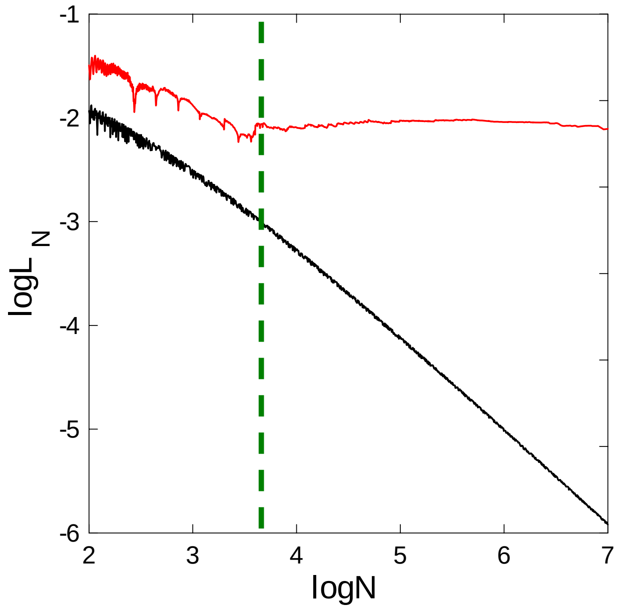
$$
\begin{aligned}
x_1' &= x_1 + x_2 \\
x_2' &= x_2 - \nu \sin(x_1 + x_2) - \mu\,[1 - \cos(x_1 + x_2 + x_3 + x_4)] \\
x_3' &= x_3 + x_4 \\
x_4' &= x_4 - \kappa \sin(x_3 + x_4) - \mu\,[1 - \cos(x_1 + x_2 + x_3 + x_4)]
\end{aligned}
\qquad (\mathrm{mod}\ 2\pi)
$$

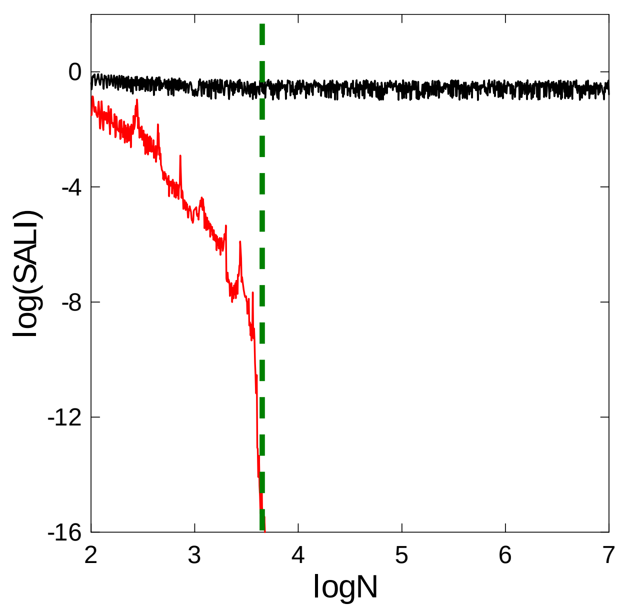


For ν=0.5, κ=0.1, μ=0.1 we consider the orbits:

ordered orbit C *with initial conditions x_1=0.5, x_2=0, x_3=0.5, x_4=0.*

chaotic orbit D *with initial conditions x_1=3, x_2=0, x_3=0.5, x_4=0.*





Applications – 4D Accelerator map

We consider the 4D symplectic map

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \cos\omega_1 & -\sin\omega_1 & 0 & 0 \\ \sin\omega_1 & \cos\omega_1 & 0 & 0 \\ 0 & 0 & \cos\omega_2 & -\sin\omega_2 \\ 0 & 0 & \sin\omega_2 & \cos\omega_2 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 + x_1^2 - x_3^2 \\ x_3 \\ x_4 - 2x_1x_3 \end{pmatrix}$$

describing the instantaneous sextupole 'kicks' experienced by a particle as it passes through an accelerator (Turchetti & Scandale 1991, Bountis & Tompaidis 1991, Vrahatis et al. 1996, 1997).

x_1 and x_3 are the particle's deflections from the ideal circular orbit, in the horizontal and vertical directions respectively.

x_2 and x_4 are the associated momenta

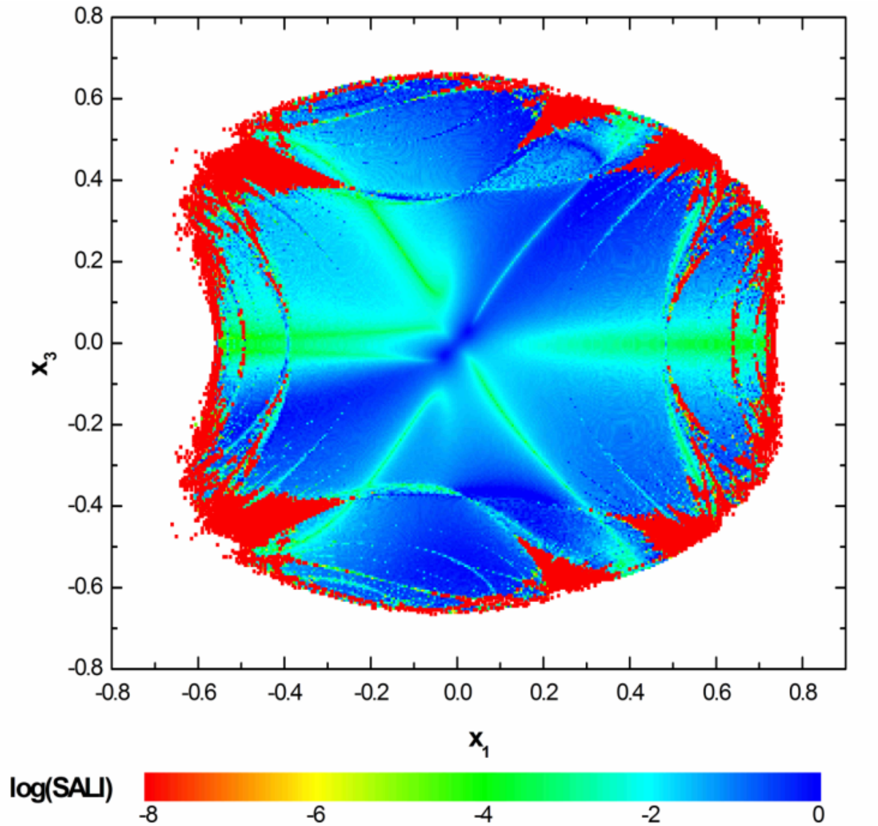
ω_1, ω_2 are related to the accelerator's tunes q_x, q_y by

$$\omega_1 = 2\pi q_x, \quad \omega_2 = 2\pi q_y$$

Our problem is to estimate the region of stability of the particle's motion, the so-called dynamic aperture of the beam (Bountis & Skokos, 2006, Nucl. Inst Meth. Phys Res. A, 561, 173).

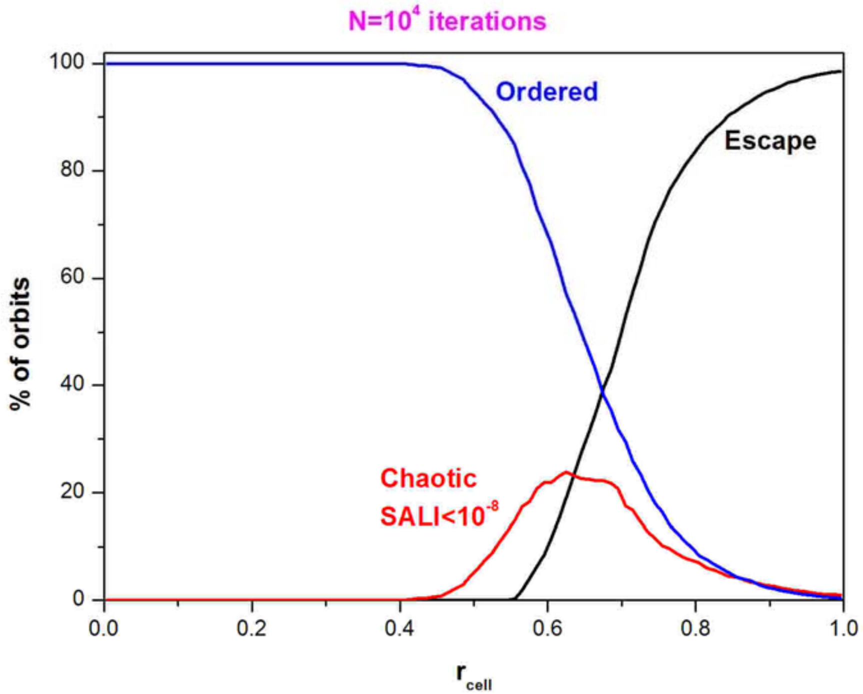
4D Accelerator map – "Global" study

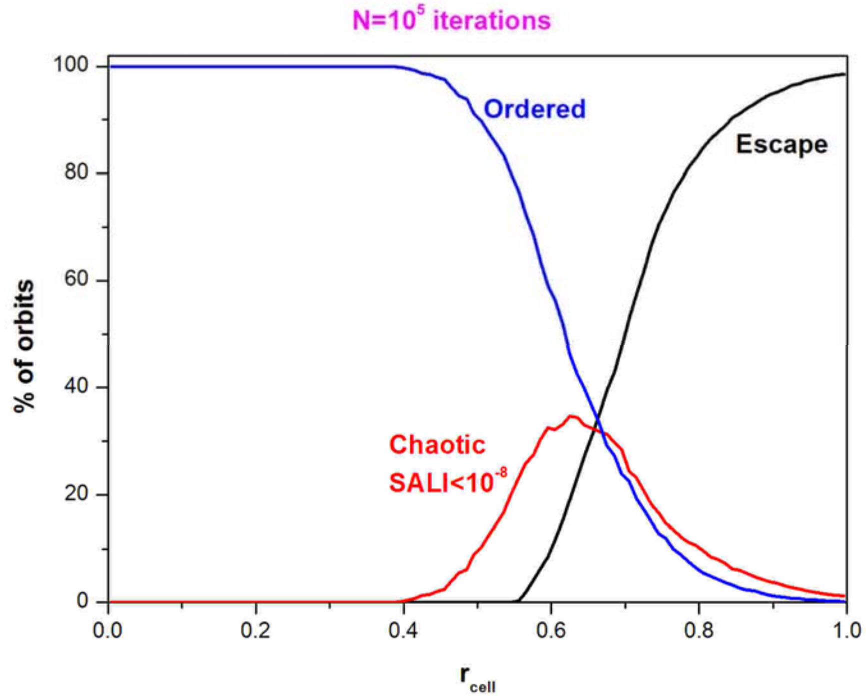
Regions of different values of the SALI on the subspace $x_2(0)=x_4(0)=0$, after 10^4 iterations (q_x=0.61803 q_y=0.4152)



4D Accelerator map – "Global" study

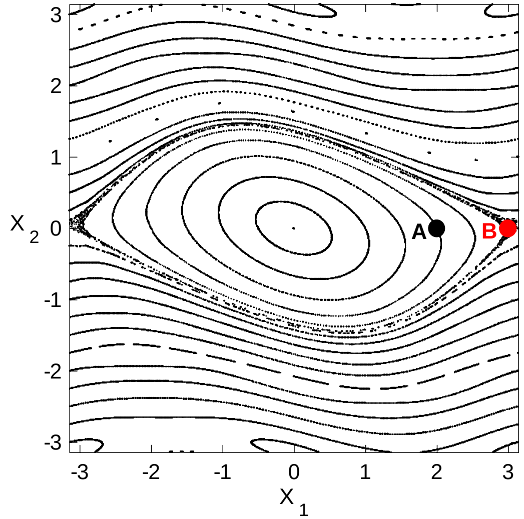
We consider 1,922,833 orbits by varying all x_1, x_2, x_3, x_4 within spherical shells of width 0.01 in a hypersphere of radius 1. (q_x=0.61803 q_y=0.4152)



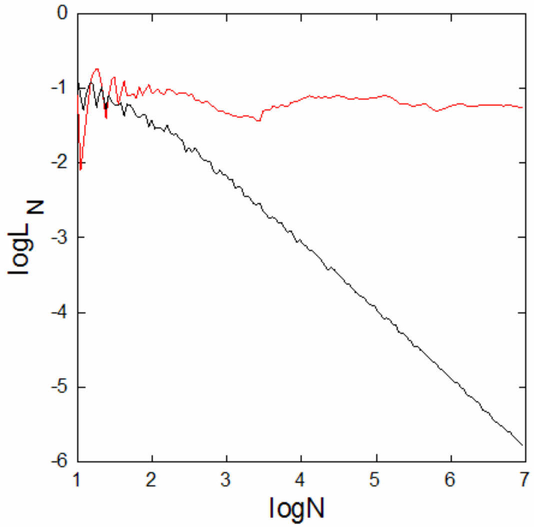


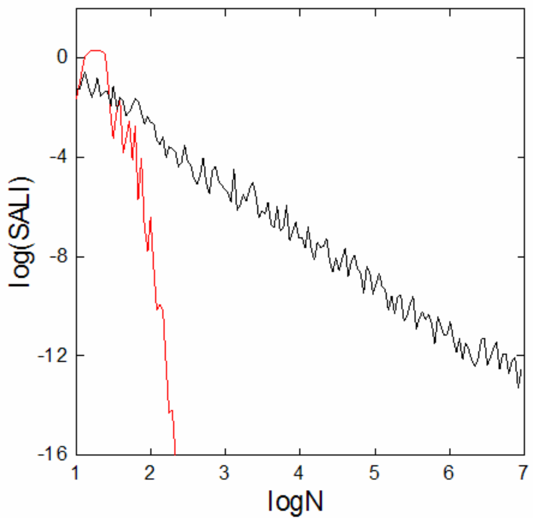
Applications – 2D map

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_2 - \nu \sin(x_1 + x_2) \end{aligned} \quad (\text{mod } 2\pi)$$



For ν=0.5 we consider the orbits:
ordered orbit A *with initial conditions $x_1=2$, $x_2=0$.*
chaotic orbit B *with initial conditions $x_1=3$, $x_2=0$.*





Behavior of SALI

2D maps

SALI$\rightarrow$0 both for regular and chaotic orbits

following, however, completely different time rates which allows us to distinguish between the two cases.

Hamiltonian flows and multidimensional maps

SALI$\rightarrow$0 for chaotic orbits

SALI$\rightarrow$constant $\neq$ 0 for regular orbits

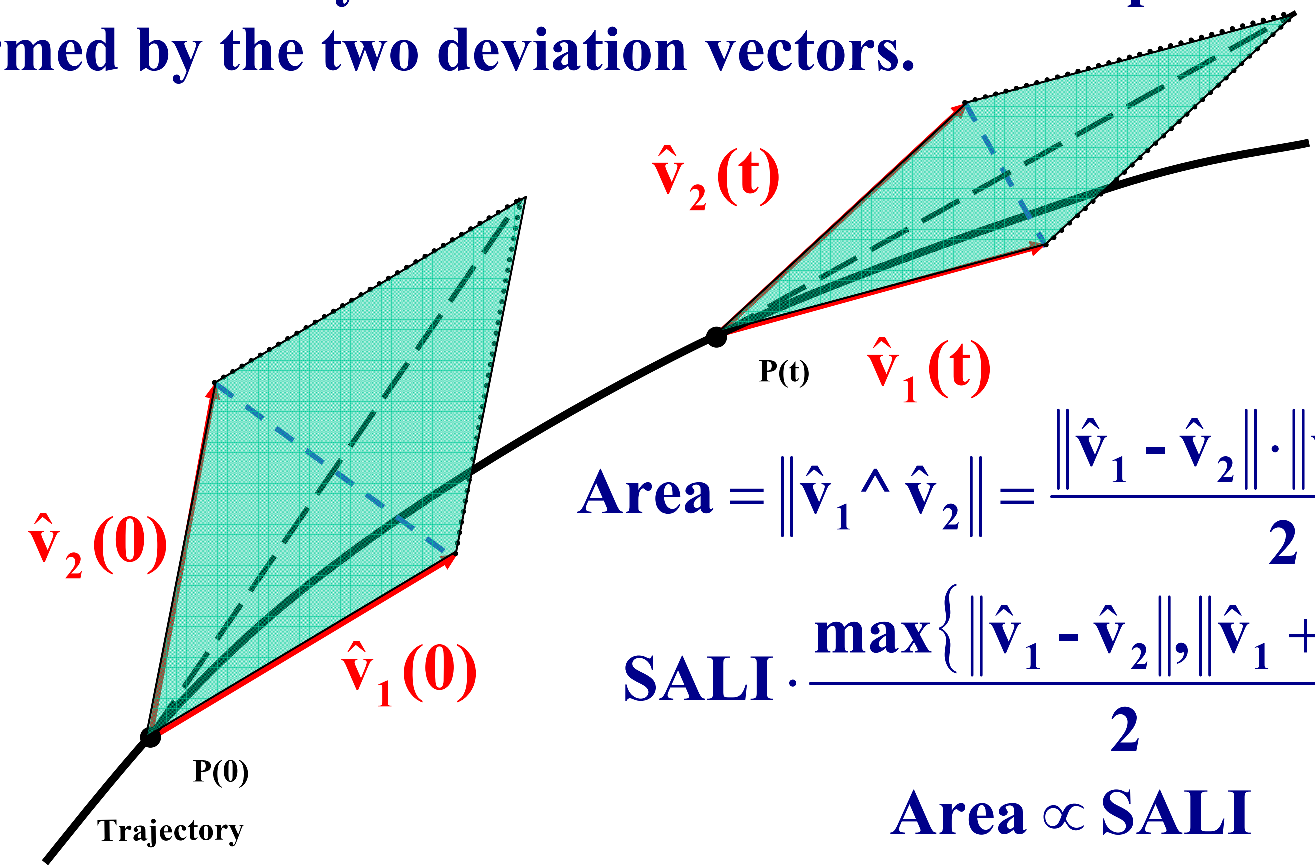
Questions

Can we generalize SALI so that the new index:

- **Can rapidly reveal the nature of chaotic orbits with $\sigma_1 \approx \sigma_2$ (SALI$\propto e^{-(\sigma_1-\sigma_2)t}$)?**
- **Depends on several Lyapunov exponents for chaotic orbits?**
- **Exhibits power-law decay for regular orbits depending on the dimensionality of the tangent space of the reference orbit as for 2D maps?**

Definition of Generalized Alignment Index (GALI)

SALI effectively measures the 'area' of the parallelogram formed by the two deviation vectors.



$$\text{Area} = \left\| \hat{v}_1 \wedge \hat{v}_2 \right\| = \frac{\left\| \hat{v}_1 - \hat{v}_2 \right\| \cdot \left\| \hat{v}_1 + \hat{v}_2 \right\|}{2} =$$

$$\text{SALI} \cdot \frac{\max\left\{ \left\| \hat{v}_1 - \hat{v}_2 \right\|, \left\| \hat{v}_1 + \hat{v}_2 \right\| \right\}}{2} \Rightarrow$$

$$\text{Area} \propto \text{SALI}$$

Definition of GALI

In the case of an N degree of freedom Hamiltonian system or a 2N symplectic map we follow the evolution of

k deviation vectors with 2≤k≤2N,

and define (Skokos et al., 2007, Physica D, 231, 30) the Generalized Alignment Index (GALI) of order k :

$$\boxed{\mathrm{GALI}_k(t) = \left\| \hat{v}_1(t) \wedge \hat{v}_2(t) \wedge \ldots \wedge \hat{v}_k(t) \right\|}$$

where

$$\hat{v}_1(t) = \frac{v_1(t)}{\left\| v_1(t) \right\|}$$

Wedge product

We consider as a basis of the 2N-dimensional tangent space of the Hamiltonian flow the usual set of orthonormal vectors:

$$\hat{e}_1 = (1,0,0,...,0),\ \hat{e}_2 = (0,1,0,...,0),\ ...,\ \hat{e}_{2N} = (0,0,0,...,1)$$

Then for k deviation vectors we have:

$$\begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \vdots \\ \hat{v}_k \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1\,2N} \\ v_{21} & v_{22} & \cdots & v_{2\,2N} \\ \vdots & \vdots & & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{k\,2N} \end{bmatrix} \cdot \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \\ \hat{e}_{2N} \end{bmatrix}$$

$$\hat{v}_1 \wedge \hat{v}_2 \wedge \cdots \wedge \hat{v}_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2N} \begin{vmatrix} v_{1\,i_1} & v_{1\,i_2} & \cdots & v_{1\,i_k} \\ v_{2\,i_1} & v_{2\,i_2} & \cdots & v_{2\,i_k} \\ \vdots & \vdots & & \vdots \\ v_{k\,i_1} & v_{k\,i_2} & \cdots & v_{k\,i_k} \end{vmatrix} \hat{e}_{i_1} \wedge \hat{e}_{i_2} \wedge \cdots \wedge \hat{e}_{i_k}$$

Computation of GALI

For k deviation vectors:

$$\begin{bmatrix} \hat{\mathbf{v}}_1 \\ \hat{\mathbf{v}}_2 \\ \vdots \\ \hat{\mathbf{v}}_k \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1\,2N} \\ v_{21} & v_{22} & \cdots & v_{2\,2N} \\ \vdots & \vdots & & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{k\,2N} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \vdots \\ \hat{\mathbf{e}}_{2N} \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \vdots \\ \hat{\mathbf{e}}_{2N} \end{bmatrix}$$

the 'norm' of the wedge product is given by:

$$\left\| \hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2 \wedge \cdots \wedge \hat{\mathbf{v}}_k \right\| = \left\{ \sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2N} \begin{vmatrix} v_{1\,i_1} & v_{1\,i_2} & \cdots & v_{1\,i_k} \\ v_{2\,i_1} & v_{2\,i_2} & \cdots & v_{2\,i_k} \\ \vdots & \vdots & & \vdots \\ v_{k\,i_1} & v_{k\,i_2} & \cdots & v_{k\,i_k} \end{vmatrix}^2 \right\}^{1/2} = \sqrt{\det(\mathbf{A} \cdot \mathbf{A}^T)}$$

Computation of GALI

From **Singular Value Decomposition (SVD) of A^T** we get:

$$A^T = U \cdot W \cdot V^T$$

where U is a column-orthogonal 2N×k matrix **($U^T \cdot U = I$)**, V^T is a k×k orthogonal matrix **($V \cdot V^T = I$)**, and W is a diagonal k×k matrix with positive or zero elements, the so-called **singular values.** So, we get:

$$\det(A \cdot A^T) = \det(V \cdot W^T \cdot U^T \cdot U \cdot W \cdot V^T) = \det(V \cdot W \cdot I \cdot W \cdot V^T) =$$

$$\det(V \cdot W^2 \cdot V^T) = \det(V \cdot \mathrm{diag}(w_1^2, w_2^2, \ldots w_k^2) \cdot V^T) = \prod_{i=1}^{k} w_i^2$$

Thus, $GALI_k$ is computed by:

$$\boxed{GALI_k = \sqrt{\det(A \cdot A^T)} = \prod_{i=1}^{k} w_i \Rightarrow \log\left(GALI_k\right) = \sum_{i=1}^{k} \log\left(w_i\right)}$$

Behavior of $GALI_k$ for chaotic motion

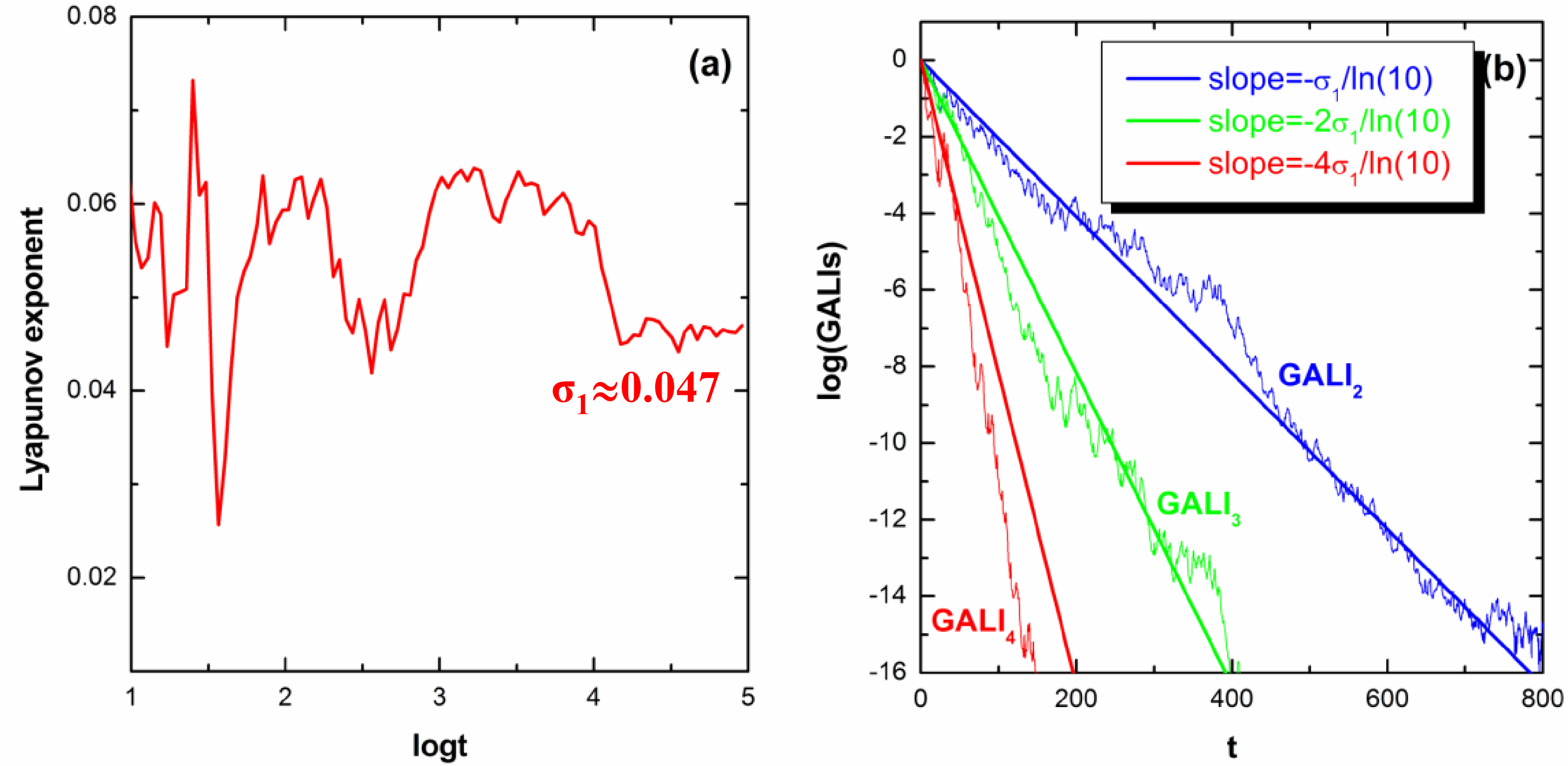
$GALI_k$ ($2 \leq k \leq 2N$) tends exponentially to zero with exponents that involve the values of the first k largest Lyapunov exponents $\sigma_1, \sigma_2, \ldots, \sigma_k$:

$$\boxed{GALI_k(t) \propto e^{-\left[(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3)+\ldots+(\sigma_1-\sigma_k)\right]t}}$$

The above relation is valid even if some Lyapunov exponents are equal, or very close to each other.

Behavior of $GALI_k$ for chaotic motion

2D Hamiltonian (Hénon-Heiles system)

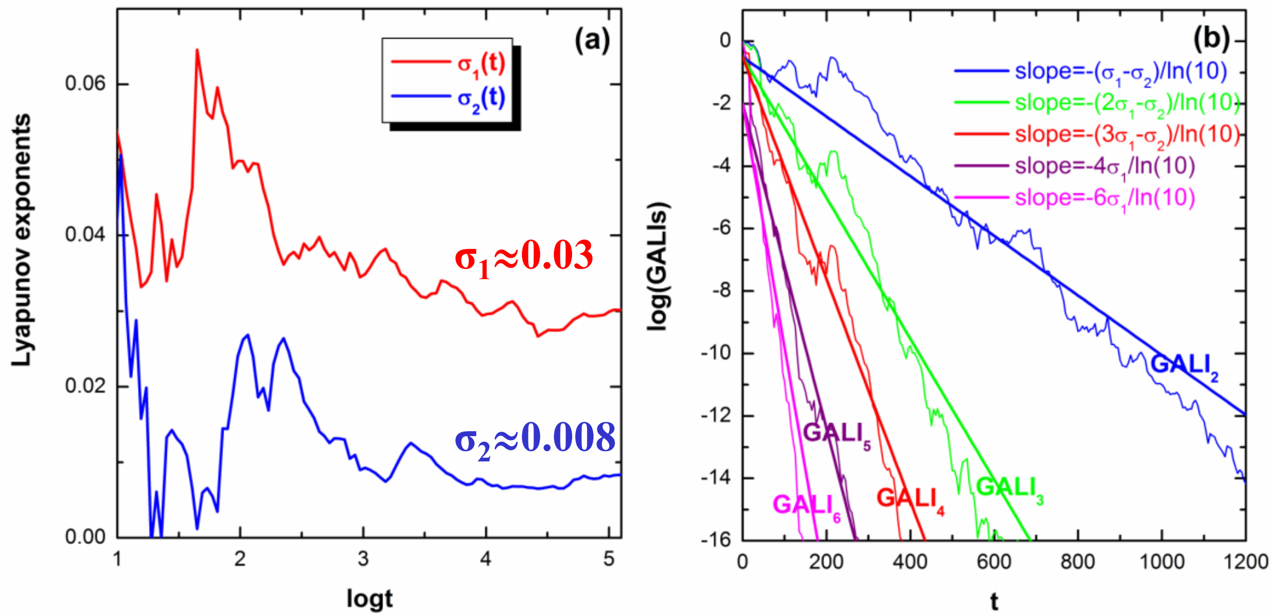


Behavior of GALI_k for chaotic motion

3D system:

$$H_3 = \sum_{i=1}^{3} \frac{\omega_i}{2}(q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3$$

with ω_1=1, $\omega_2=\sqrt{2}$, $\omega_3=\sqrt{3}$, H_3=0.09.

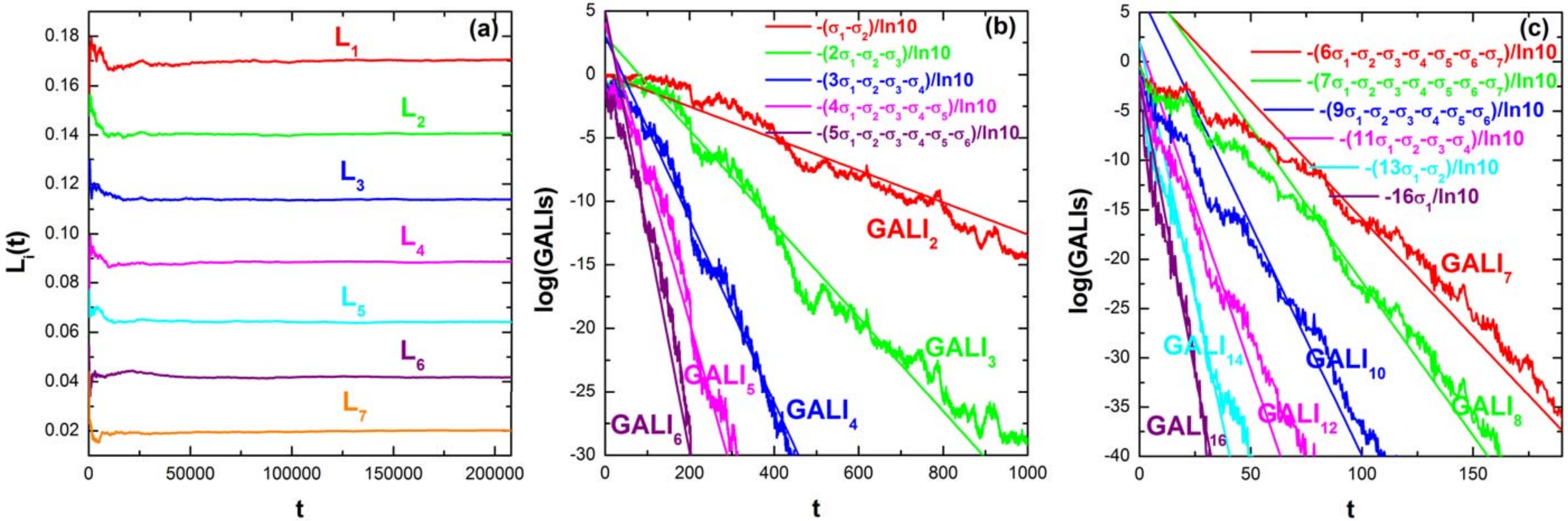


Behavior of $GALI_k$ for chaotic motion

N particles Fermi-Pasta-Ulam (FPU) system:

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=0}^{N}\left[\frac{1}{2}(q_{i+1} - q_i)^2 + \frac{\beta}{4}(q_{i+1} - q_i)^4\right]$$

with fixed boundary conditions, N=8 and β=1.5.



Behavior of $GALI_k$ for regular motion

If the motion occurs on an s-dimensional torus with s≤N then the behavior of $GALI_k$ is given by (Skokos et al., 2008, EPJ-ST, 165, 5):

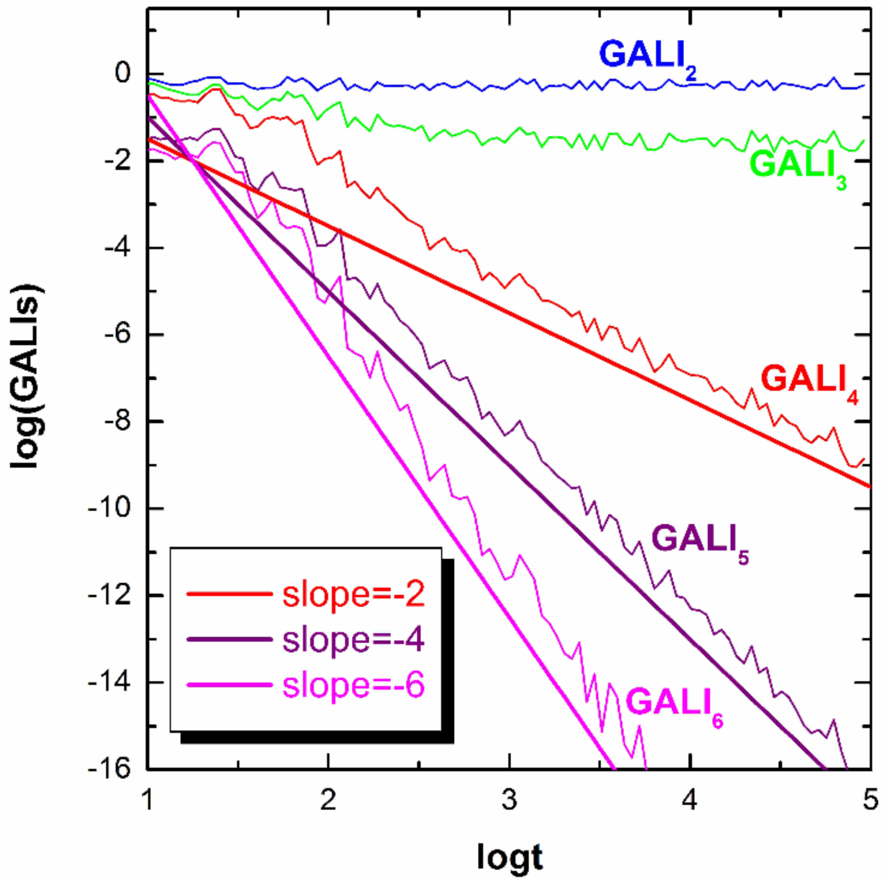
$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if} \quad 2 \leq k \leq s \\ \dfrac{1}{t^{k-s}} & \text{if} \quad s < k \leq 2N-s \\ \dfrac{1}{t^{2(k-N)}} & \text{if} \quad 2N-s < k \leq 2N \end{cases}$$

while in the common case with s=N we have :

$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if} \quad 2 \leq k \leq N \\ \dfrac{1}{t^{2(k-N)}} & \text{if} \quad N < k \leq 2N \end{cases}$$

Behavior of $GALI_k$ for regular motion

3D Hamiltonian



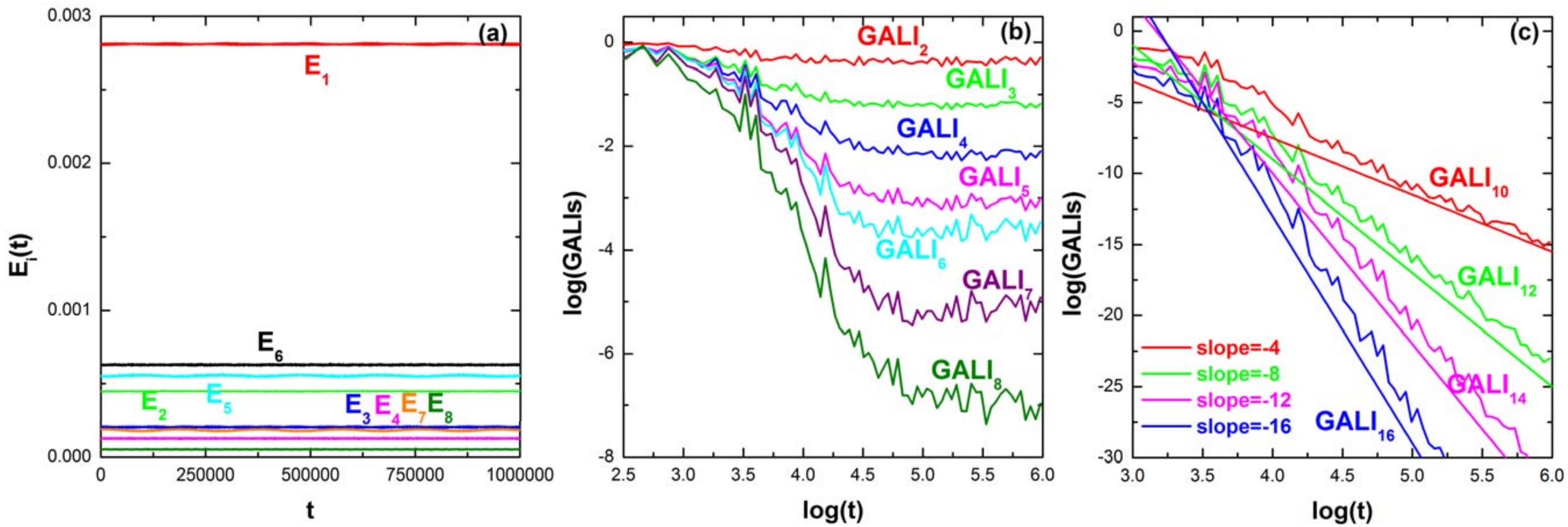
Behavior of $GALI_k$ for regular motion

N=8 FPU system: The unperturbed Hamiltonian (β=0) is written as a sum of the so-called harmonic energies E_i:

$$E_i = \frac{1}{2}\left(P_i^2 + \omega_i^2 Q_i^2\right), \; i = 1, \ldots, N$$

with:

$$Q_i = \sqrt{\frac{2}{N+1}} \sum_{i=1}^{N} q_i \sin\left(\frac{ki\pi}{N+1}\right), \; P_i = \sqrt{\frac{2}{N+1}} \sum_{i=1}^{N} p_i \sin\left(\frac{ki\pi}{N+1}\right), \; \omega_i = 2\sin\left(\frac{i\pi}{2(N+1)}\right)$$



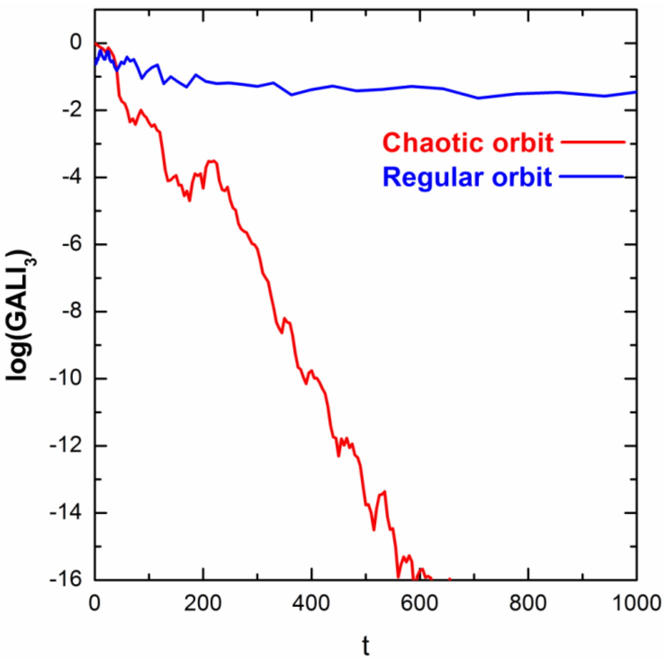
Global dynamics

• **$GALI_2$ (practically equivalent to the use of SALI)**

• **$GALI_N$**

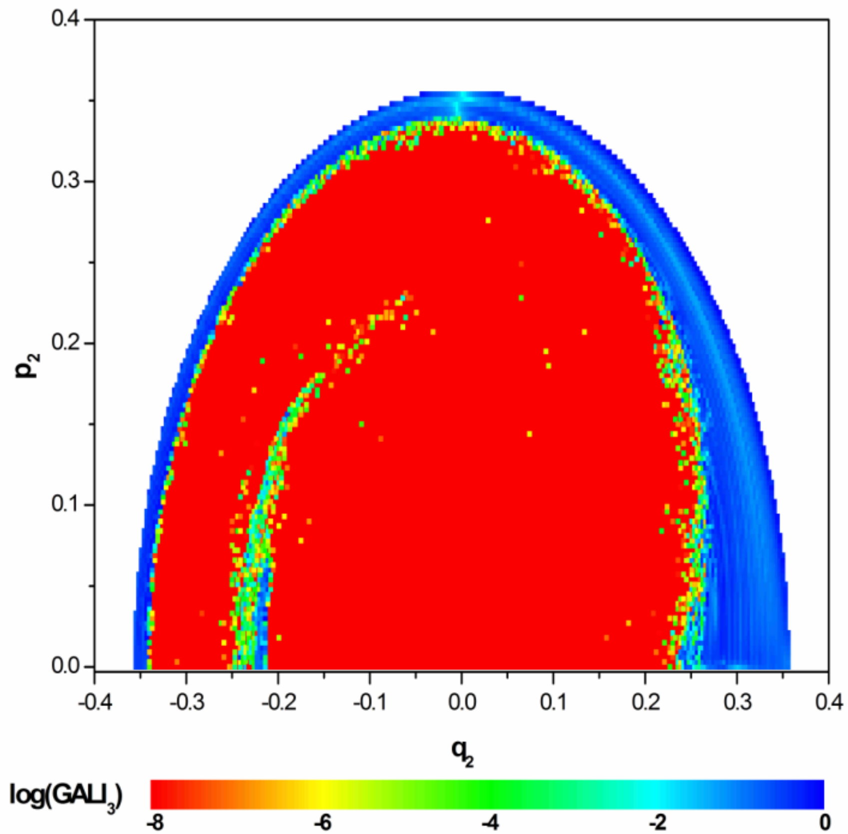
Chaotic motion: $GALI_N \rightarrow 0$ (exponential decay)

Regular motion: $GALI_N \rightarrow \text{constant} \neq 0$



3D Hamiltonian

Subspace $q_3=p_3=0$, $p_2\geq0$ for $t=1000$.



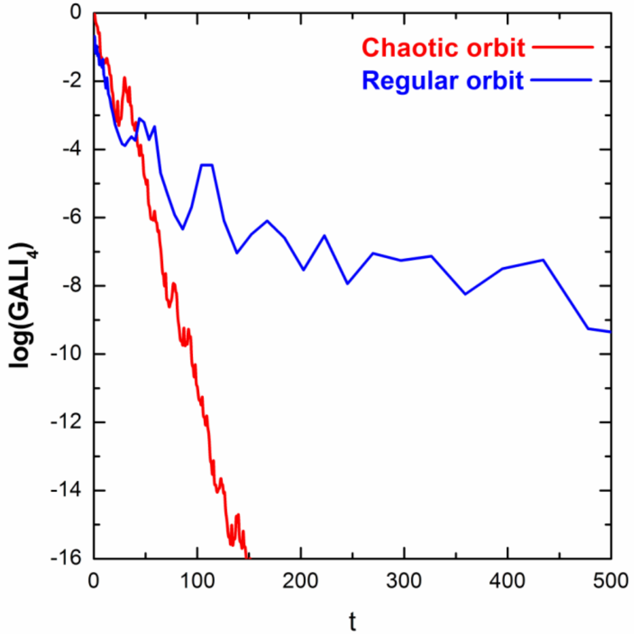
Global dynamics

GALI$_k$ with k>N

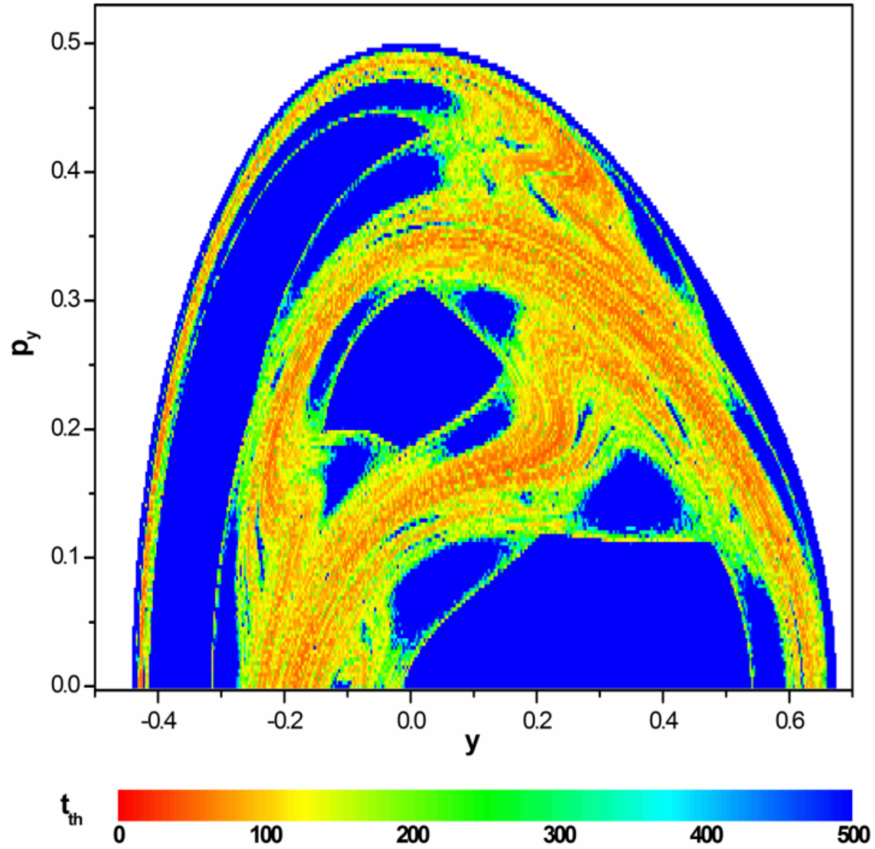
The index tends to zero both for regular and chaotic orbits but with completely different time rates:

Chaotic motion: exponential decay

Regular motion: power law

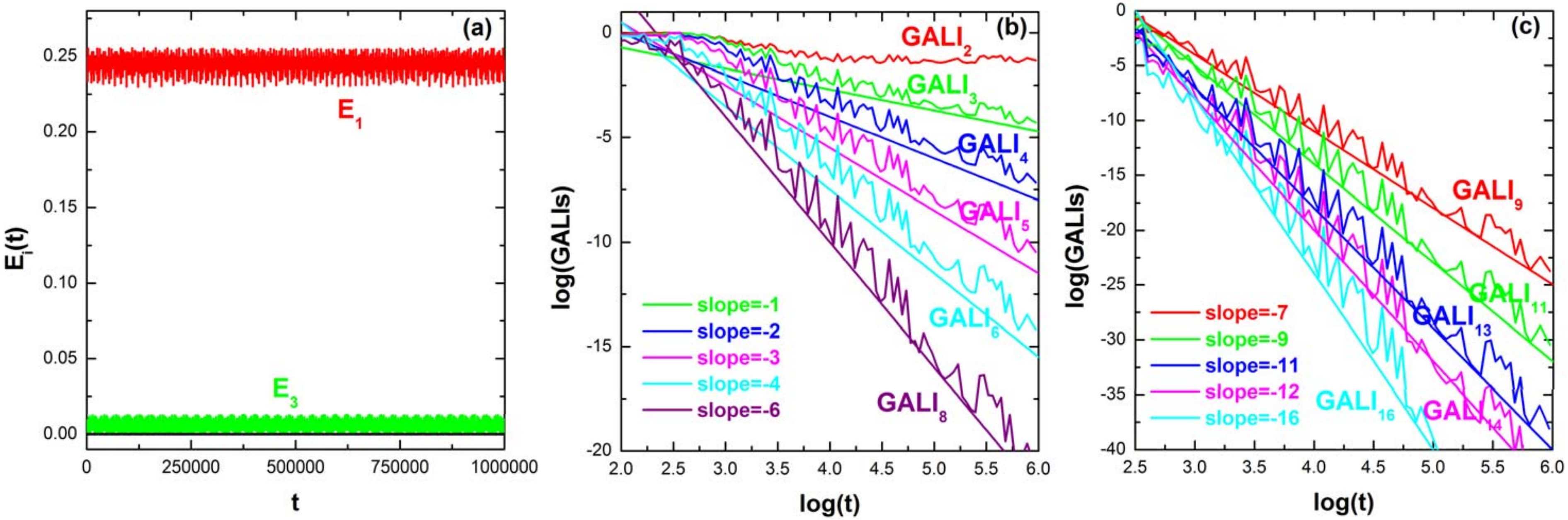


2D Hamiltonian (Hénon-Heiles)
Time needed for GALI$_4$<10^{-12}



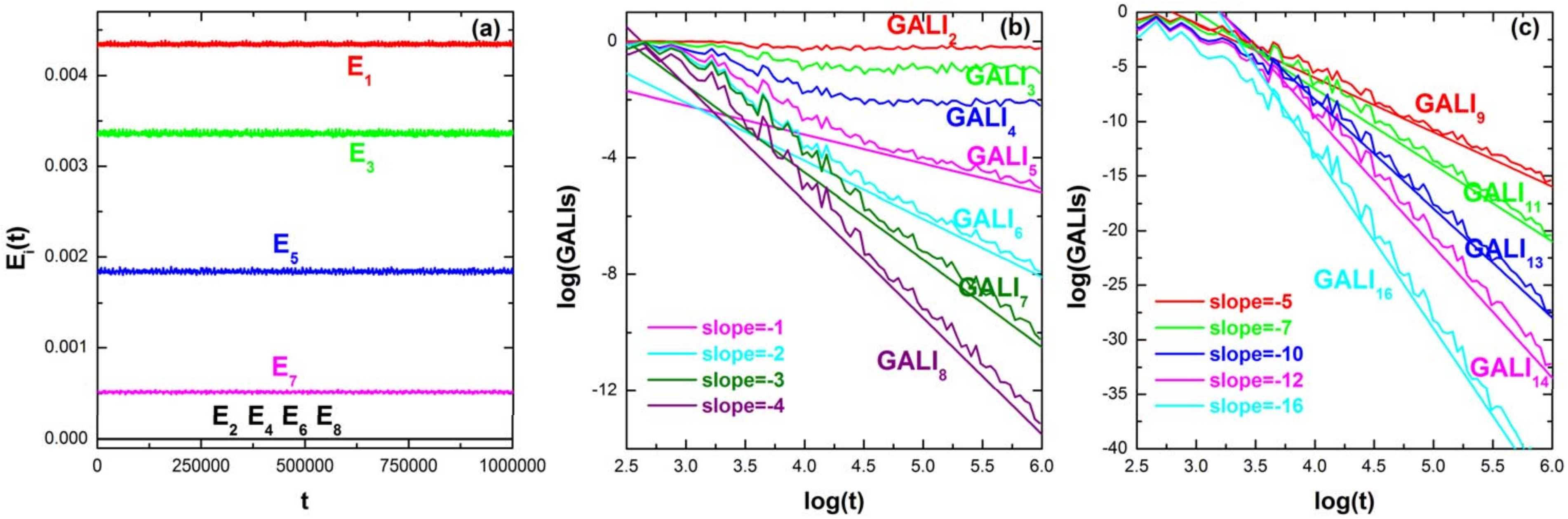
Regular motion on low-dimensional tori

A regular orbit lying on a 2-dimensional torus for the N=8 FPU system.



Regular motion on low-dimensional tori

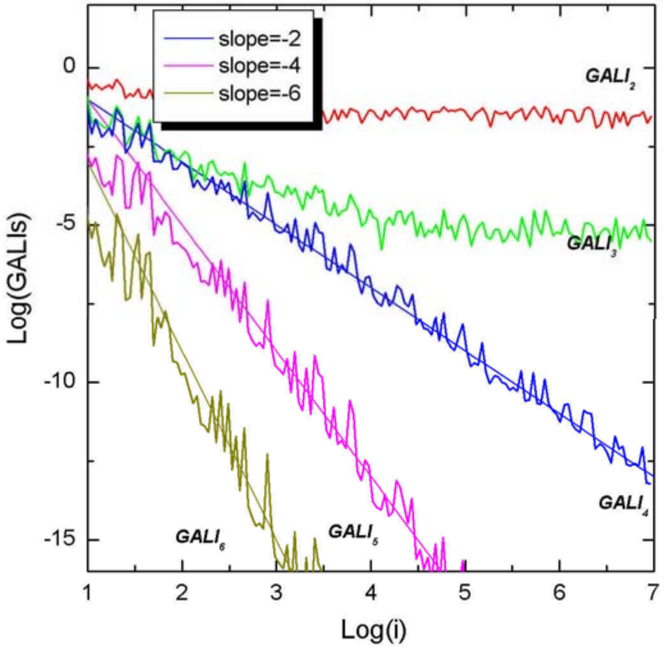
A regular orbit lying on a 4-dimensional torus for the N=8 FPU system.



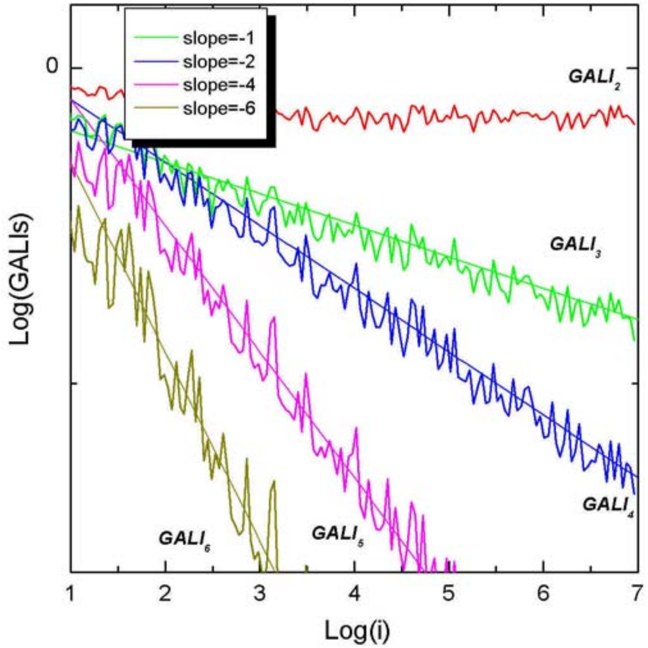
Low-dimensional tori - 6D map

$$\begin{aligned}
x'_1 &= x_1 + x'_2 \\
x'_2 &= x_2 + \tfrac{K_1}{2\pi}\sin(2\pi x_1) - \tfrac{B}{2\pi}\left\{\sin[2\pi(x_5 - x_1)] + \sin[2\pi(x_3 - x_1)]\right\} \\
x'_3 &= x_3 + x'_4 \\
x'_4 &= x_4 + \tfrac{K_2}{2\pi}\sin(2\pi x_3) - \tfrac{B}{2\pi}\left\{\sin[2\pi(x_1 - x_3)] + \sin[2\pi(x_5 - x_3)]\right\} \quad \text{(mod 1)} \\
x'_5 &= x_5 + x'_6 \\
x'_6 &= x_6 + \tfrac{K_3}{2\pi}\sin(2\pi x_5) - \tfrac{B}{2\pi}\left\{\sin[2\pi(x_3 - x_5)] + \sin[2\pi(x_1 - x_5)]\right\}
\end{aligned}$$

3D torus



2D torus



Behavior of $GALI_k$

Chaotic motion:

$GALI_k \rightarrow 0$ exponential decay

$$GALI_k(t) \propto e^{-[(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3)+...+(\sigma_1-\sigma_k)]t}$$

Regular motion:

$GALI_k \rightarrow$ constant $\neq 0$ or $GALI_k \rightarrow 0$ power law decay

$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if} \quad 2 \leq k \leq s \\ \dfrac{1}{t^{k-s}} & \text{if} \quad s < k \leq 2N-s \\ \dfrac{1}{t^{2(k-N)}} & \text{if} \quad 2N-s < k \leq 2N \end{cases}$$

Conclusions

- **Generalizing the SALI method we define the Generalized ALignment Index of order k ($GALI_k$) as the volume of the generalized parallelepiped, whose edges are k unit deviation vectors. $GALI_k$ is computed as the product of the singular values of a matrix (SVD algorithm).**
- **Behaviour of $GALI_k$:**
  - ✓ **Chaotic motion: it tends exponentially to zero with exponents that involve the values of several Lyapunov exponents.**
  - ✓ **Reguler motion: it fluctuates around non-zero values for $2 \leq k \leq s$ and goes to zero for $s < k \leq 2N$ following power-laws, with s being the dimensionality of the torus.**
- **$GALI_k$ indices :**
  - ✓ **can distinguish rapidly and with certainty between regular and chaotic motion**
  - ✓ **can be used to characterize individual orbits as well as "chart" chaotic and regular domains in phase space.**
  - ✓ **are perfectly suited for studying the global dynamics of multidimentonal systems**
  - ✓ **can identify regular motion in low–dimensional tori**

References

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