Lecture 2 – Master equation and system size expansion

February 7, 2020

Outline

1. The master equation for time continuous processes.
2. Birth and death processes - Intro.
3. Simulating stochastic orbits: the Gillespie algorithm.
4. Stochastic predictions: the van Kampen system size expansion.
5. Theory vs. simulations.

The Master Equation

This is basically the continuous time version of Markov chains.

Derivation from the Chapman-Kolmogorov (CK) equation

The CK equation is

$$p(n, t + \Delta t | n_0, t_0) = \sum_{n'} p(n, t + \Delta t | n', t) p(n', t | n_0, t_0).$$

Now, we assume that

$$p(n, t + \Delta t | n', t) = \begin{cases} 1 - \kappa_n(t)\Delta t + \mathcal{O}(\Delta t)^2, & \text{if } n = n', \\ w_{nn'}(t)\Delta t + \mathcal{O}(\Delta t)^2, & \text{if } n \neq n'. \end{cases} \quad (1)$$

$w_{nn'}(t)$ is the *transition rate* and is only defined for $n \neq n'$

Now, by normalisation,

$$\begin{aligned} 1 &= \sum_n p(n, t + \Delta t | n', t) \\ &= 1 - \kappa_{n'}(t)\Delta t + \mathcal{O}(\Delta t)^2 + \sum_{n \neq n'} w_{nn'}(t)\Delta t + \mathcal{O}(\Delta t)^2 \end{aligned}$$

$$\Rightarrow \kappa_{n'}(t) = \sum_{n \neq n'} w_{nn'}(t).$$

Alternatively, switching the indices,

$$\kappa_n(t) \quad = \quad \sum_{n' \neq n} w_{n'n}(t). \tag{2}$$

Therefore, using (1), we see that the CK equation reads

$$\begin{aligned} p(n, t + \Delta t | n_0, t_0) \quad &= \quad (1 - \kappa_n(t)\Delta t + \ldots) p(n, t | n_0, t_0) \\ &+ \quad \sum_{n' \neq n} w_{nn'}(t) p(n', t | n_0, t_0) \Delta t + \mathcal{O}(\Delta t)^2, \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & \frac{p(n, t + \Delta t | n_0, t_0) - p(n, t | n_0, t_0)}{\Delta t} \\ = & -\kappa_n(t) p(n, t | n_0, t_0) + \sum_{n' \neq n} w_{nn'}(t) p(n', t | n_0, t_0) + \mathcal{O}(\Delta t). \end{aligned}$$

Now let $\Delta t \to 0$:

$$\frac{dp(n,t|n_0,t_0)}{dt} = -\kappa_n(t)p(n,t|n_0,t_0) + \sum_{n' \neq n} w_{nn'}(t)p(n',t|n_0,t_0).$$

Using (2), the middle term may be rewritten to find

$$\frac{dp(n,t|n_0,t_0)}{dt} = -\sum_{n' \neq n} w_{n'n}(t)p(n,t|n_0,t_0) + \sum_{n' \neq n} w_{nn'}(t)p(n',t|n_0,t_0).$$

If we had started, not from the CK equation, but from the first relation which defines Markov processes:

$$p(n,t+\Delta t) = \sum_{n'} p(n,t+\Delta t|n',t)p(n',t)$$

then exactly the same sequence of steps would lead to

$$\frac{dp(n,t)}{dt} = -\sum_{n' \neq n} w_{n'n}(t)p(n,t) + \sum_{n' \neq n} w_{nn'}(t)p(n',t).$$

This equation could also have been obtained by multiplying the equation for $p(n, t|n_0, t_0)$ by $p(n_0, t_0)$ and summing over all initial states, since

$$p(n, t) = \sum_{n_0} p(n, t|n_0, t_0)p(n_0, t_0).$$

Therefore both $p(n, t|n_0, t_0)$ and $p(n, t)$ satisfy the same equation. We will frequently write it for $p(n, t)$, with the understanding that this could also be thought of as the conditional probability.

The equation is the desired *master equation*,

$$\frac{dp(n, t)}{dt} = \sum_{n' \neq n} w_{nn'}(t)p(n', t) - \sum_{n' \neq n} w_{n'n}(t)p(n, t). \tag{3}$$

The interpretation of the master equation is straightforward. The first term is just the probability of going from $n' \to n$, and the second the probability of going from $n \to n'$.

In words, the rate of change of being in a state n is equal to the probability of making a transition into n, minus the probability of transitioning out of n.

In applications the $w_{nn'}$ are assumed to be given (this specifies the model) and we wish to determine the $p(n, t)$.

Comments:

• Notice that because of the Markov property, $w_{nn'}$ only depends on the current state of the system (n'), and does not depend on previous states (i.e. how the system got to n').

• In the derivation we have assumed that $w_{nn'}$ depends on time (which it does in general, just as for Markov chains), but usually we are only interested in situations where it is time-independent.

More informal derivations of the master equation

The master equation is essentially a balance equation; the rate of moving into the state n minus the rate of moving out of the same state is the rate of change of $p(n,t)$. That is,

Rate of change of $p(n,t)$ =

[Rate due to transitions into the state n from all the other states n'] -

[Rate due to transitions out of the state n into all other states n']

When expressed this way, and assuming the process is Markov, the master equation

$$\frac{dp(n,t)}{dt} = \sum_{n' \neq n} w_{nn'}(t)p(n',t) - \sum_{n' \neq n} w_{n'n}(t)p(n,t),$$

appears very reasonable.

Another derivation, which is also less rigorous than the original, starts from the system described as a Markov chain, and moves to continuous time by taking the duration of the time-step to zero:

So, let us start from

$$P_n(t+1) = \sum_{n'} Q_{nn'}(t) P_{n'}(t),$$

where the columns of the transition matrix add to unity:

$$\sum_{n'} Q_{n'n}(t) = 1.$$

So, if we write

$$P_n(t+1) - P_n(t) = \sum_{n'} Q_{nn'}(t) P_{n'}(t) - \sum_{n'} Q_{n'n}(t) P_n(t),$$

then the terms in the sums with $n' = n$ can be cancelled to give

$$P_n(t+1) - P_n(t) = \sum_{n' \neq n} Q_{nn'}(t) P_{n'}(t) - \sum_{n' \neq n} Q_{n'n}(t) P_n(t),$$

Now, we take the time step to be Δt, rather than unity, and divide through by the time step;

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \sum_{n' \neq n} \frac{Q_{nn'}}{\Delta t} P_{n'}(t) - \sum_{n' \neq n} \frac{Q_{n'n}}{\Delta t} P_n(t).$$

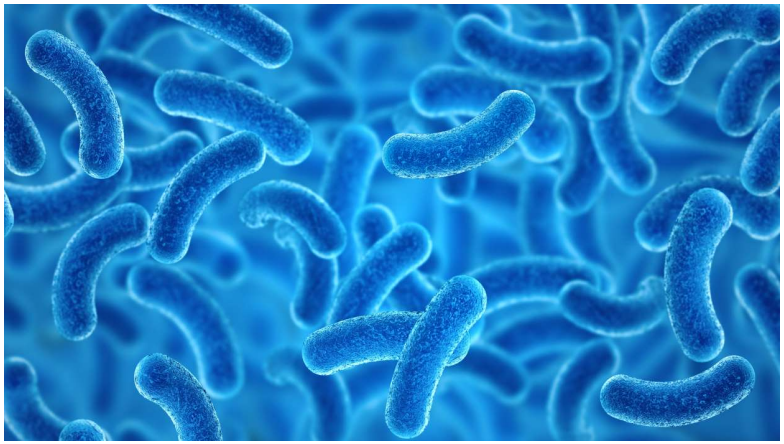
Then, it is clear that taking the time step to zero, reduces the left-hand side to a differential of the probability, with respect to time. Furthermore instead of assuming that exactly one sampling event happens per time step, we assume that *on average* one event happens in the time step. To achieve this set

$$Q_{nn'}(t) = w_{nn'}(t)\Delta t + \mathcal{O}(\Delta t)^2, \quad n \neq n'.$$

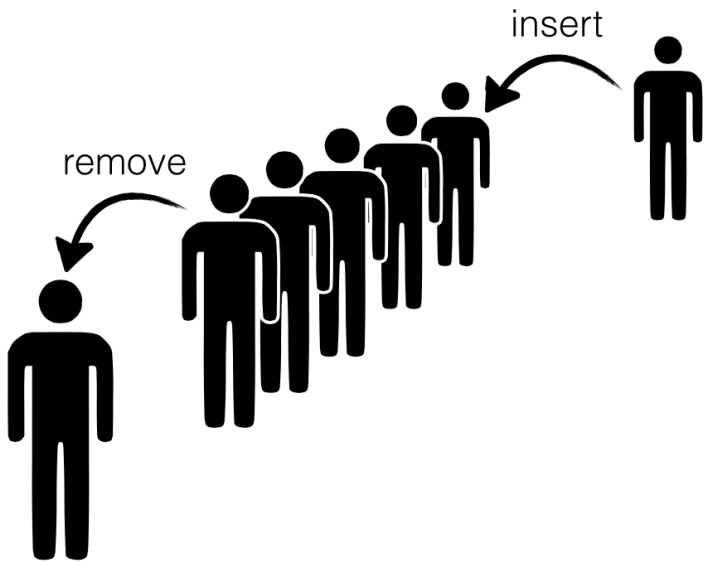
So letting $\Delta t \to 0$ we obtain the master equation:

$$\frac{dP_n}{dt} = \sum_{n' \neq n} w_{nn'}(t) P_{n'}(t) - \sum_{n' \neq n} w_{n'n}(t) P_n(t).$$

The birth and death stochastic model



- Bacteria dynamics.
- Demography.
- Queueing theory.



1. Simple model of population growth.
2. Individuals enter/exit the community.

Exact numerical scheme: the Gillespie algorithm

Consider a given chemical equation, R_1. Assume S_i, with $i = 1, 2$, to label the involved reactants (called Substrates in the original paper by Gillespie):

$$S_1 + S_2 \xrightarrow{c} 2S_1$$

In words: the (individual!) molecule of type S_1 can combine with an (individual!) molecule of type S_2 to result into two molecules of type S_1.

The probability that such a reaction will take place in the forthcoming time interval dt is controlled by:

- the number of molecules of type S_1 and S_2 and the number of possible combinations that yield to an encounter between a molecule of type S_1 and another of type S_2.
- the average probability that given a pair of molecules S_1 ed S_2, the reaction R_1 takes over.

Assume n_1 e n_2 to label the number of molecules of type 1 e 2 respectively. Then, $h = n_1 n_2$ is the number of independent combinations that result in a pair S_1 - S_2.
On the other hand c measures the probability per unit of time of reacting. Hence:

$$P_1 = chdt = cn_1 n_2 dt$$

is the probability that the reaction R_1 takes over in a given time interval dt.

So far so good! What is going to happen if we have instead a system of reactions?

How are we going to sort out which reaction is going to happen first?

Let us consider first the case where two reactions are at play, namely R_1 and R_2, specified as follows:

$$\begin{aligned} S_1 + S_2 &\xrightarrow{c_1} 2S_1 \\ 2S_1 &\xrightarrow{c_2} S_3 \end{aligned}$$

Answering to two questions is mandatory at this point:

- When is the next reaction going to occur?
- Which reaction is going to happen?

Focus on the general framework.

Imagine to have k type of molecules partitioned in the following families $(n_1, n_2, n_3 \ldots)$ and assume that those molecules can react according to M distinct reaction channels, labelled with R_i con $i = 1, .., M$.

We need to calculate the quantity:

$$P(\tau, i)d\tau$$

i.e. the probability that given the system in the state $(n_1, n_2, n_3...)$ at time t:

- the next reaction occurs in the time interval from $t + \tau$ to $t + \tau + d\tau$.
- it is the reaction R_i.

The core of the algorithm is to evaluate the, presently unknown, quantity $P(\tau, i)d\tau$.

The key idea is to split such a probability into two distinct contributions, as outlined below:

- the probability $P_0(\tau)$ that, given the state $(n_1, n_2, ...)$ at time t, no reaction would eventually occur in the time interval $(t, t+\tau)$.
- the probability P_i that the reaction i occurred in the time interval $(t+\tau, t+\tau+d\tau)$.

The second quantity can be readily evaluated. We know that

$$P_i = c_i h_i d\tau$$

where h_i refers to the number of possible combinations of the chemicals as specified by reaction R_i. c_i is instead the average probability per unit of time that the molecules could react and so give birth to the prescribed products.

To evaluate $P_0(\tau)$, the probability that no reaction occurs in in $(t, t+\tau)$, imagine to partition the inspected time interval τ into K sub-intervals, each of size $\epsilon = \tau/K$. The probability that no reaction occurred in the first interval $(t, t+\epsilon)$ is:

$$\Pi_{j=1}^{M} [1 - c_j h_j \epsilon] = 1 - \sum_{j=1}^{M} c_j h_j \epsilon + O(\epsilon)$$

On the other hand this is also the probability that no reaction would occur in the next time interval $(t+\epsilon, t+2\epsilon)$. Since we have K consecutive intervals, one can write:

$$\begin{aligned} P_0(\tau) &= \left[1 - \sum_{j=1}^{M} c_j h_j \epsilon + O(\epsilon) \right]^K \\ &= \left[1 - \sum_{j=1}^{M} c_j h_j \tau/K + O(K^{-1}) \right]^K \end{aligned}$$

Perform now the limit for $K \to \infty$. We eventually obtain:

$$P_0(\tau) = \exp\left(-\sum_{j=1}^{M} c_j h_j \tau\right)$$

from which the fundamental result follows:

The sougth probability $P(\tau, i)$

$$P(\tau, i) = P_0(\tau) a_i = a_i \exp(-a_0 \tau)$$

where we have introduced the compact notation:

$$a_i = c_i h_i$$

and

$$a_0 = \sum_{j=1}^{M} c_j h_j$$

The above expression for $P(\tau, i)$ holds for $0 < \tau < \infty$ and $i = 1, .., M$.

Starting from this setting one can construct an exact algorithm that enables one to track the dynamics of a large ensemble of microscopic constituents that have to obey to an assigned set of chemical rules (or, equivalently, whose probability $P(n, t)$ has to obey to a given Master equation).

The core idea of the computational scheme (Gillespie algorithm) is to implement a Monte Carlo strategy that is able to simulate the stochastic process represented by $P(\tau, i)$. In the following we discuss the sequential steps that we are going to consider.

- STEP 0. At time $t = 0$ assign the initial values to the variables $n_1, n_2, ...$ and to the parameters c_i. Calculate the quantities $h_i c_i$ which in practice determine $P(\tau, i)$. One can also define the time of observation $t_1 < t_2 < ...$ and the stopping time t_s.
- STEP 1. Make use of a dedicated Monte Carlo technique to generate a random pair (τ, i), which obeys to the joint probability density function $P(\tau, i)$.
- STEP 2. Make use of the values as generated above to advance the system in time by a quantity τ, while adjusting the values of the population sizes n_i implicated in the selected reaction i. After this operation is being taken to completion, calculate again the quantities $h_i c_i$ for those reactions that have experienced a change in the chemicals amount.
- STEP 3. If time t is less than t_s or if there are no reactants left into the system $(h_i = 0)$ stop the simulations. Otherwise, start again from STEP 1.

Clearly the crucial step is:

- STEP 2. Make use of a dedicated Monte Carlo technique to generate a random pair (τ, i), which obeys to the joint probability density function $P(\tau, i)$.

to which the following slides are entirely devoted.

We should generate the pair (τ, i) in accordance with the distribution $P(\tau, i)$, as calculated below. We shall illustrate the so called direct method.

To this end we shall make use of our ability to generate random numbers r obeying to a uniform distribution. Notice that τ is a continuous variable, while i is discrete.

First let us write:

$$P(\tau, i) = P_1(\tau) P_2(i|\tau)$$

The probability $P_1(\tau)$ follows from:

$$P_1(\tau) = \sum_{i=1}^{M} P(\tau, i)$$

Hence, inserting in the preceding relation:

$$P_2(i|\tau) = P(\tau, i) / \sum_{i=1}^{M} P(\tau, i)$$

Recalling the above expression for $P(\tau, i)$ yields:

$$\begin{aligned} P_1(\tau) &= a_0 \exp(-a_0 \tau) \\ P_2(i|\tau) &= a_i / a_0 \end{aligned}$$

where $0 \leq \tau < \infty$ and $i = 1, 2, ..M$.

Both probability density functions are normalized in their respective domain of definition.

$$\begin{aligned} \int_0^\infty P_1(\tau) &= \int_0^\infty a_0 \exp(-a_0\tau) = 1 \\ \sum_{j=1}^{M} P_2(i|\tau) &= \sum_{j=1}^{M} a_i/a_0 = 1 \end{aligned}$$

The idea of the direct method is to generate a random number τ in agreement with $P_1(\tau)$ and then an integer i as dictated by $P_2(i|\tau)$. The resulting pair (τ, i) will therefore obey to $P(\tau, i)$.

As we shall outline in the following, it is possible to generate a random quantity τ which obeys to $P_1(\tau)$: (i) by extracting a random number r_1 from a uniform distribution and (ii) by calculating:

$$\tau = (1/a_0)\log(1/r_1)$$

Analogously (no proof given here), one can obtain an integer random i which obeys to $P_2(i|\tau)$ by extracting a random (real) number r_2 from a uniform distribution and selecting i as the integer that fulfills the double inequivalence:

$$\sum_{j=1}^{i-1} a_j < r_2 a_0 < \sum_{j=1}^{i} a_j$$

Finally, we discuss the origin of the formula for τ. It follows from the inversion technique, a Monte Carlo method which enables one to generate random numbers from a generic pdf, by using uniformly distributed random numbers.

Assume, we wish to generate the random number x distributed as $P(x)$. By definition, $P(x')dx'$ is the probability that x falls in the interval delimited by x' and $x' + dx'$. Consider $F(x)$ defined as:

$$F(x) = \int_{-\infty}^{x} P(x')dx'$$

clearly $F(x_0)$ is the probability that x is smaller than x_0. Function $F(x)$ measures the probability for x to be smaller than x_0. $F(x)$ is the probability distribution, distinct from the probability density function $P(x)$.

The inversion method, consists in extracting a uniformly distributed random number r and then select x such that $F(x) = r$, namely:

$$x = F^{-1}(r)$$

where $F^{-1}(\cdot)$ is the inverse of the distribution function associated to the pdf $P(\cdot)$.

Calculate in fact the probability that x as generated according the the above prescriptions would fall in the interval $[x', x' + dx']$. By construction, this probability is identical to the probability that r falls in between $F(x')$ e $F(x' + dx')$. Since r is uniformly distributed, such a probability reads:

$$F(x' + dx') - F(x') = F'(x')dx' = P(x')dx'$$

Assume one needs to generate a random number distributed as the pdf:

$$P(x) = A \exp(-Ax)$$

Then $F(x) = 1 - \exp(-Ax)$ e so, by imposing $F(x) = r$ one readily obtains

$$x = (1/A) \log(1/r)$$

i.e. the formula evoked before. Notice that in the derivation we have replaced $1 - r$ with the statistically equivalent quantity r.

On the implementation: back to the birth-death model

$$E \xrightarrow{b} X$$
$$X \xrightarrow{d} E$$

- X stands for one individual
- E is one empty space (vacancy)
- n number of element of type X
- $n_E = N - n$

The initial condition:

```
time=zeros(1,tmax);
nX=zeros(1,tmax);

nL(1,1)=X;
```

The main loop:

```
for i=2:tmax,
Calculate the transition probability
a1 = b (N-X)/N;
a2 = d X/N;
a0=a1+a2;

Gillespie recipe
r1=rand(1,1); r2=rand(1,1);
tau=-1/a0*log(r1); r2=a0*r2;
ind=1;

Update the population amount

Save the results
end
```

Recall that to obtain a random integer i which obeys to $P_2(i|\tau)$ one can extract a random (real) number r_2 from a uniform distribution and then select i as the integer that fulfills the double inequivalence:

$$\sum_{j=1}^{i-1} a_j < r_2 a_0 < \sum_{j=1}^{i} a_j \tag{4}$$

In practice the values of a_j are summed iteratively until the obtained sum becomes larger than $r_2 a_0$. The corresponding integer j (the number of elements summed up) is the index i we are looking for.

```
while(ind),
prob=a1;
if(r2<prob),
X=X+1; ind=0; break;
end
prob=prob+a2;
if(r2<prob),
X=X-1; ind=0; break;
end
end
```

Perturbative approach to characterize the fluctuations

The birth and death process introduced above yields the following master equation for the probability $P(n,t)$:

$$\begin{aligned}\frac{dP(n,t)}{dt} &= -T(n-1|n)P(n,t) + T(n|n+1)P(n+1,t) \\ &\quad -T(n+1|n)P(n,t) + T(n|n-1)P(n-1,t)\end{aligned}$$

Here n stands for the (discrete) number of individual of type X. The transition probabilities can be readily obtained as:

$$\begin{aligned}T(n-1|n) &= d\frac{n}{N} \\ T(n+1|n) &= b\left(1-\frac{n}{N}\right)\end{aligned}$$

where N is the total number of microscopic constituents, including the empties.

Recovering the deterministic (mean field) limit

Focus on the time evolution of the mean quantity:

$$\frac{\langle n \rangle}{N} = \frac{1}{N} \sum_n n P(n,t)$$

To this end, multiply by n both sides of the Master Equation and sum over all possible states.

Left hand side

$$\sum_n n \frac{dP(n,t)}{dt} = \frac{d}{d(t/N)} \sum_n \frac{n}{N} P(n,t) = \frac{d\langle n \rangle}{d\tau}$$

where $\tau = t/N$.

Right hand side (first two terms)

$$\begin{aligned}\sum_n \; n \;\; & (T(n|n+1)P(n+1,t) - T(n-1|n)P(n,t)) \\ = \;\; & \sum_{n'}(n'-1)T(n'-1|n')P(n',t) - \sum_n nT(n-1|n)P(n,t) \\ = \;\; & -\sum_n T(n-1|n)P(n,t)\end{aligned}$$

Recalling the definition of $T(n-1|n)$ one obtains:

$$-\sum_n d\frac{n}{N}P(n,t) = -d\frac{\langle n \rangle}{N}$$

Similarly, for the other terms

Right hand side (last two terms)

$$\begin{aligned}\sum_n \; n \;\; & (T(n|n-1)P(n-1,t) - T(n+1|n)P(n,t)) \\ = \;\; & \sum_{n'}(n'+1)T(n'+1|n')P(n',t) - \sum_n nT(n+1|n)P(n,t) \\ = \;\; & \sum_n T(n+1|n)P(n,t)\end{aligned}$$

Recalling the definition of $T(n+1|n)$ one obtains:

$$\sum_n b\left(1-\frac{n}{N}\right)P(n,t) = b\left(1-\frac{\langle n\rangle}{N}\right)$$

Collecting all terms together:

$$\frac{d\langle n\rangle}{d\tau} = b\left(1 - \frac{\langle n\rangle}{N}\right) - d\frac{\langle n\rangle}{N}$$

and introducing $\phi = \langle n\rangle/N$:

$$\frac{d\phi}{d\tau} = b(1-\phi) - d\phi$$

The above ODE governs the evolution of the (continuum) average distribution. Fluctuations have been dropped by performing the ensemble average. The solution of the ODE is:

$$\phi(\tau) = \frac{b}{b+d}\left[1 - \left(1 - \phi_0\frac{b+d}{b}\right)\exp[-(b+d)\tau]\right]$$

The system converges asymptotically to the stable fixed point $\phi^* = b/(b+d)$

Accounting for fluctuations: the van Kampen expansion

The van Kampen ansatz consists in splitting the finite size concentration n/N into two contributions:

$$\frac{n}{N} = \phi + \frac{\xi}{\sqrt{N}}$$

where ξ is a stochastic variable. The quantity $1/\sqrt{N}$ is small for large or moderate system sizes N: it hence plays the role of small parameter in a perturbative expansion. Start by re-writing the master equation in the following compact form:

$$\begin{aligned} \frac{dP}{dt} &= \left(\mathcal{E}^{+1} - 1\right) T(n-1|n)P(n,t) \\ &+ \left(\mathcal{E}^{-1} - 1\right) T(n+1|n)P(n,t) \end{aligned}$$

where $\mathcal{E}^{\pm 1} f(n) = f(n \pm 1)$, $f(n)$ being an arbitrary function of the discrete variable n.

Let us begin by considering the left hand side of the master equation. Introduce $\Pi(\xi, t) \equiv P(n(\phi, \xi), t)$. Hence:

$$\frac{\partial \Pi}{\partial t} = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial n}\frac{\partial n}{\partial t}$$

and thus:

$$\frac{\partial P}{\partial t} = \frac{\partial \Pi}{\partial t} - \frac{\partial P}{\partial n} N \frac{d\phi}{dt}$$

On the other hand:

$$\frac{\partial \Pi}{\partial \xi} = \frac{\partial P}{\partial \xi} = \frac{\partial P}{\partial n}\frac{\partial n}{\partial \xi} = \frac{\partial P}{\partial n}\sqrt{N}$$

Combining the two above expressions:

Right hand side of ME

$$\frac{\partial P}{\partial t} = \frac{1}{N}\frac{\partial \Pi}{\partial \tau} - \frac{1}{\sqrt{N}}\frac{d\phi}{d\tau}\frac{\partial \Pi}{\partial \xi}$$

Consider:

$$\mathcal{E}^{+1} f(n) = f(n+1) = f\left(N\phi + \sqrt{N}\xi + 1\right) = f\left(N\phi + \sqrt{N}\left(\xi + \frac{1}{\sqrt{N}}\right)\right)$$

Hence:

$$\mathcal{E}^{+1} f\left(N\phi + \sqrt{N}\xi\right) = f\left(N\phi + \sqrt{N}\left(\xi + \frac{1}{\sqrt{N}}\right)\right)$$

or equivalently:

$$\mathcal{E}^{+1} f\left(\xi\right) = f\left(\xi + \frac{1}{\sqrt{N}}\right) \simeq f\left(\xi\right) + \frac{1}{\sqrt{N}}\partial_\xi f + \frac{1}{2N}\partial_\xi^2 f$$

and:

$$\left(\mathcal{E}^{+1} - 1\right) f\left(\xi\right) = \left(\frac{1}{\sqrt{N}}\frac{\partial}{\partial\xi} + \frac{1}{2N}\frac{\partial^2}{\partial\xi^2}\right) f\left(\xi\right)$$

Summing up, the above step-operators $\mathcal{E}^{\pm 1}$ (as they are called) admits a straightforward expansion with respect to $1/\sqrt{N}$.

$$\mathcal{E}^{\pm 1} = 1 \pm \frac{1}{\sqrt{N}} \frac{\partial}{\partial \xi} + \frac{1}{2N} \frac{\partial^2}{\partial \xi^2} + \dots$$

Hence the first term in the right hand side of the master equation reads:

$$\begin{aligned} & \left(\mathcal{E}^{+1} - 1\right) T(n-1|n) P(n,t) \\ = & \left(\frac{1}{\sqrt{N}} \frac{\partial}{\partial \xi} + \frac{1}{2N} \frac{\partial^2}{\partial \xi^2} \right) d \left(\phi + \frac{\xi}{\sqrt{N}} \right) \Pi(\xi, t) \end{aligned}$$

By organizing the various terms in the above expression one gets:

First terms in the right hand side

$$\frac{1}{\sqrt{N}}\left[d\phi\frac{\partial}{\partial\xi}\Pi\right]+\frac{1}{N}d\left[\frac{\partial}{\partial\xi}(\xi\Pi)+\frac{1}{2}\phi\frac{\partial^2}{\partial\xi^2}\Pi\right]+...$$

up to $1/N$ contributions. Similarly for the other contribution in the master equation:

Last terms in the right hand side

$$-\frac{1}{\sqrt{N}}\left[b(1-\phi)\frac{\partial}{\partial\xi}\Pi\right]+\frac{1}{N}b\left[\frac{\partial}{\partial\xi}(\xi\Pi)+\frac{1}{2}(1-\phi)\frac{\partial^2}{\partial\xi^2}\Pi\right]+...$$

At the leading order, $1/\sqrt{N}$

Confronting the terms at the leading order in the master equation one obtains:

$$\frac{d\phi}{d\tau} = b(1-\phi) - d\phi$$

namely the equation which rules the dynamics of the examined system in the deterministic limit.

At the next-to-leading order, $1/N$

Confronting the terms at the next-to-leading order in the master equation one obtains the following Fokker-Planck equation :

$$\frac{\partial \Pi}{\partial \tau} = (d+b)\frac{\partial}{\partial \xi}(\xi \Pi) + \frac{1}{2}(d\phi + b(1-\phi))\frac{\partial^2}{\partial \xi^2}\Pi$$

The solution of the above one-dimensional Fokker-Planck equation is a Gaussian whose first and second moments can be readily computed. Multiply both sides of the Fokker-Planck equation by ξ and integrate over the real axis in $d\xi$. A simple calculation yields:

First moment of Π

$$\frac{d\langle \xi \rangle}{d\tau} = -(b+d)\langle \xi \rangle$$

where $\langle \xi \rangle = \int \xi \Pi d\xi$. The solution reads $\langle \xi \rangle = \langle \xi \rangle_0 \exp\left[-(d+b)\tau\right]$. Asymptotically, $\langle \xi \rangle \to 0$. Finite size fluctuations are hence described by a Gaussian centered in zero for sufficiently long times.

A similar reasoning applies to the second moment. The latter is defined as $\langle \xi^2 \rangle = \int \xi^2 \Pi d\xi$ and it can be shown to obey to the following differential equation:

Second moment of Π

$$\frac{d\langle \xi^2 \rangle}{d\tau} = -2(b+d)\langle \xi^2 \rangle + [(d-b)\phi + b]$$

Assume we are interested in the statistics of the fluctuations around the deterministic equilibrium, when $\phi \to \phi^* = b/(b+d)$. Imposing stationarity ($\frac{d\langle \xi^2 \rangle}{d\tau} = 0$) yields:

$$\langle \xi^2 \rangle_{st} \equiv \sigma^2 = \frac{db}{(d+b)^2}$$

The distribution of fluctuations is hence predicted to be:

$$\Pi_{st}(\xi) \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{\xi^2}{2\sigma^2}\right)$$