Higher-order networks

An introduction to simplicial complexes

Lesson IV

Franqui Chair Lessons

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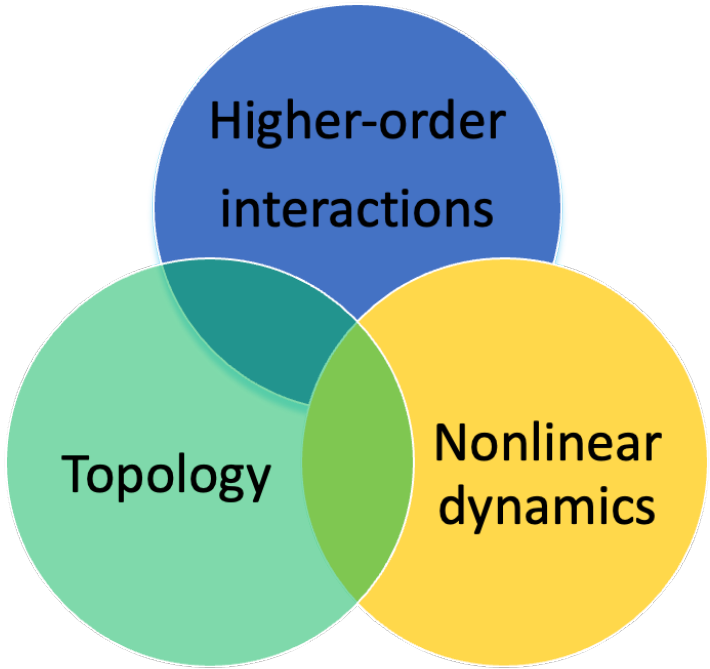
Complexity challenge



Lesson IV:
Dirac synchronization, Global Topological Synchronisation and more

- Dirac synchronisation
    - Phenomenology and Theory

- Global topological synchronisation and Master Stability Function

    - Global synchronisation on graphs
    - Global synchronisation on simplicial and cell complexes

- Turing patterns coupled by the Dirac operator

Addendum:

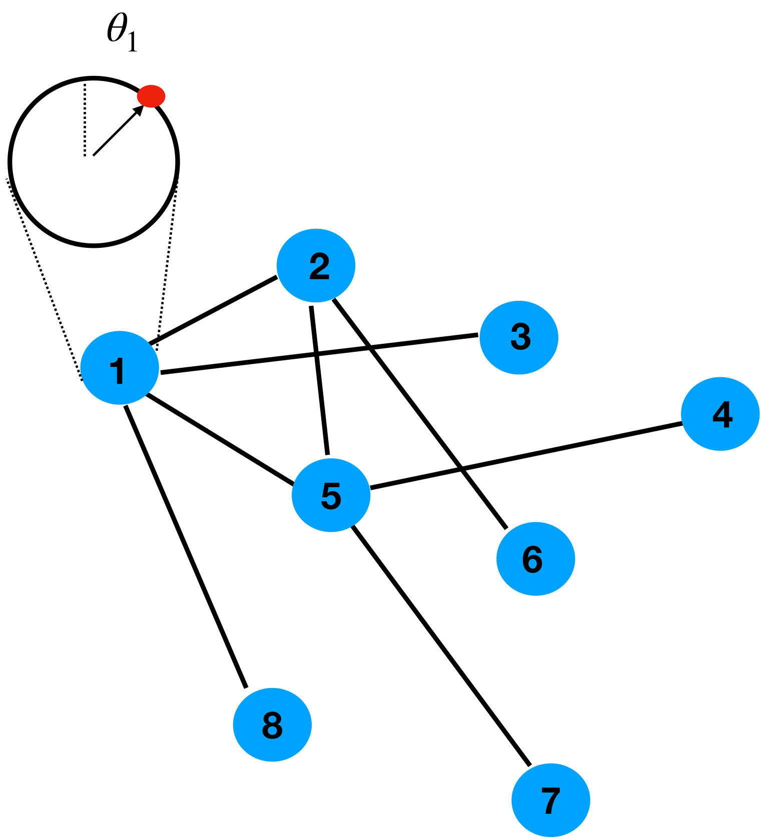
- Triadic percolation and non-linear dynamics of the giant component

Topological signals

- Citations in a collaboration network
- Speed of wind at given locations
- Currents at given locations in the ocean
- Fluxes in biological transportation networks
- Synaptic signal
- Edge signals in the brain

Topological signals
are co-chains or vector fields

Kuramoto model on a network

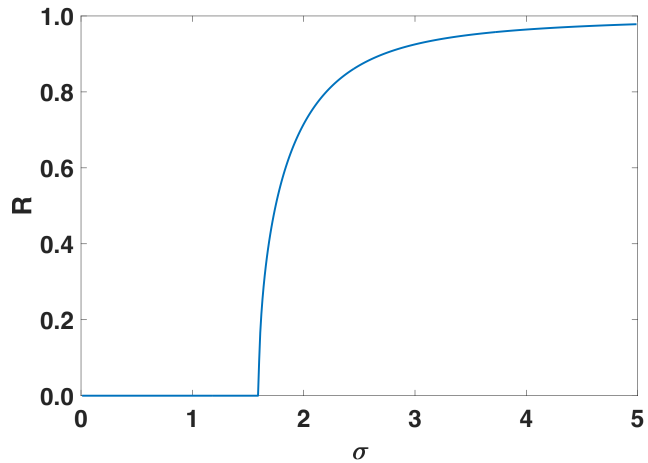


The Kuramoto model

$$\dot{\theta}_r = \omega_r + \sigma \sum_{j=1}^{N} a_{rj} \sin\left(\theta_j - \theta_r\right)$$

With $\omega \sim \mathcal{N}(\Omega,1)$

describes synchronization of node phases of $\sigma > \sigma_c$

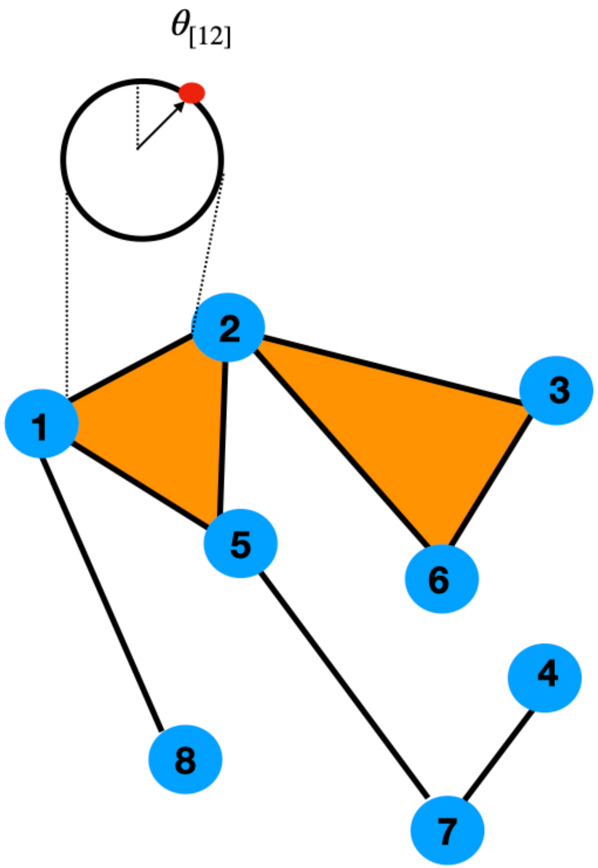


Order parameter

$$R = \frac{1}{N}\left|\sum_{r=1}^{N} e^{i\theta_r}\right|$$

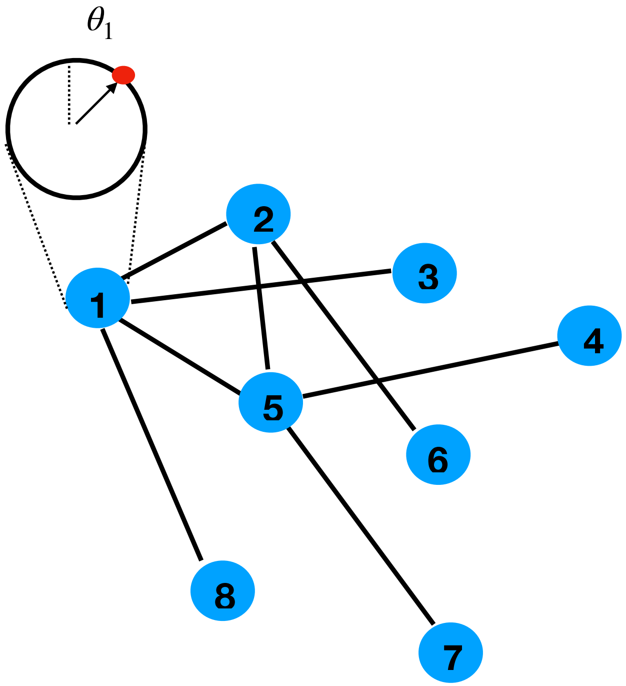
In the Standard Kuramoto model the free dynamics of the synchronised state is uniform over the whole (connected) network

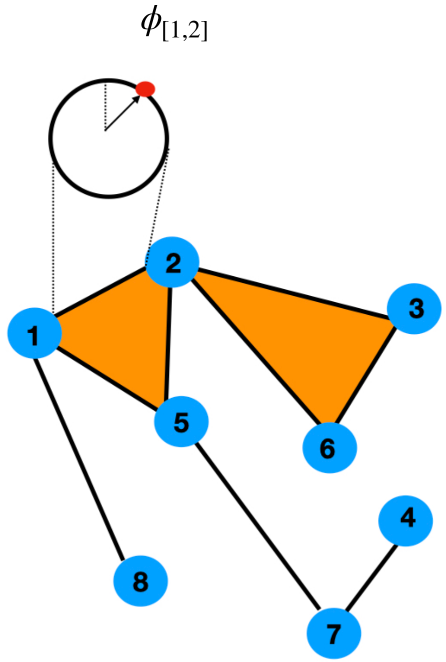
The Topological Kuramoto model



How to define
the Topological Kuramoto model
coupling higher dimensional
topological signals?

Topological Kuramoto model





Standard Kuramoto model

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

Topological Higher-order Kuramoto model

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]} \boldsymbol{\phi},$$

A. P. Millan, J.J. Torres and G. Bianconi PRL (2020)

The Topological Kuramoto Model

Learns Topology

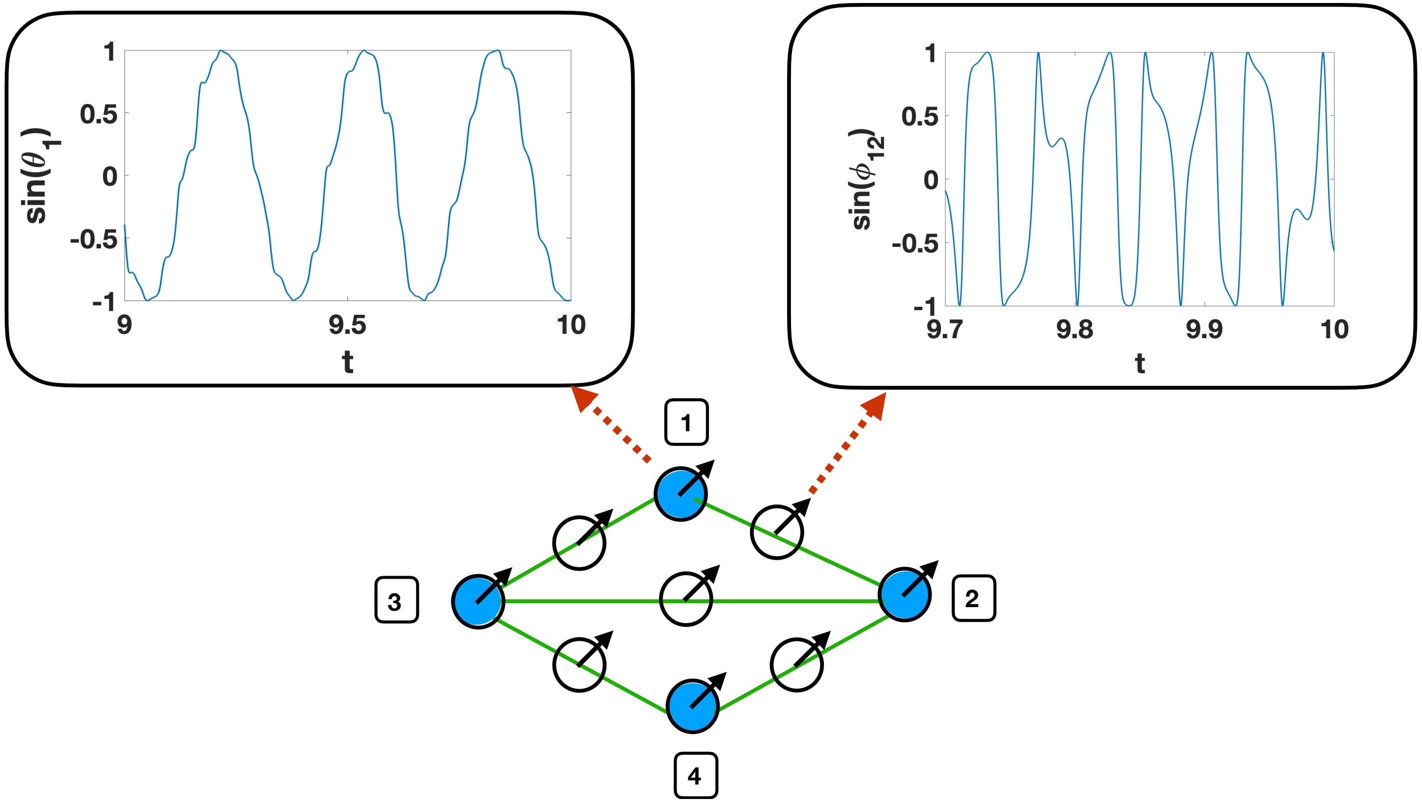
In the Topological Kuramoto model the free dynamics of the synchronized state is localised on the n-dimensional holes

$$\frac{d\langle \mathbf{u}_{harm}, \boldsymbol{\phi} \rangle}{dt} = \langle \mathbf{u}_{harm}, \hat{\boldsymbol{\omega}} \rangle$$

The free dynamics is localised on harmonic components

Dirac synchronisation

Simplicial complexes and networks can sustain dynamical variables (signals) not only defined on nodes but also defined on higher order simplices these signals are called *topological signals*



Dirac operator on graph

Dirac operator on a graph

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{[1]} \\ \mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix}$$

Action of the Dirac operator on the topological spinor

$$\mathbf{D}\boldsymbol{\Psi} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{[1]} \\ \mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \chi \\ \psi \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{[1]}\psi \\ \mathbf{B}_{[1]}^{\top}\chi \end{pmatrix}$$

Topological synchronisation on nodes and links

Topological synchronization of nodes and links of a network

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[1]}^{\top} \sin \mathbf{B}_{[1]} \boldsymbol{\phi},$$

Can be written in terms of the Dirac operator as

$$\dot{\boldsymbol{\Phi}} = \boldsymbol{\Omega} - \sigma \mathbf{D} \sin \mathbf{D} \boldsymbol{\Phi},$$

where

$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\phi} \end{pmatrix}, \quad \boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\omega} \\ \hat{\boldsymbol{\omega}} \end{pmatrix}$$

Normalised Dirac operator on a network

$$\hat{\mathbf{D}} = \begin{pmatrix} \mathbf{0} & \bar{\mathbf{B}}^*_{[1]} \\ \bar{\mathbf{B}}_{[1]}/2 & \mathbf{0} \end{pmatrix}$$

with $$\bar{\mathbf{B}}^*_{[1]} = \mathbf{G}_{[0]}\bar{\mathbf{B}}^\top_{[1]}\mathbf{G}^{-1}_{[1]} = \mathbf{K}_0^{-1}\mathbf{B}_{[1]}$$

$$\hat{\mathbf{D}}^2 = \mathscr{L} = \begin{pmatrix} \hat{\mathbf{L}}_{[0]} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{L}}_{[1]} \end{pmatrix}$$

with

$$\hat{\mathbf{L}}_{[0]} = \bar{\mathbf{B}}^*_{[1]}\bar{\mathbf{B}}_{[1]}, \hat{\mathbf{L}}_{[1]} = \bar{\mathbf{B}}_{[1]}\bar{\mathbf{B}}^*_{[1]}$$

F. Baccini, F. Geraci and G. Bianconi (2022)

Modified dynamics using the normalised Dirac operator

Topological synchronization of nodes and links of a network can be modified by considering the weighted coboundary operator and its dual

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \bar{\mathbf{B}}^{*}_{[1]} \sin \bar{\mathbf{B}}_{[1]} \boldsymbol{\theta},$$
$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \bar{\mathbf{B}}_{[1]} \sin \bar{\mathbf{B}}^{*}_{[1]} \boldsymbol{\phi},$$

that can be written in terms of the normalized Dirac operator as

$$\dot{\boldsymbol{\Phi}} = \boldsymbol{\Omega} - \sigma \hat{\mathbf{D}} \sin \hat{\mathbf{D}} \boldsymbol{\Phi} \,.$$

Dirac Synchronization

Dirac Synchronization allows to couple locally and topologically signals defined on nodes and links.

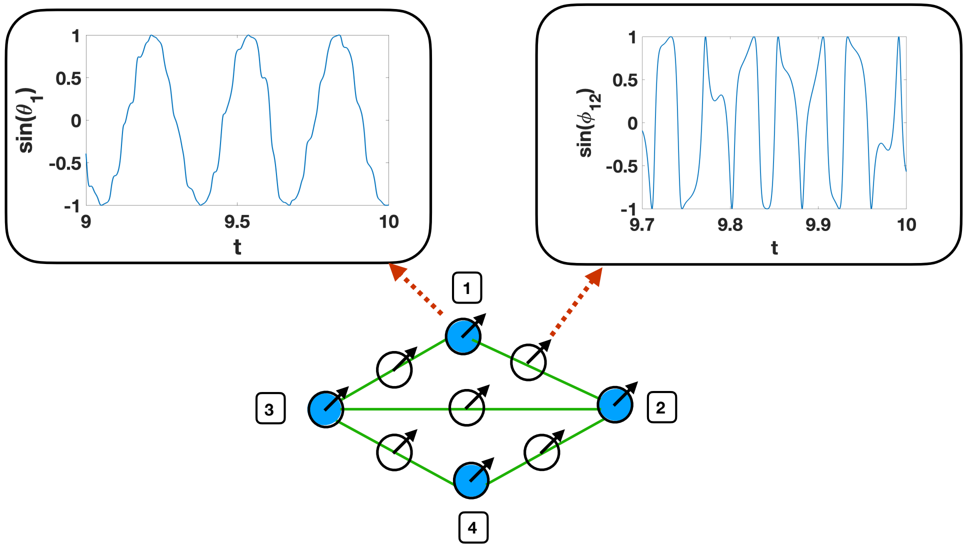
Given $\mathbf{\Phi} = (\boldsymbol{\theta}, \boldsymbol{\phi})^\top$

Dirac synchronisation obeys

$$\dot{\mathbf{\Phi}} = \mathbf{\Omega} - \sigma\hat{\mathbf{D}}\sin((\hat{\mathbf{D}} - \gamma z \hat{\mathbf{D}}^2)\mathbf{\Phi})$$

where

$$\gamma = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$$



L. Calmon, J. Restrepo, J.J. Torres and G. Bianconi (2022)

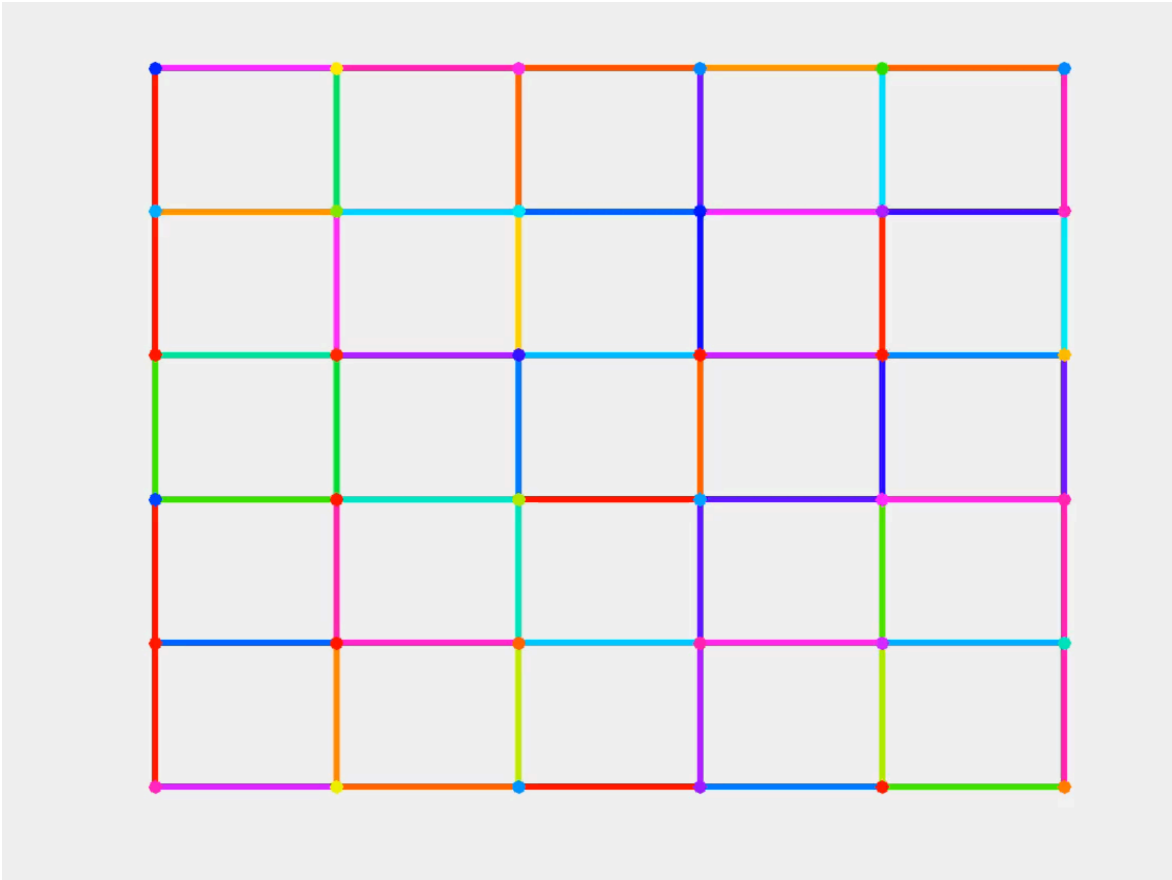
In the Dirac Synchronization the free dynamics of the synchronized state is localised on the links around

1-dimensional holes (since we are in a network)

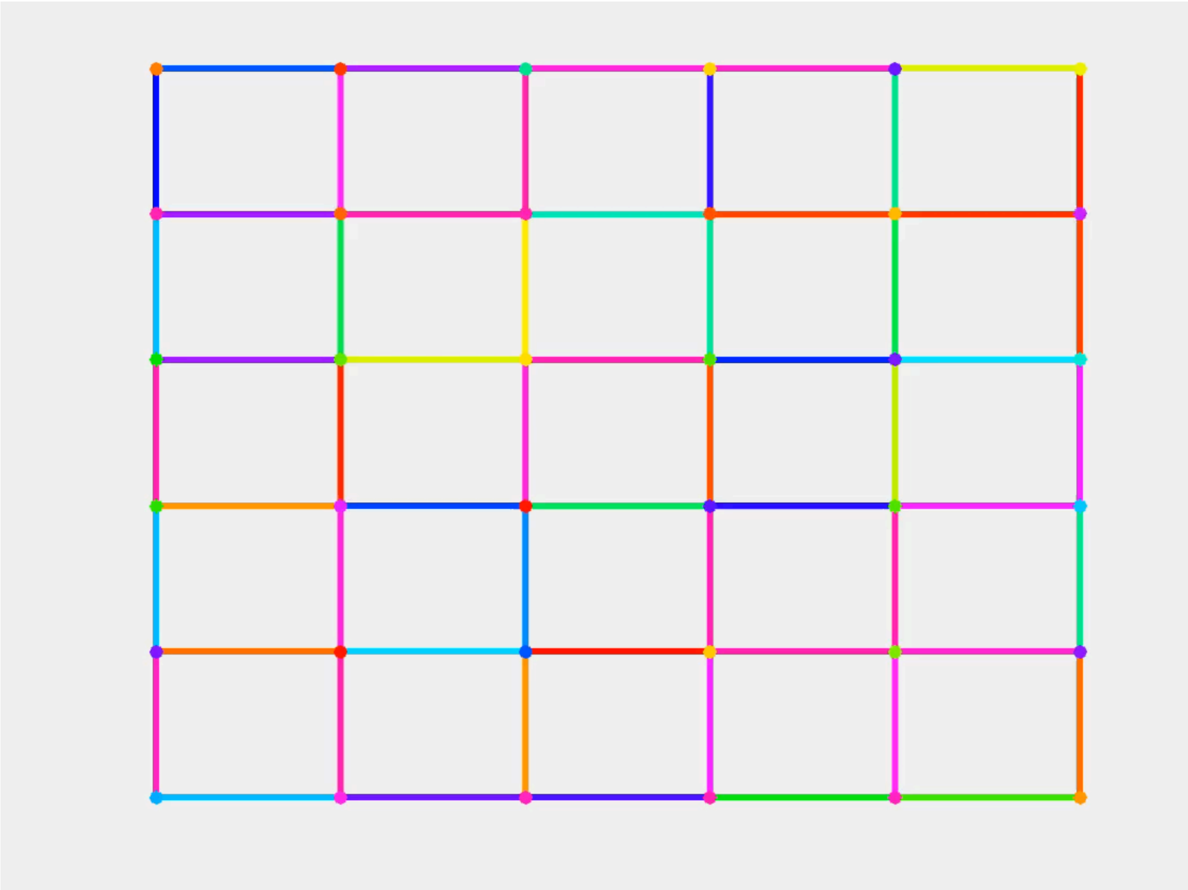
$$\frac{d\langle \mathbf{u}_{harm}, \boldsymbol{\phi}\rangle}{dt} = \langle \mathbf{u}_{harm}, \hat{\boldsymbol{\omega}}\rangle$$

The free dynamics is localised on harmonic components

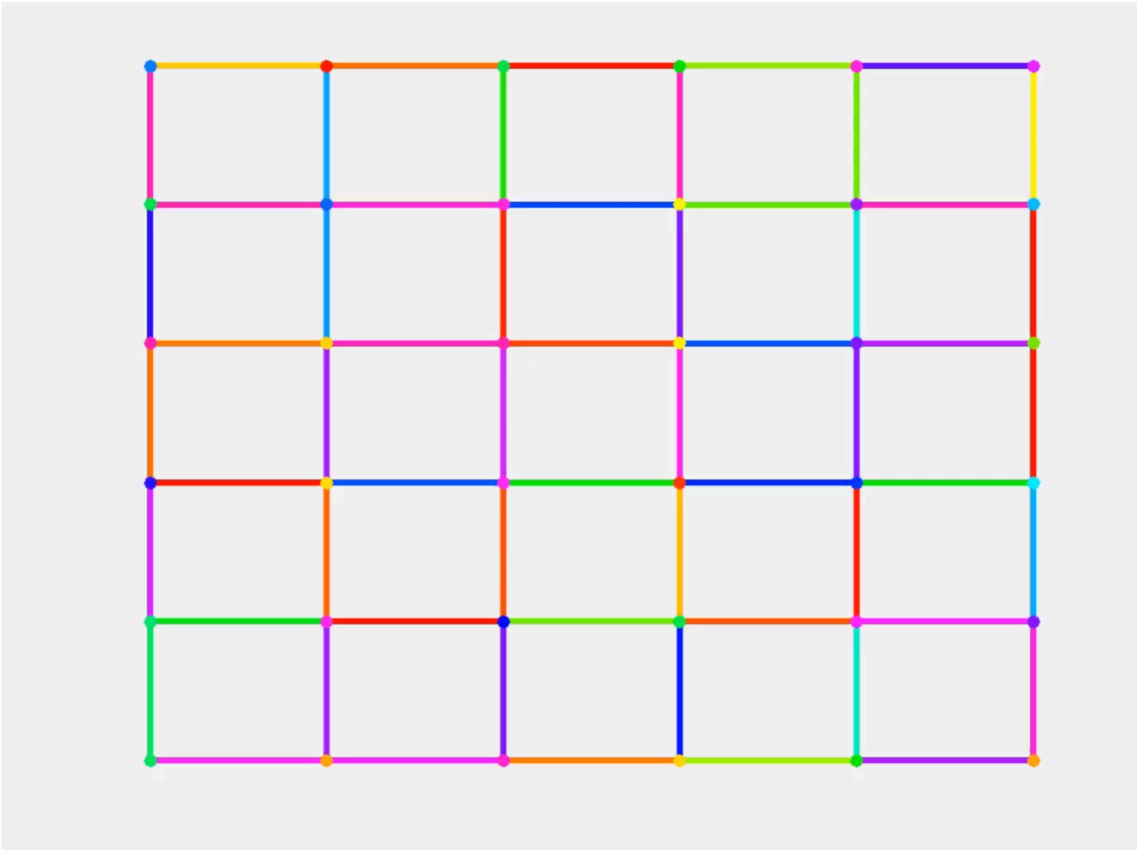
Dirac synchronisation $\sigma = 0.5$



Dirac synchronisation $\sigma = 5$



Dirac synchronisation $\sigma = 10$



Component-wise expression of the dynamical equations

The expression for the dynamical equations of Dirac synchronization read

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \bar{\mathbf{B}}^*_{[1]} \sin(\bar{\mathbf{B}}_{[1]}\boldsymbol{\theta} + z\hat{\mathbf{L}}_{[1]}\boldsymbol{\phi})$$
$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \bar{\mathbf{B}}_{[1]} \sin(\bar{\mathbf{B}}^*_{[1]}\boldsymbol{\phi} - z\hat{\mathbf{L}}_{[0]}\boldsymbol{\theta}),$$

which can be also expressed as

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{K}^{-1}\mathbf{B}_{[1]} \sin(\mathbf{B}^\top_{[1]}\boldsymbol{\theta}/2 + z\mathbf{B}^\top_{[1]}\mathbf{K}^{-1}\mathbf{B}_{[1]}\boldsymbol{\phi}/2)$$
$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \frac{1}{2}\sigma \mathbf{B}^\top_{[1]} \sin(\mathbf{K}^{-1}\mathbf{B}_{[1]}\boldsymbol{\phi} - z\mathbf{K}^{-1}\mathbf{B}_{[1]}\mathbf{B}^\top_{[1]}\boldsymbol{\theta}/2),$$

Projections

The phases of the links can be projected onto the nodes by defining

$$\boldsymbol{\psi} = \mathbf{K}_0^{-1}\mathbf{B}_{[1]}\boldsymbol{\phi}$$

$$\tilde{\boldsymbol{\omega}} = \mathbf{K}_0^{-1}\mathbf{B}_{[1]}\hat{\boldsymbol{\omega}}$$

And considering the projected equations

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma\mathbf{K}_0^{-1}\mathbf{B}\sin\left(\mathbf{B}^\top(\boldsymbol{\theta} + z\boldsymbol{\psi})/2)\right),$$

$$\dot{\boldsymbol{\psi}} = \tilde{\boldsymbol{\omega}} - \sigma\frac{1}{2}\hat{\mathbf{L}}_0\sin\left(\boldsymbol{\psi} - z\hat{\mathbf{L}}_0\boldsymbol{\theta}/2\right),$$

Where $\hat{\mathbf{L}}_{[\mathbf{0}]} = \mathbf{K}_0^{-1}\mathbf{B}_{[1]}\mathbf{B}_{[1]}^\top$

Coupled phases

Let us introduced the coupled nodes and link phases defined as

$$\alpha_r = (\theta_r + z\psi_r)/2,$$
$$\beta_r = z(\theta_r - \Theta_r)/2 - \psi_r$$

Where

$$\Theta_r = \sum_{s=1}^{N} \frac{a_{rs}}{k_r}\theta_s \, .$$

The dynamical equations for read then

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{K}_0^{-1}\mathbf{B}\sin\left(\mathbf{B}^\top(\boldsymbol{\theta} + z\boldsymbol{\psi})/2)\right),$$
$$\dot{\boldsymbol{\psi}} = \tilde{\boldsymbol{\omega}} - \sigma\frac{1}{2}\hat{\mathbf{L}}_0 \sin\left(\boldsymbol{\psi} - z\hat{\mathbf{L}}_0\boldsymbol{\theta}/2\right),$$

$$\dot{\theta}_r = \omega_r + \sigma\frac{1}{k_r}\sum_{s=1}^{N} a_{rs}\sin\left(\alpha_s - \alpha_r\right),$$
$$\dot{\psi}_r = \tilde{\omega}_r - \sigma\frac{1}{2k_r}\sum_{s=1}^{N} a_{rs}\left[\sin\left(\beta_s\right) - \sin\left(\beta_r\right)\right].$$

Node and links are "entangled "

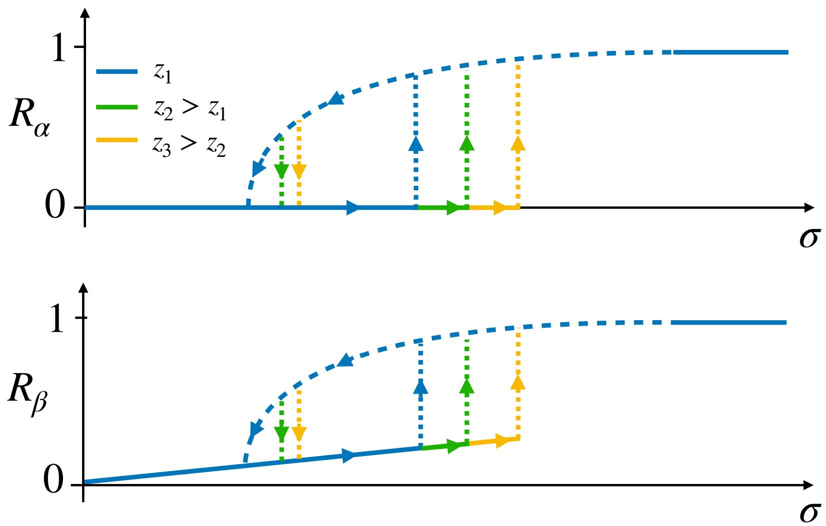
- **Node and links signals are entangled.**

- The order parameters depend on linear combinations of nodes and link signals

$$X_\alpha = R_\alpha e^{\mathrm{i}\eta_\alpha} = \frac{1}{N}\sum_{r=1}^{N} e^{\mathrm{i}\alpha_r},$$

$$X_\beta = R_\beta e^{\mathrm{i}\eta_\beta} = \frac{1}{N}\sum_{r=1}^{N} e^{\mathrm{i}\beta_r},$$

- **The synchronization transition is discontinuous**



L. Calmon, J. Restrepo, J.J. Torres and G. Bianconi (2022)

Equations for the angles $\boldsymbol{\alpha}, \boldsymbol{\beta}$

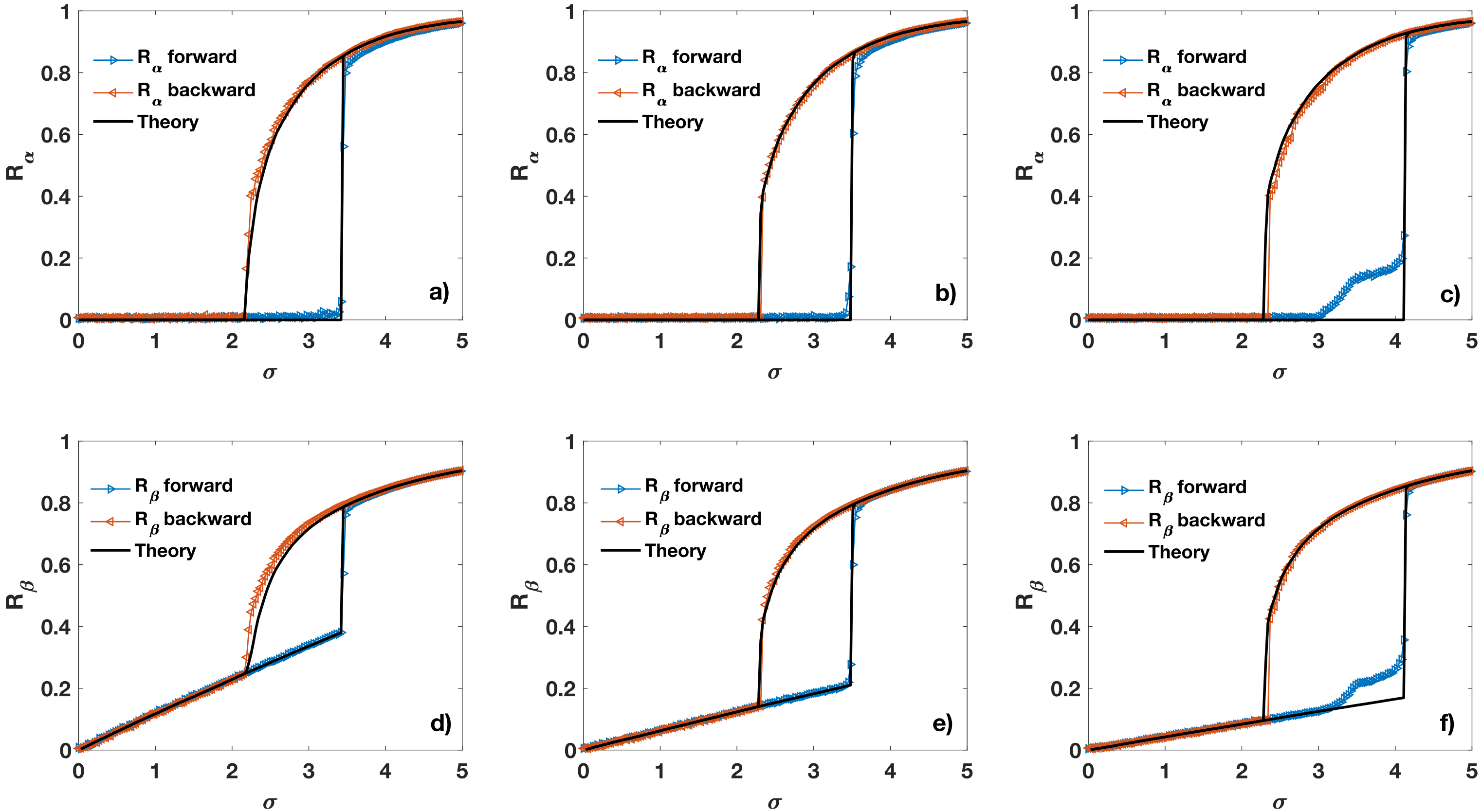
- The closed equations for the angles α_r, β_r are given by

$$\begin{pmatrix} \dot{\alpha}_r \\ \dot{\beta}_r \end{pmatrix} = \boldsymbol{\kappa}_r + \frac{\sigma}{2}\mathbf{Im}\left[\mathbf{X}\begin{pmatrix} e^{-\mathrm{i}\alpha_\mathrm{r}} \\ e^{-\mathrm{i}\beta_\mathrm{r}} \end{pmatrix}\right]$$

- With

$$\boldsymbol{\kappa}_r = \begin{pmatrix} \frac{1}{2}\omega_r + \frac{z}{2}\hat{\omega}_r - \frac{1}{2}\hat{\Omega} - \frac{z}{4}\sigma\mathbf{Im}\left(X_\beta\right) \\ \frac{z\hat{c}}{2}\omega_r - \hat{\omega}_r - \frac{z}{2}\hat{\Omega} + \frac{1}{2}\sigma\mathbf{Im}\left(X_\beta\right) \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} X_\alpha & -\frac{z}{2} \\ zX_\alpha & 1 \end{pmatrix}.$$

Dependence on z



Upper synchronisation threshold

In Dirac synchronisation

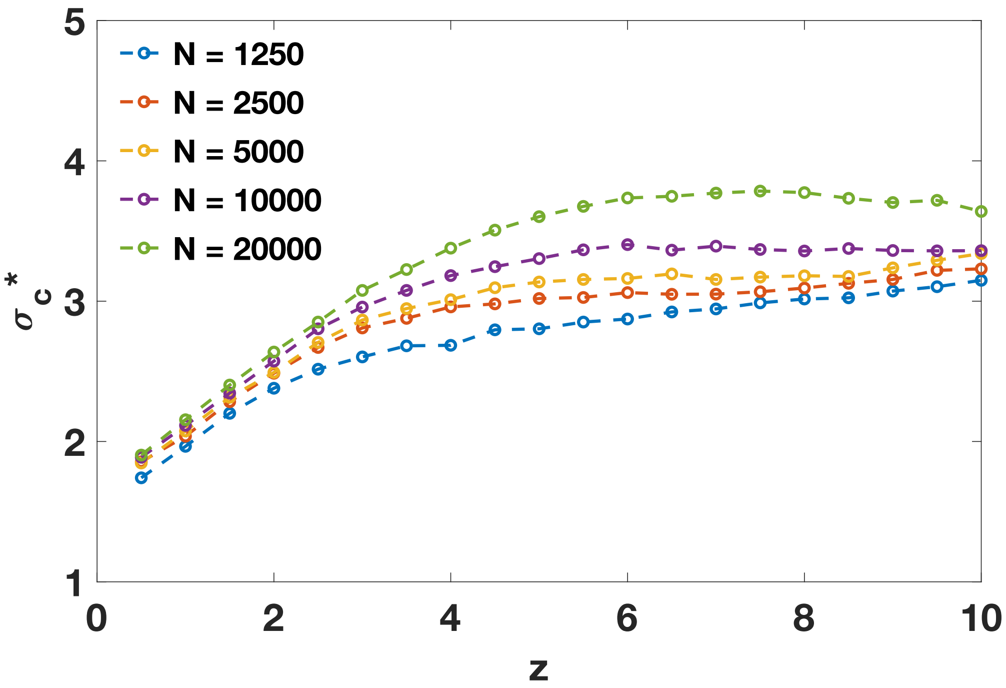
the

hysteresis loop

of the

first order transition

is stable.



Linearised Dynamics

The linearised dynamics is dictated by the Dirac operator

$$\dot{\mathbf{\Phi}} = \mathbf{\Omega} - \sigma(\hat{\mathbf{D}}^2 + z\boldsymbol{\gamma}\hat{\mathbf{D}}^3)\mathbf{\Phi},$$

Let us now decompose $\mathbf{\Phi}, \mathbf{\Omega}$ on the eigenvectors of the Dirac operator $\mathbf{W}_\lambda$ obtaining

$$\mathbf{\Phi} = \sum_\lambda c_\lambda W_\lambda \quad \mathbf{\Omega} = \sum_\lambda \omega_\lambda W_\lambda$$

Linearised Dynamics

The harmonic component of the signal oscillates freely

$$\dot{c}_{harmonic} = \hat{\boldsymbol{\Omega}}_{harmonic}$$

The other modes freeze asymptotically **at a stable focu**s in time and obey

$$\begin{pmatrix} \dot{c}_\lambda \\ \dot{c}_{-\lambda} \end{pmatrix} = \begin{pmatrix} \omega_\lambda \\ \omega_{-\lambda} \end{pmatrix} - \sigma \begin{pmatrix} \lambda^2 & -z\lambda^3 \\ z\lambda^3 & \lambda^2 \end{pmatrix} \begin{pmatrix} c_\lambda \\ c_{-\lambda} \end{pmatrix}$$

Where $\lambda \neq 0$ indicates a positive eigenvalue of the Dirac operator

Linearised Dynamics (continuation)

The dynamical equation for the harmonic mode

has solution

$$c_{harm}(t) = c_{harm}(0) + \omega_{harm} t$$

Therefore the harmonic modes

undergo an unperturbed motion

Linearised Dynamics (continuation)

The dynamical equation for the other modes

has solution

$$\begin{pmatrix} c_\lambda(t) \\ c_{-\lambda}(t) \end{pmatrix} = A(t)\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} + B(t)\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$$

with

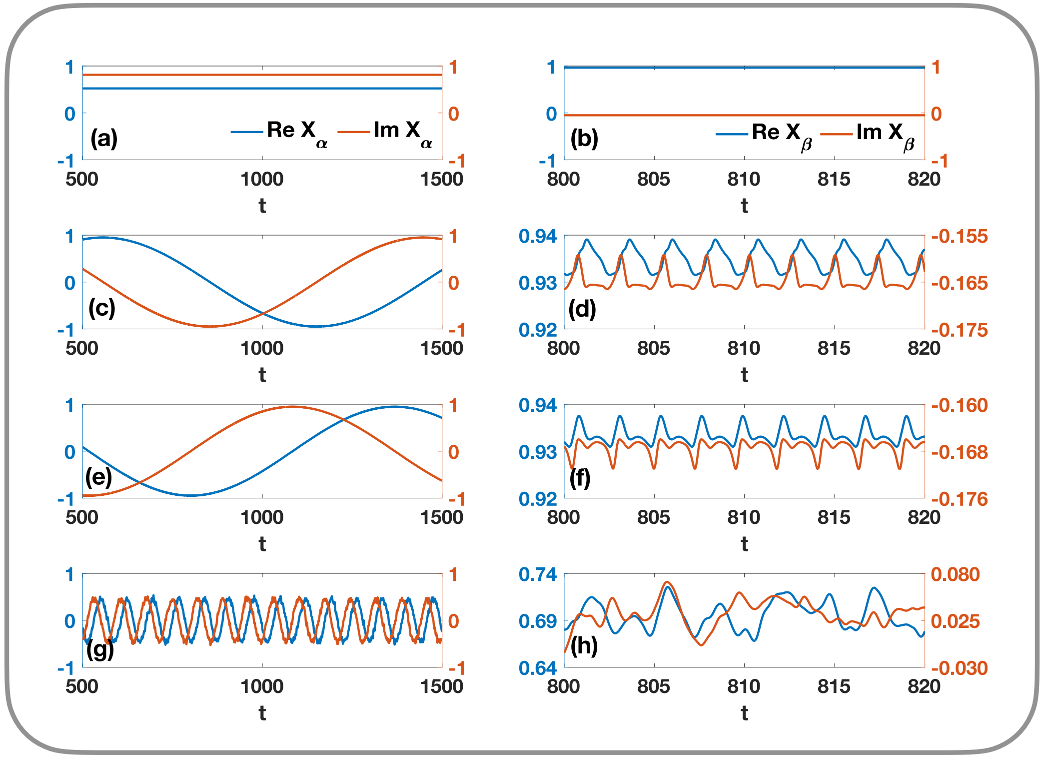
$$A(t) = \frac{\omega_\lambda + \mathrm{i}\omega_{-\lambda}}{2\sigma(\lambda^2 + \mathrm{i}z\lambda^3)}\left(1 - e^{-\sigma(\lambda + \mathrm{i}z\lambda^3)\mathrm{t}}\right) + A(0)e^{-\sigma(\lambda + \mathrm{i}z\lambda^3)\mathrm{t}}$$

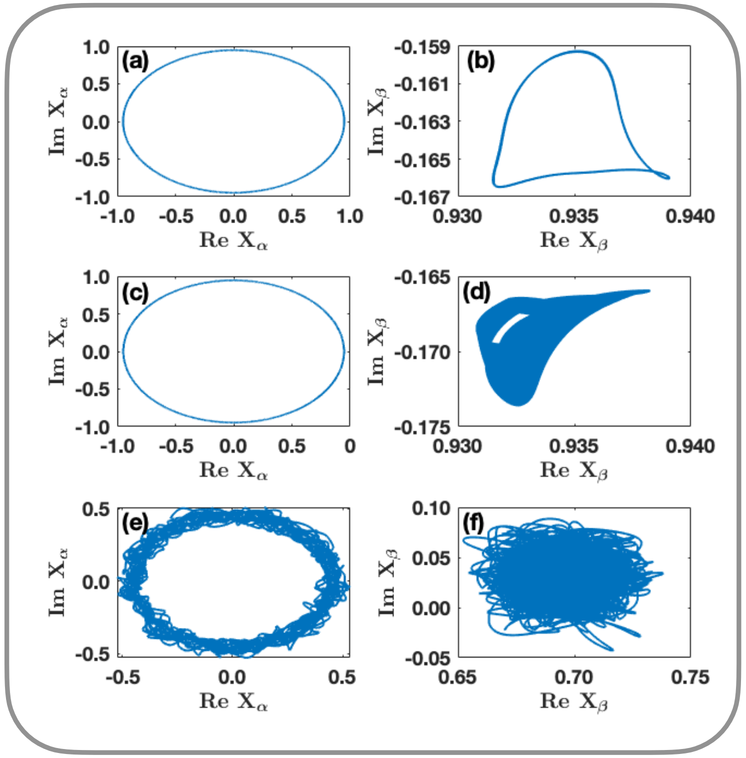
$$B(t) = \frac{\omega_\lambda - \mathrm{i}\omega_{-\lambda}}{2\sigma(\lambda^2 + \mathrm{i}z\lambda^3)}\left(1 - e^{-\sigma(\lambda - \mathrm{i}z\lambda^3)\mathrm{t}}\right) + B(0)e^{-\sigma(\lambda - \mathrm{i}z\lambda^3)\mathrm{t}}$$

Therefore while the non-harmonic modes display a stable focus.

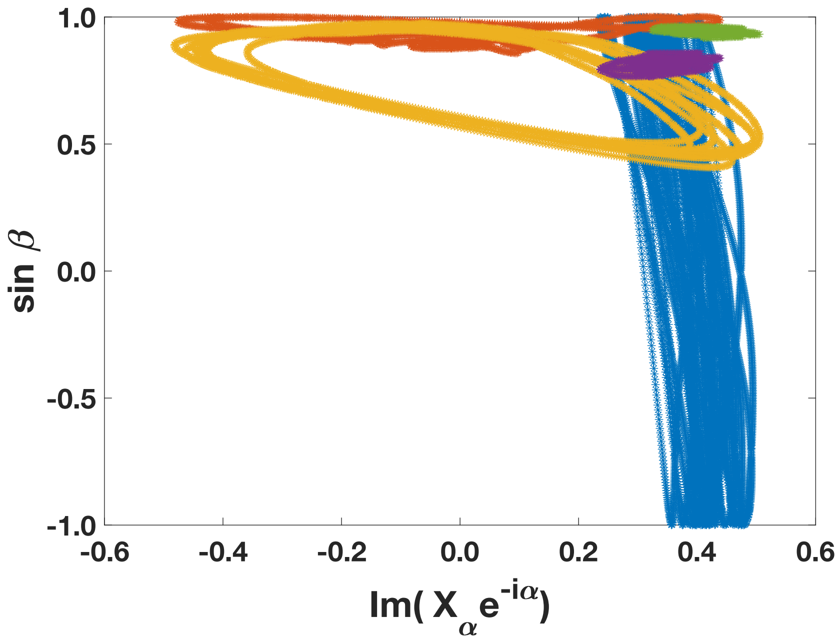
Dirac Synchronization is rhythmic

One of the two complex order parameters develops spontaneous low frequency rhythms





Classification of phases



In Dirac synchronisation each node is assigned to phases α_r, β_r each node can be classified in four classes:

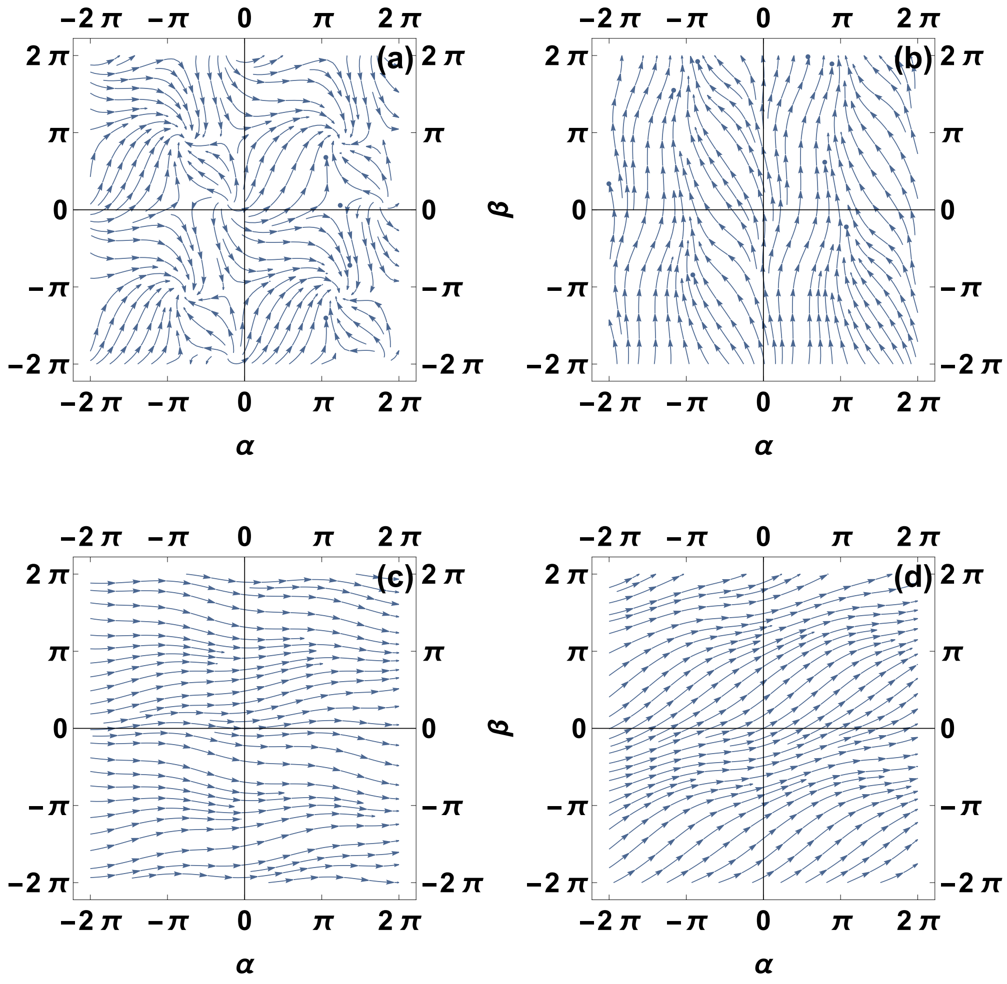
- Both α_r, β_r frozen
- α_r frozen while β_r drifting
- α_r drifting while β_r frozen
- Both α_r, β_r drifting

Classification of phases

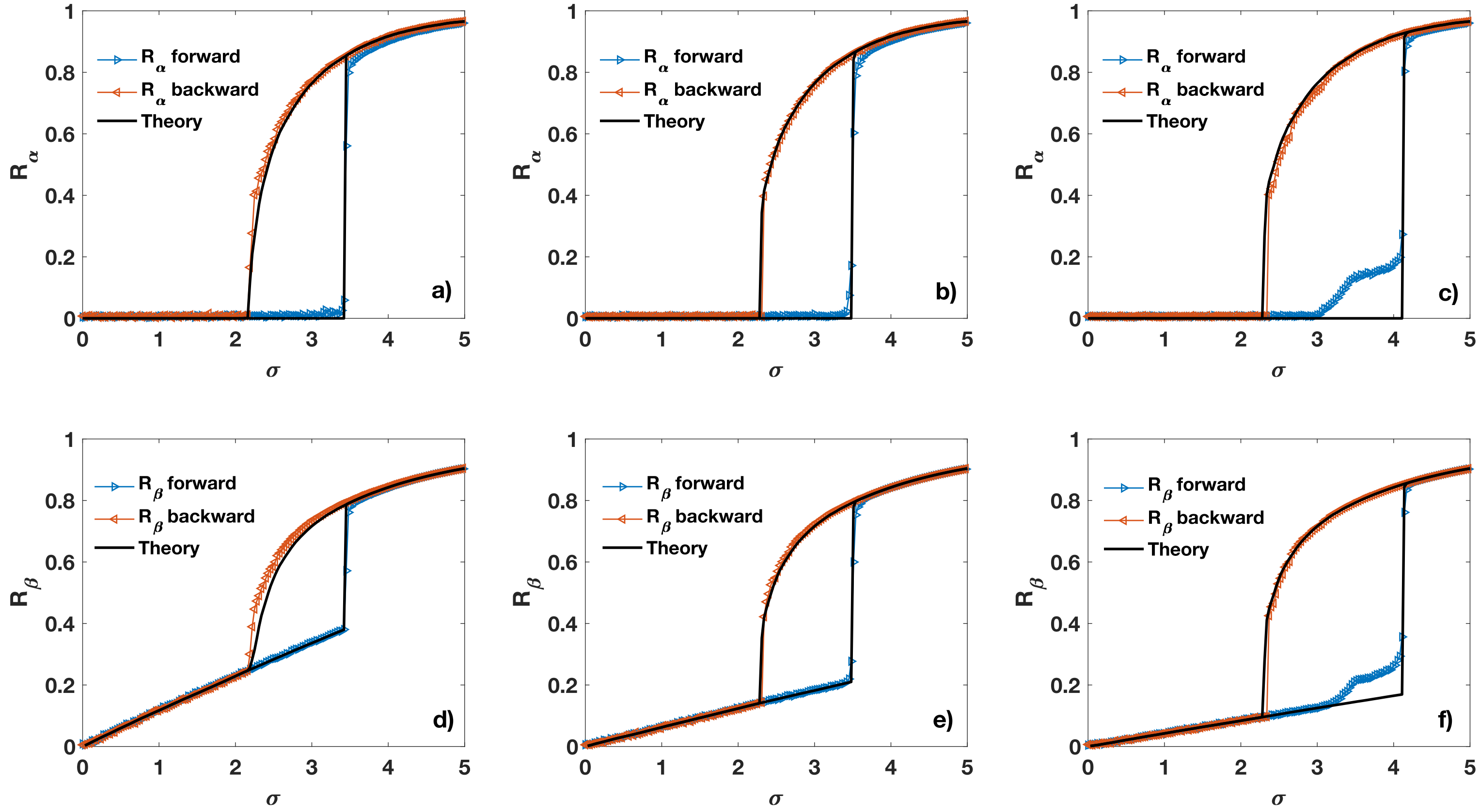
In Dirac synchronisation each node is assigned to phases α_r, β_r each node can be classified in four classes

(streamplots shown in the figure):

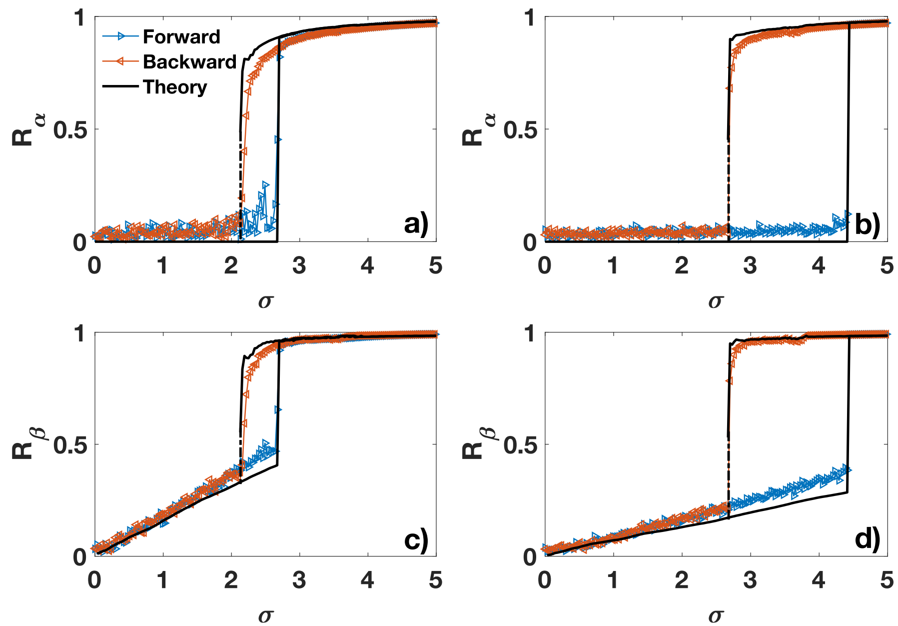
- Both α_r, β_r frozen
- α_r frozen while β_r drifting
- α_r drifting while β_r frozen
- Both α_r, β_r drifting
- **From this classification we can derive and approximated predicted phase diagram**



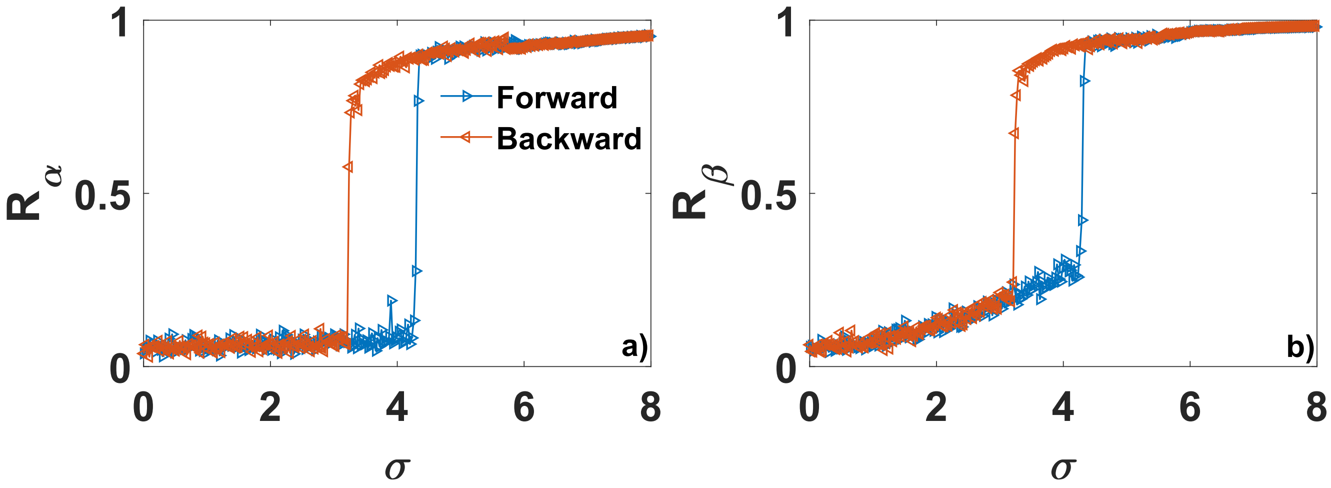
Dependence on z comparison with theory on fully connected network



Dirac synchronisation on Poisson networks



Dirac synchronisation on C.elegans

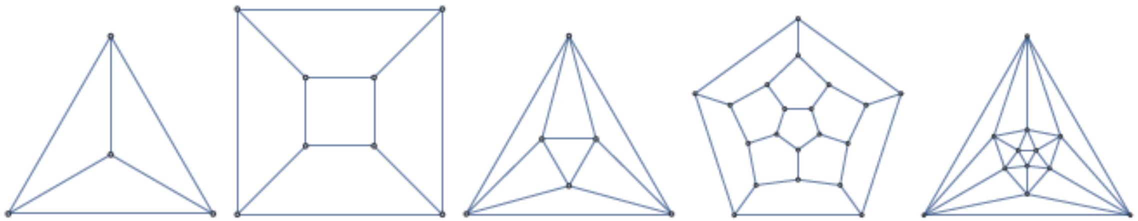


Global synchronisation of topological signals on simplicial and cell complexes

Further Topological and Combinatorial properties of higher-order networks

Cell complexes

(a) (b) (c) (d) (e)



(a) (b) (c) (d) (e)

A cell complex $\hat{\mathcal{K}}$ has the following two properties:

(a) it is formed by a set of cells that is closure-finite, meaning that every cell is covered by a finite union of open cells;

(b) given two cells of the cell complex $\alpha \in \hat{\mathcal{K}}$ and $\alpha' \in \hat{\mathcal{K}}$ then either their intersection belongs to the cell complex, i.e. $\alpha \cap \alpha' \in \hat{\mathcal{K}}$ or their intersection is a null set, i.e. $\alpha \cap \alpha' = \emptyset$.

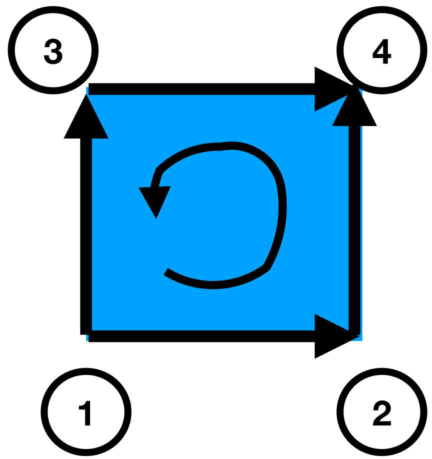
Boundary matrix of a cell complex

The boundary matrix of a cell complex has matrix elements

$$B_{[m]}(\alpha_r^{m-1}, \alpha_s^m) = \begin{cases} 1 & \textbf{if } \alpha_r^{m-1} \sim \alpha_s^m \\ -1 & \textbf{if } \sigma_r^{m-1} \nsim \sigma_s^m \\ 0 & \textbf{otherwise} \end{cases}$$

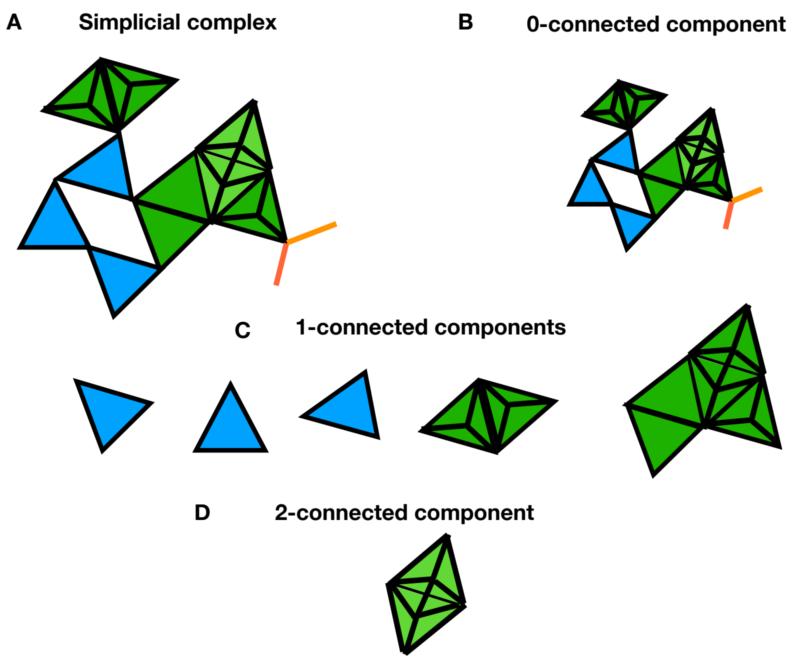
Example

$$\mathbf{B}_{[1]} = \begin{array}{c|cccc} & [1,2] & [1,3] & [3,4] & [2,4] \\ \hline [1] & -1 & -1 & 0 & 0 \\ [2] & 1 & 0 & 0 & -1 \\ [3] & 0 & 1 & -1 & 0 \\ [4] & 0 & 0 & 1 & 1 \end{array}, \quad \mathbf{B}_{[2]} = \begin{array}{c|c} & [1,2,4,3] \\ \hline [1,2] & 1 \\ [1,3] & -1 \\ [3,4] & -1 \\ [2,4] & 1 \end{array}$$



Geometrical properties of simplicial complexes

m-connected components

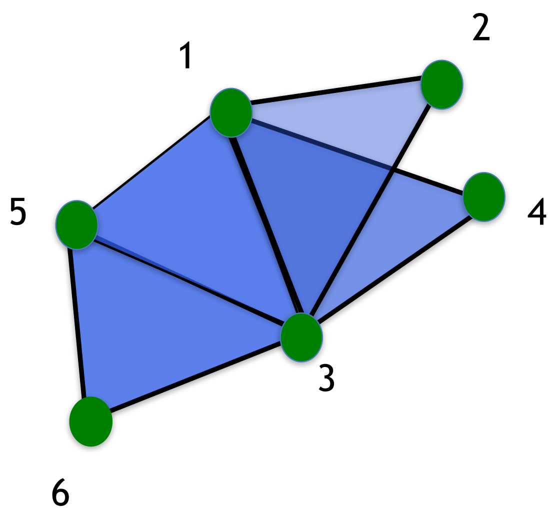


Generalized degree

The generalized degree $k_{m',m}(\alpha)$of a m-face α

is given by the number

of m'-dimensional simplices incident to the m-face α.



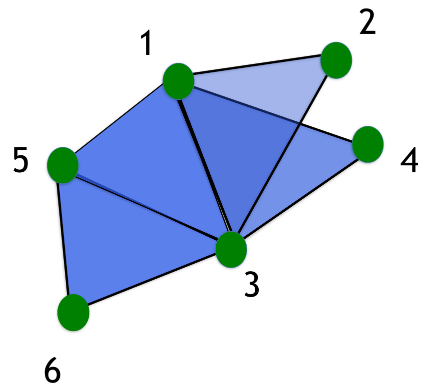
r	$k_{2,0}([r])$
[1]	3
[2]	1
[3]	4
[4]	1
[5]	2
[6]	1

$[r,s]$	$k_{2,1}([r,s])$
[1,2]	1
[1,3]	3
[1,4]	1
[1,5]	1
[2,3]	1
[3,4]	1
[3,5]	2
[3,6]	1
[5,6]	1

Incidence number

To each (d-1)-face α we associate the incidence number

$$n_\alpha = k_{d,d-1}(\alpha) - 1$$



[Bianconi & Rahmede (2016)]

(i,j)	$n_{(i,j)}$
(1,2)	0
(1,3)	2
(1,4)	0
(1,5)	0
(2,3)	0
(3,4)	0
(3,5)	1
(3,6)	0
(5,6)	0

Discrete manifolds

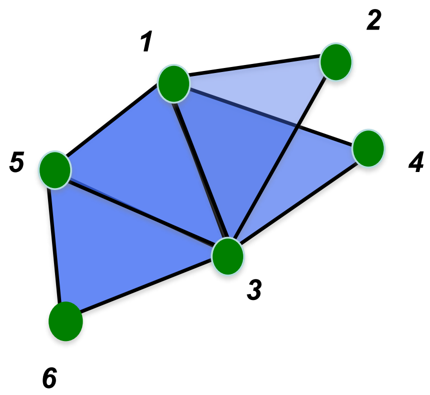
COMBINATORIAL CONDITIONS FOR DISCRETE MANIFOLDS

A discrete manifold $\mathcal{M}$ of dimension d is a pure simplicial complex that satisfies the following two conditions:

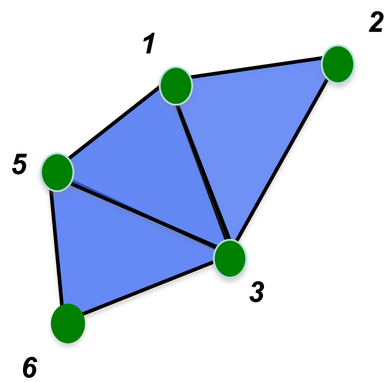
- it is $(d-1)$-connected;
- every two d-simplices α, α' belonging to the simplicial complex $\mathcal{K}$ either overlap on a $(d-1)$-face of $\mathcal{K}$, i.e. $\alpha \cap \alpha' \in S_{d-1}(\mathcal{K})$ or do not overlap, i.e. $\alpha \cap \alpha' = \emptyset$.
- all its $(d-1)$-faces α have an incidence number $n_\alpha \in \{0, 1\}$.

Discrete manifolds

If n_α takes only values $n_\alpha \in \{0,1\}$
each (d-1)-face is incident at most to two d-dimensional simplices.

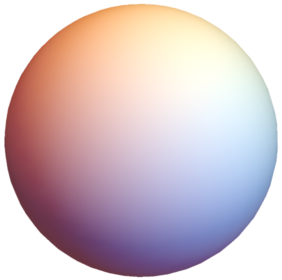


NOT A MANIFOLD



MANIFOLD

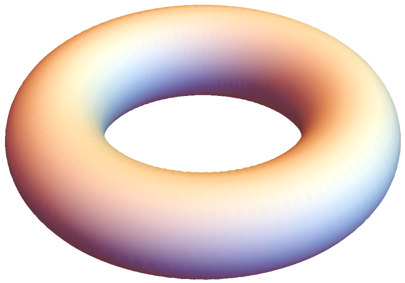
Example key manifolds and their Betti numbers



n-dimensional hypersphere

Betti numbers

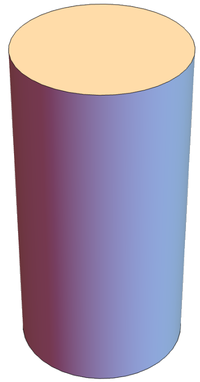
$$\beta_0 = \beta_{n-1} = 1$$
$$\beta_k = 0 \text{ for } 0 < k < n-1$$



n-dimensional torus (cell complex)

Betti numbers

$$\beta_k = \binom{n-1}{k}$$



n-dimensional cylider

Betti numbers

$$\beta_0 = \beta_{n-2} = 1$$
$$\beta_k = 0 \text{ for } k \neq 0, k \neq n-2$$

Global synchronisation on graphs

Uncoupled dynamics of identical node oscillators

Consider coupled identical oscillators defined on the nodes, captured by the 0-cochain $\mathbf{X} \in C^0$ with value $\mathbf{x}_r \in \mathbb{R}^d$ on each node i .

In absence of interactions these nodes obey the same dynamics

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r)$$

with arbitrary non-linear function $\mathbf{f}(\mathbf{x})$.

Global synchronisation on graphs

Consider the coupling of the oscillators implemented with the graph Laplacian leading to the coupled dynamics

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r) - \sigma \sum_{\beta} [L_{[0]}]_{rs} \, \mathbf{h}(\mathbf{x}_s)$$

with arbitrary non-linear functions $\mathbf{f}(\mathbf{x}), \mathbf{h}(\mathbf{x})$.

The global synchronisation is a state in which

$$\mathbf{x}_r = \mathbf{x}_s \ \forall r, s \in Q_0(\mathscr{K})$$

Global synchronisation state of topological signals

The global synchronisation is a state in which

$$\mathbf{x}_r = \mathbf{x}_s \ \forall r, s \in Q_0(\mathscr{K})$$

The coupled dynamics

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r) - \sigma \sum_{\beta} [L_{[0]}]_{rs} \ \mathbf{h}(\mathbf{x}_s)$$

admits always a global synchronisation state in which all the node haves the same dynamics.

In fact the harmonic eigenvector of the graph Laplacian is constant

$$\mathbf{u}^{harm} \propto \mathbf{1}$$

Synchronised state

The globally synchronised state $\mathbf{x}_r = \mathbf{x}^\star(t)\ \forall r \in Q_0(\mathscr{K})$.

Satisfies the dynamics

$$\frac{d\mathbf{x}^\star}{dt} = \mathbf{f}(\mathbf{x}^\star)$$

i.e. it describes a stationary state of the coupled oscillators.

Under which conditions is this solution stable for the coupled oscillators?

Master Stability Function for graphs

- The Master Stability Function establishes the dynamical conditions ensuring the stability of global synchronisation.
- It depends on the non-zero spectrum of the graph Laplacian.
- It is based on an expansion around a stable solution of the uncoupled dynamics.

Master Stability Function for graphs

Expanding for $\delta \mathbf{x}_r = \mathbf{x}_r - \mathbf{x}^\star$ we obtain

$$\frac{d\delta \mathbf{x}_r}{dt} = \mathbf{J_f}(\mathbf{x}^\star)\delta \mathbf{x}_r - \sigma \sum_{s=1}^{N} L_{[0]}(r,s)\mathbf{J_h}(\mathbf{x}^\star)\delta \mathbf{x}_s$$

This equation can be projected on the eigenmodes $\boldsymbol{\eta}_i$ of the graph Laplacian obtaining

$$\frac{d\boldsymbol{\eta}_i}{dt} = \left[\mathbf{J_f}(\mathbf{x}^\star) - \sigma\lambda_i \mathbf{J_h}(\mathbf{x}^\star)\right]\boldsymbol{\eta}_i$$

Therefore the synchronised state is stable if the maximum Lyapunov exponent of the above equation obeys $\Lambda_{max}(\lambda_i) < 0 \forall i$

Global synchronisation of higher-order topological signals

Uncoupled dynamics of topological signals

Consider coupled identical oscillators defined on the n-simplices, captured by the n-cochain $\mathbf{X} \in C^n$ with $n > 0$ and values $\mathbf{x}_r \in \mathbb{R}^d$ on each n-simplex r .

In absence of interactions these simplices obey the same dynamics

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r)$$

To insure invariance of the uncoupled equations upon change of orientation of each simplex we must impose that $\mathbf{f}(\mathbf{x})$ is an **odd** function, i.e. $\mathbf{f}(\mathbf{x}) = -\mathbf{f}(-\mathbf{x})$.

Proof

Consider the uncoupled dynamics $$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r)$$

Upon change of orientation of the simplex r we have $\mathbf{x}_r \rightarrow -\mathbf{x}_r$.

Therefore the dynamics becomes $$\frac{d\mathbf{x}_r}{dt} = -\mathbf{f}(-\mathbf{x}_r)$$

Imposing invariance of the dynamics under this change of orientation implies that the function $\mathbf{f}(\mathbf{x})$ must be odd, i.e. $\mathbf{f}(\mathbf{x}) = -\mathbf{f}(-\mathbf{x})$.

Coupled identical topological signals

- The coupled dynamics obeys

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r) - \sigma \sum_{\beta} [L_{[n]}]_{rq} \, \mathbf{h}(\mathbf{x}_q)$$

- where in order to ensure invariance under change of orientation of the simplifies $\mathbf{h}(\mathbf{x})$ should be an **odd** function.

Global synchronisation state of topological signals

Recall that for higher order topological signals, the signs of the signal is determined by the orientation of the simplex, i.e.

$$\mathbf{x}(\alpha_r) = -\mathbf{x}(-\alpha_r)$$

For instance a positive sign of an edge flux is relative to the orientation chosen for that edge.

It follows that the state of global synchronisation is a state in which

$\mathbf{x}_r = u_r \bar{\mathbf{x}}$ **with** $u_r \in \{1, -1\}\ \forall r \in Q_n(\mathcal{K})$

Global topological synchronisation

- It follows that the coupled dynamics

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r) - \sigma \sum_q [L_{[n]}]_{rq} \, \mathbf{h}(\mathbf{x}_q)$$

- can lead to global synchronisation only if the kernel of the Hodge Laplacian $\mathbf{L}_{[n]}$ admits an eigenvector $\mathbf{u}$ with elements of constant absolute value.

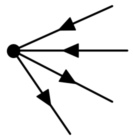
- Therefore for identical higher-order oscillators there are not only dynamical but also topological constraints to global synchronisation

Topological conditions for global synchronisation

- Assume $\mathbf{u}$ is a vector of elements $|u_r| = 1$.
- Global synchronisation can only happen if there is one such vector $\mathbf{u}$ in the kernel of the Hodge Laplacian $\mathbf{L}_{[n]}$.
- Therefore we must have $\mathbf{B}_{[n]}\mathbf{u} = \mathbf{0}, \mathbf{u}^{\top}\mathbf{B}_{[n+1]} = \mathbf{0}$

Topological constraints for global synchronisation

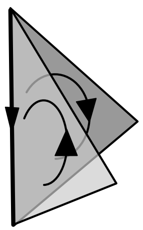
$\sigma^{(0)}$ *0-simplex*



$$\mathbf{B}_1 = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & \dots & 0 \\ \times & \times & \times & \times & \times & \dots & \times \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \end{pmatrix}$$

$$\mathbf{B}_1 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \times \\ \vdots \end{pmatrix}$$

$\sigma^{(1)}$ *1-simplex*



$$\mathbf{B}_2 = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ \times & \times & \times & \dots & \times \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$\mathbf{B}_2 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \times \\ \vdots \end{pmatrix}$$

Assume $\mathbf{u}$ is a vector of elements $|u_r| = 1.$

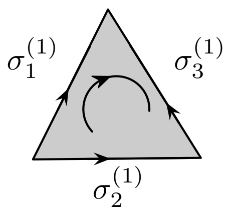
The condition $\mathbf{B}_{[n]}\mathbf{u} = \mathbf{0}$

This implies that:

The simplicial or cell complex must be balanced

Topological constraints for global synchronisation

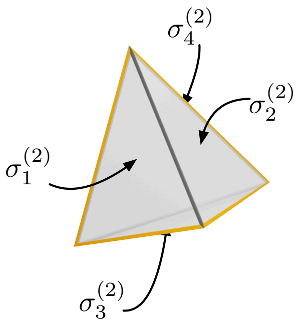
$\sigma^{(2)}$ *2-simplex*



$$\mathbf{B}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$(1,1,1)\mathbf{B}_2 = -1 \neq 0$$

$\sigma^{(3)}$ *3-simplex*



$$\mathbf{B}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(1,1,1,1)\mathbf{B}_3 = 0$$

Assume $\mathbf{u}$ is a vector of elements $|u_r| = 1$.

The condition $\mathbf{u}^\top \mathbf{B}_{[n+1]} = \mathbf{0}$

This implies that:

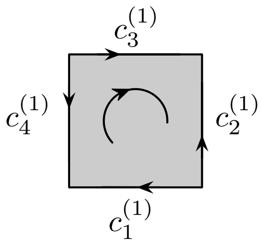
On simplicial complexes topological signals

of odd dimension can never achieve

global synchronisation

Topological constraints for global synchronisation

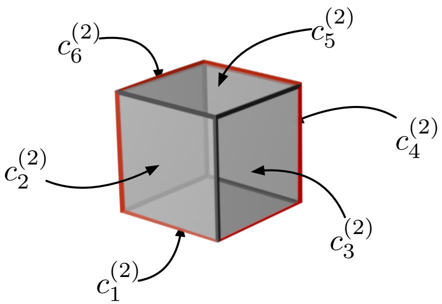
$c^{(2)}$ *2-cell*



$$\mathbf{B}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$(1,1,1,1)\mathbf{B}_2 = 0$$

$c^{(3)}$ *3-cell*



$$\mathbf{B}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(1,1,1,1,1,1)\mathbf{B}_3 = 0$$

Assume $\mathbf{u}$ is a vector of elements $|u_r| = 1$.

The condition $\mathbf{u}^\top \mathbf{B}_{[n+1]} = \mathbf{0}$

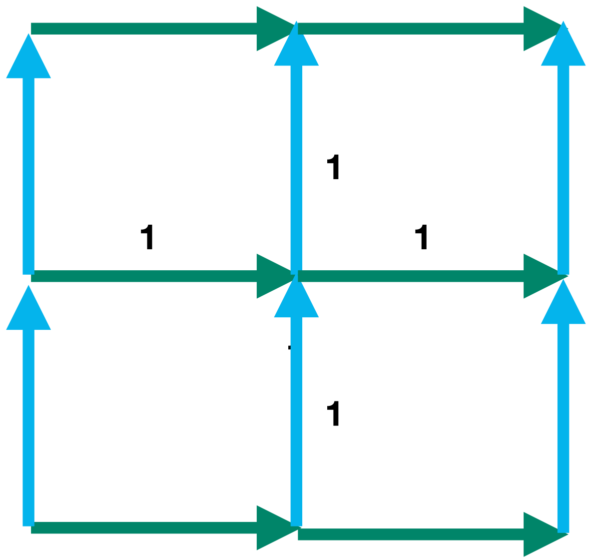
This implies that:

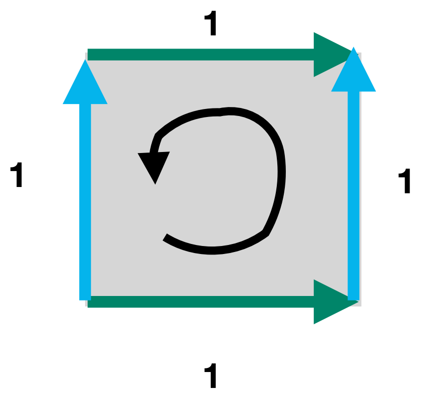
Cell complexes of any dimension can

achieve global synchronisation

overcoming topological obstruction

Square lattice with periodic boundary conditions (torus)





- Consider a square lattice with periodic boundary conditions (a torus).
- The eigenvector $\mathbf{u} = \mathbf{1}$ defined on each link of the network is in the kernel of $\mathbf{L}_{[1]}$, i.e. $\mathbf{1} \in \mathrm{ker}(\mathbf{L}_{[1]})$

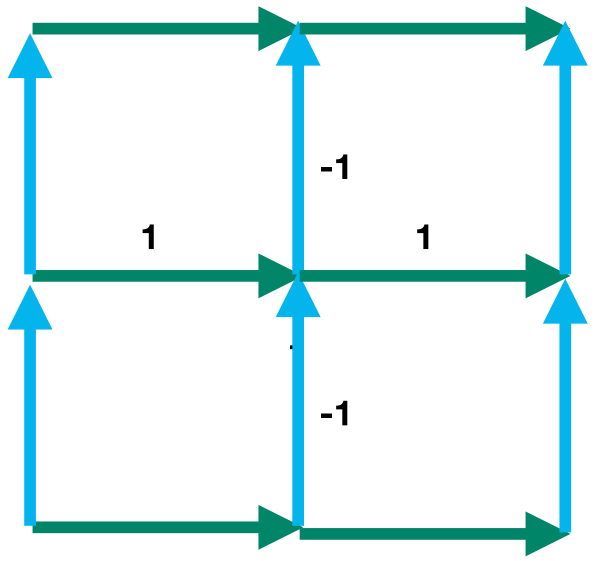
Indeed $\mathbf{B}_{[1]}\mathbf{u} = \mathbf{0}, \mathbf{u}^\top \mathbf{B}_{[2]} = \mathbf{0}$

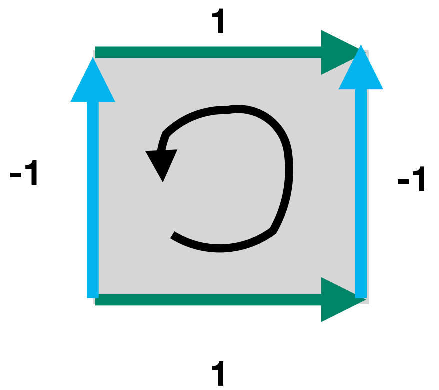
or

div $\mathbf{u} = \mathbf{0}$, curl $\mathbf{u} = \mathbf{0}$

(see figures)

Square lattice with periodic boundary conditions (torus)





- Consider a square lattice with periodic boundary conditions (a torus).
- The eigenvector $\mathbf{u}$ defined on each link of the network and elements $u_r = 1$ on each x-type link and $u_r = -1$ on each y-type link is in the kernel of $\mathbf{L}_{[1]}$, i.e.$\mathbf{u} \in$ ker$(\mathbf{L}_{[1]})$

Indeed $\mathbf{B}_{[1]}\mathbf{u} = \mathbf{0}, \mathbf{u}^\top \mathbf{B}_{[2]} = \mathbf{0}$

or

div $\mathbf{u} = \mathbf{0}$, curl $\mathbf{u} = \mathbf{0}$

(see figures)

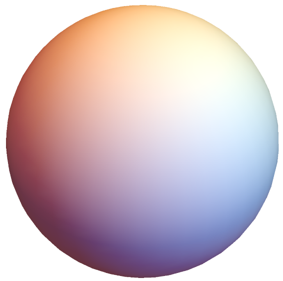
Properties of global synchronisation of topological signals

- The globally synchronised state is aligned with an harmonic eigenvector of the Hodge Laplacian, i.e. **requires topologies with holes that span the entire simplicial or cell complex**.

- Since the Hodge Laplacian has an harmonic space with dimension given by the Betti number, the same simplicial or cell complex can sustain different globalised states (see tori)

Example of manifolds sustaining global synchronisation

Synchronisation of $(n-1)$-dimensional topological signal

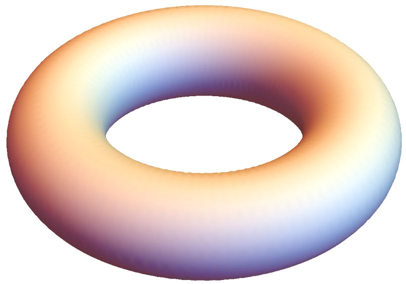


n-dimensional hypersphere

Betti numbers

$$\beta_0 = \beta_{n-1} = 1$$
$$\beta_k = 0 \text{ for } 0 < k < n-1$$

Synchronisation of any k-dimensional topological signal



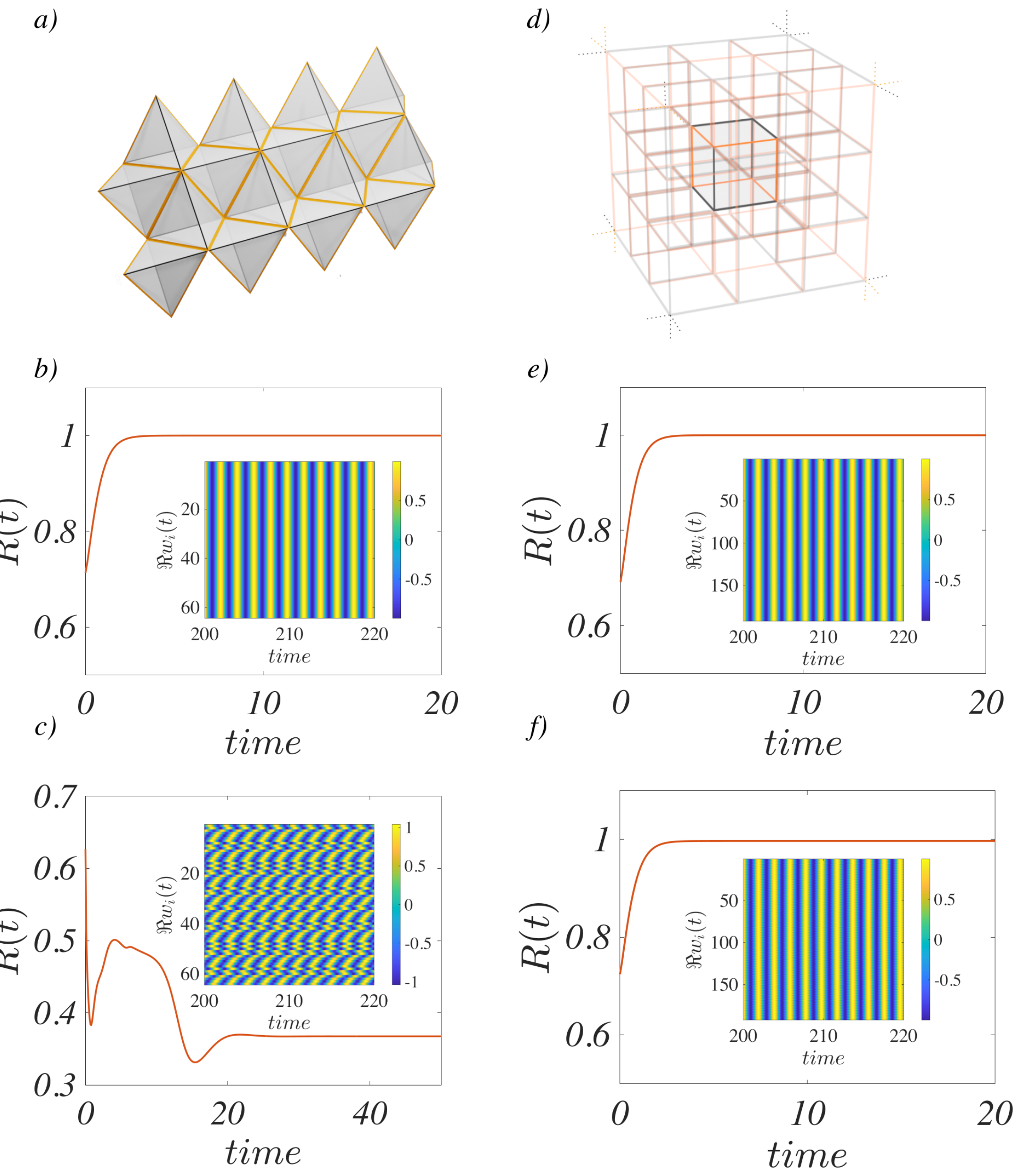
n-dimensional torus (cell complex)

Betti numbers

$$\beta_k = \binom{n-1}{k}$$

Master Stability Function for simplicial and cell complexes

- The Master Stability Function establishes the dynamical conditions ensuring the stability of global synchronisation.
- It depends on the non-zero spectrum of the Hodge Laplacian.
- It should account for the possible degeneracy of the zero eigenvalue (a dimension of the kernel greater than one)
- It is based on an expansion around a stable solution of the uncoupled dynamics.



Global Topological synchronisation

- Simulation of Stuart-Landau coupled oscillators.
- On cell complexes forming square lattices topological signals of any dimension can achieve global synchronisation
- On simplicial complexes topological signals of odd dimension can never achieve global synchronisation

Carletti, Giambagli, Bianconi (2022)

The Dirac operator on simplicial complexes

The Dirac operator allows
to study interacting topological signals of different dimensions
coexisting in the same network topology

Dirac operator

$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{B}_{[1]} & 0 \\ \mathbf{B}_{[1]}^\top & 0 & \mathbf{B}_{[2]} \\ 0 & \mathbf{B}_{[2]}^\top & 0 \end{pmatrix},$$

Topological signal "spinor" $\mathbf{s} = \bigoplus_{m=0}^{d} C^d$

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}$$

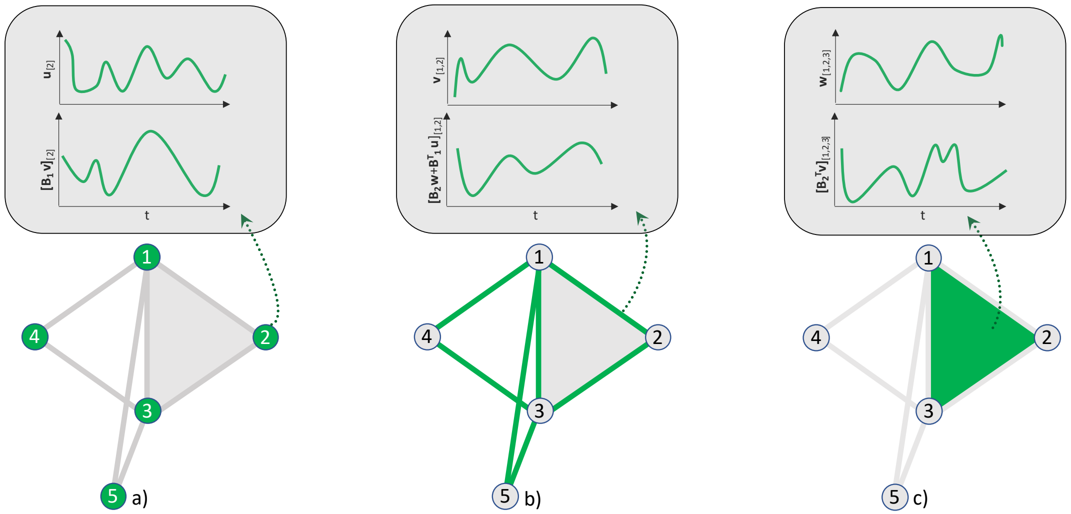
$\mathbf{x}_1$ Node signal

$\mathbf{x}_2$ Link signal

$\mathbf{x}_3$ Triangle signal

The action of the Dirac operator

The Dirac operator allows cross-talking between signals of different dimension



$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{B}_{[1]} & 0 \\ \mathbf{B}_{[1]}^{\top} & 0 & \mathbf{B}_{[2]} \\ 0 & \mathbf{B}_{[2]} & 0 \end{pmatrix}, \text{ acts on } \boldsymbol{\Phi} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix} \quad \rightarrow \quad \mathbf{D}\boldsymbol{\Phi} = \begin{pmatrix} \mathbf{B}_{[1]}\mathbf{x}_2 \\ \mathbf{B}_{[1]}^{\top}\mathbf{x}_1 + \mathbf{B}_{[2]}\mathbf{x}_3 \\ \mathbf{B}_{[2]}^{\top}\mathbf{x}_2 \end{pmatrix}$$

Dirac Turing patterns

Defining $\mathbf{\Phi} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)^\top$ describing topological signals on nodes and links, and 2-cells and the reaction-diffusion dynamics

$$\dot{\mathbf{\Phi}} = F(\mathbf{\Phi}, \mathbf{D}\mathbf{\Phi}) - \gamma\mathbf{D}\mathbf{\Phi},$$

With the matrix of diffusion coefficients given by

$$\boldsymbol{\gamma} = \begin{pmatrix} D_0 & 0 & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & D_2 \end{pmatrix}$$

Giambagli et al. (2022)

The homogeneous pattern

The homogenous pattern
$\boldsymbol{\Phi} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)^\top = \mathbf{u}^\top$ is a solution of the considered dynamics

$$\dot{\boldsymbol{\Phi}} = F(\boldsymbol{\Phi}, \mathbf{D}\boldsymbol{\Phi}) - \gamma \mathbf{D}\boldsymbol{\Phi},$$

If and only if

$$\mathbf{u} \in \ker(\mathbf{D}),$$

Giambagli et al. (2022)

The homogeneous pattern

The condition

$$\mathbf{u} \in \ker(\mathbf{D}),$$

In 2d implies that

$$\mathbf{u}_1 \in \ker \mathbf{L}_{[0]} \quad \mathbf{u}_2 \in \ker \mathbf{L}_{[1]} \quad \mathbf{u}_3 \in \ker \mathbf{L}_{[2]}$$

Which might be allowed only on some special topologies

(e.g. square lattice with periodic boundary conditions, i.e. torus)

Giambagli et al. (2022)

The homogeneous pattern

The condition

$$\mathbf{u} \in \text{ker}(\mathbf{D}),$$

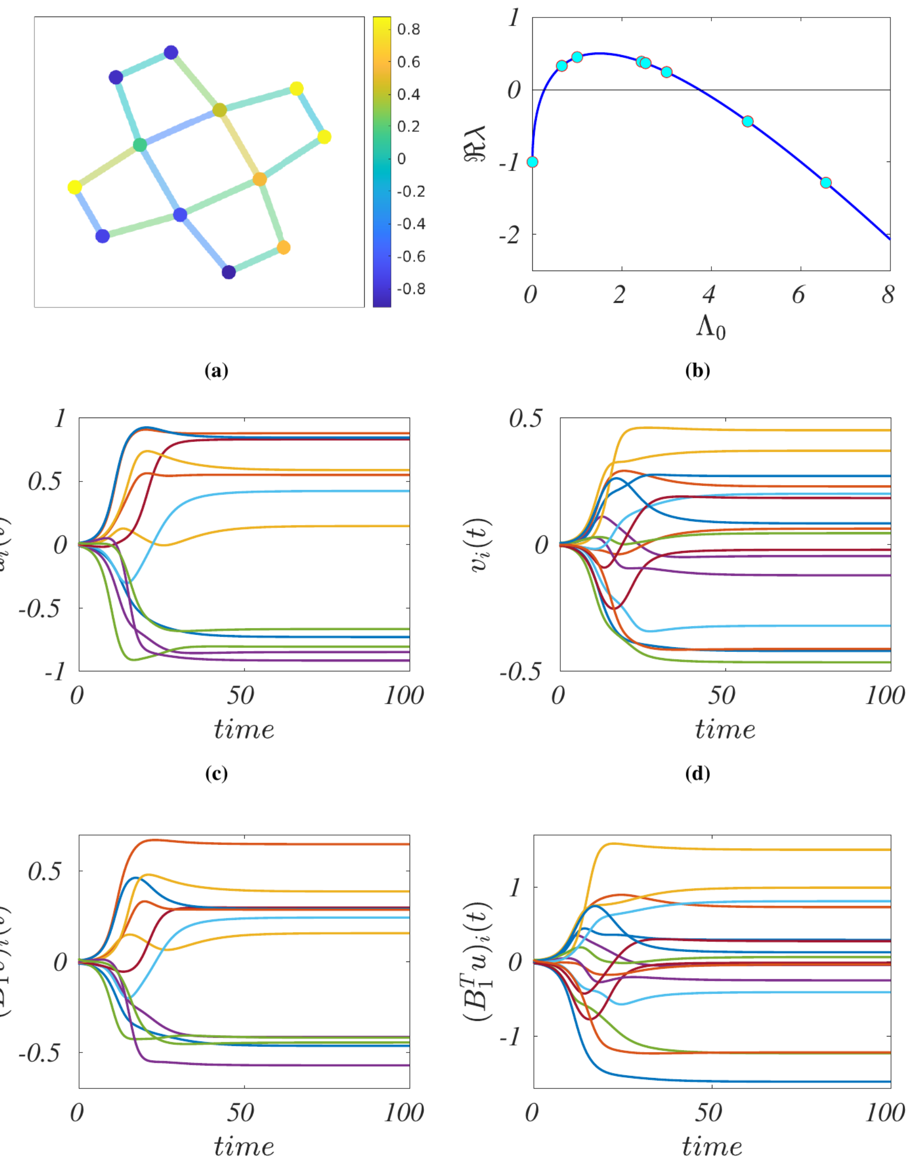
In 1d implies

$$\mathbf{u}_1 = \mathbf{1} \in \text{ker}\mathbf{L}_{[0]} \quad \mathbf{u}_2 \in \text{ker}\mathbf{L}_{[1]}$$

Which implies that the networks have d even degree of the nodes

Giambagli et al. (2022)

Dirac Turing patterns



(a) (b) (c) (d) (e) (f)

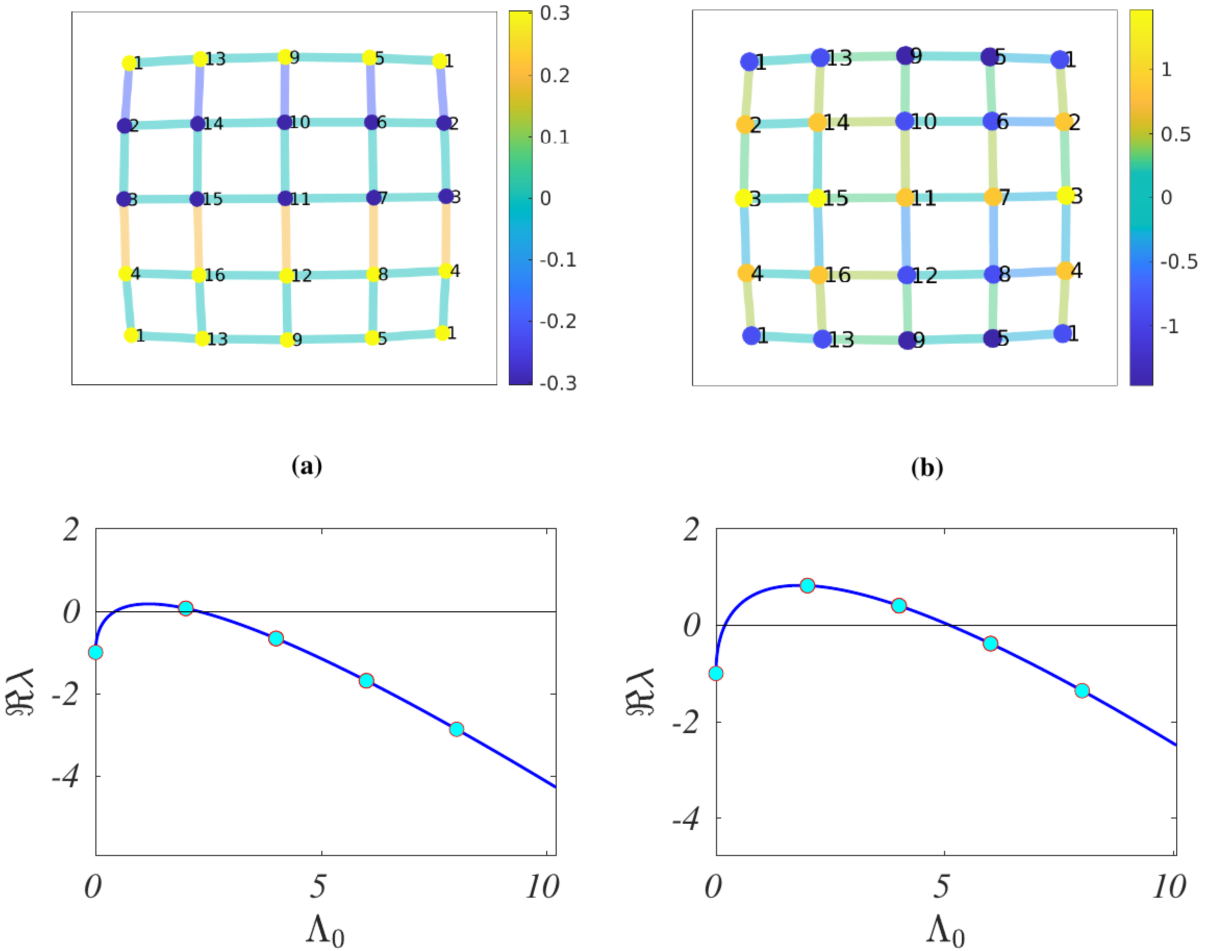
Defining $\mathbf{\Psi} = (\boldsymbol{\theta}, \boldsymbol{\phi})^\top$ describing topological signals on nodes and links and the reaction diffusion dynamics

$$\dot{\mathbf{\Phi}} = F(\mathbf{\Phi}, \mathbf{D}\mathbf{\Phi}) - \gamma \mathbf{D}\mathbf{\Phi},$$

Turing patterns on nodes and links can set in provided suitable topological and dynamical conditions.

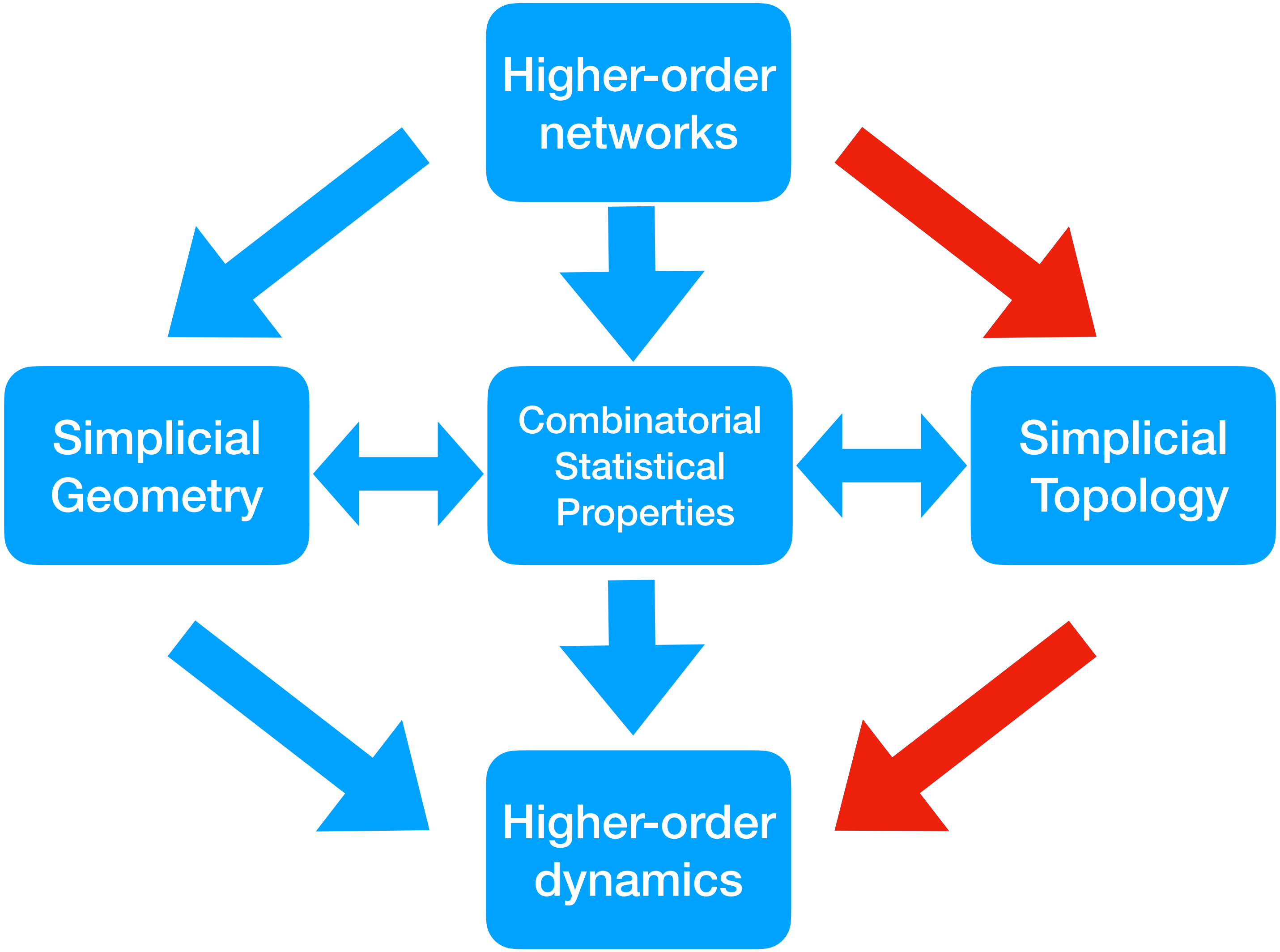
Giambagli et al. (2022)

Dirac Turing patterns



- Hypercubic tessellations of d-dimensional torus admit Turing patterns on any dimension
- The figure show Turing patterns on nodes and links on a 2D Torus.

Higher-order structure and dynamics



Lesson IV:
Dirac synchronization, Global Topological Synchronisation and more

- **Dirac synchronisation**
    - **Phenomenology and Theory**

- **Global topological synchronisation and Master Stability Function**

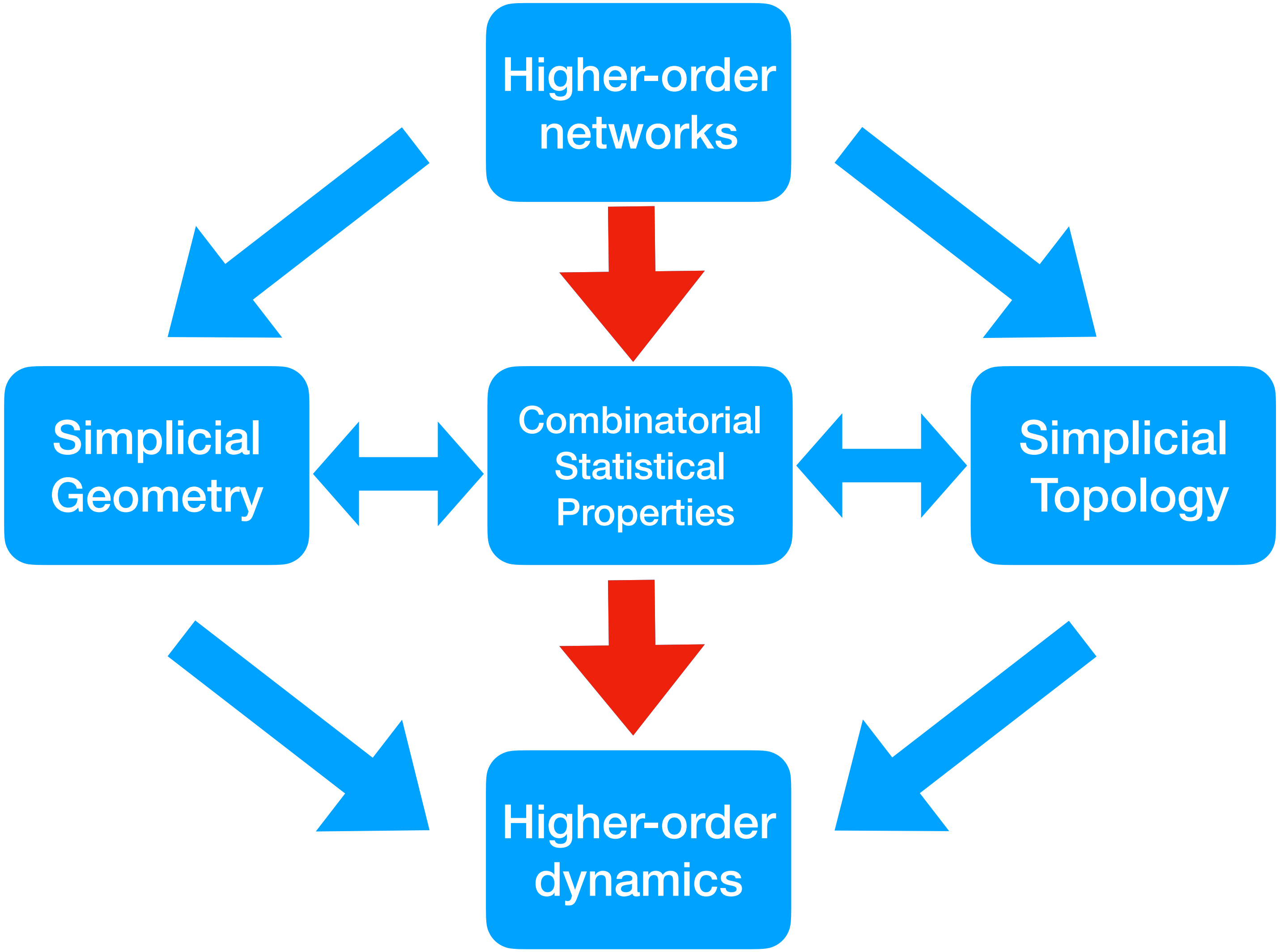
    - **Global synchronisation on graphs**
    - **Global synchronisation on simplicial and cell complexes**

- **Turing patterns coupled by the Dirac operator**

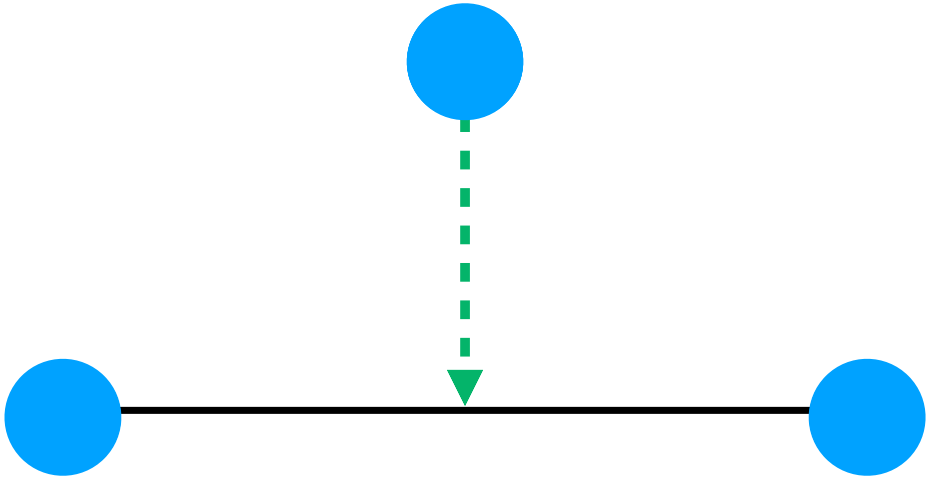
Addendum:

- **Triadic percolation and non-linear dynamics of the giant component**

Higher-order structure and dynamics



Triadic interactions



A triadic interaction occurs when a node affects the interaction between other two nodes

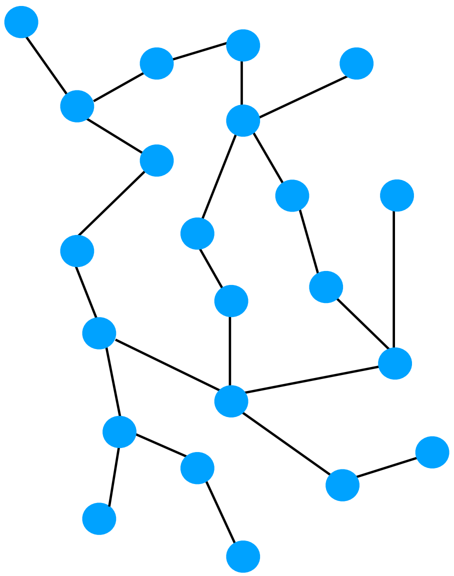
Sign of triadic interactions

A triadic interactions can be positive or negative

The presence of a third species can enhance or can inhibit the interaction between two species

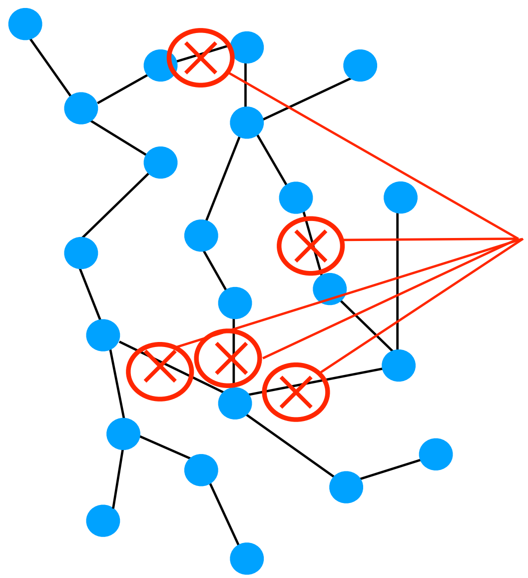
The presence of a glia can change the synaptic interactions between two neurons

Robustness of a network



We assume that a fraction 1-p of links is damaged.
We evaluate the robustness of the network by calculating the fraction R
of nodes in the giant component after this inflicted damage.

Robustness of a network



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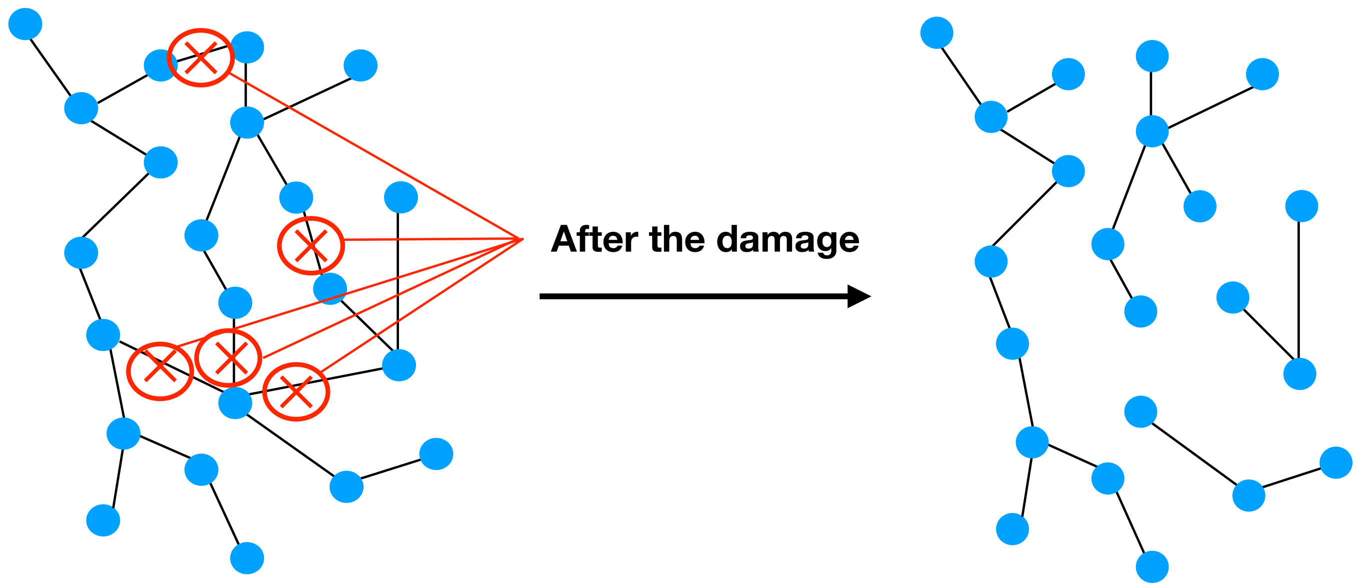
Robustness of a network



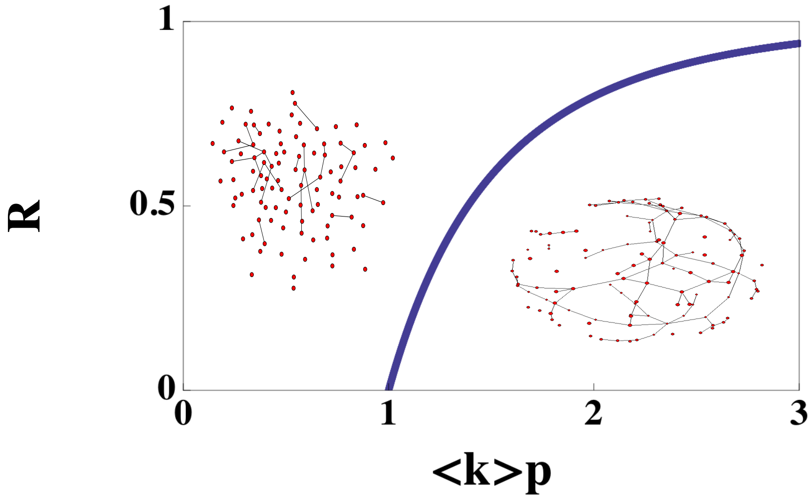
We assume that a fraction 1-p of links is damaged.
We evaluate the robustness of the network by calculating the fraction R of nodes in the giant component after this inflicted damage.

Percolation transition

As links are damaged with probability f=1-p

the fraction R of nodes in the giant component

of an infinite network has a transition from a non-zero to a zero value



$$S = 1 - G_1\left(1 - pS\right)$$

$$R = 1 - G_0(1 - pS)$$

In brain and in climate networks

however the giant component

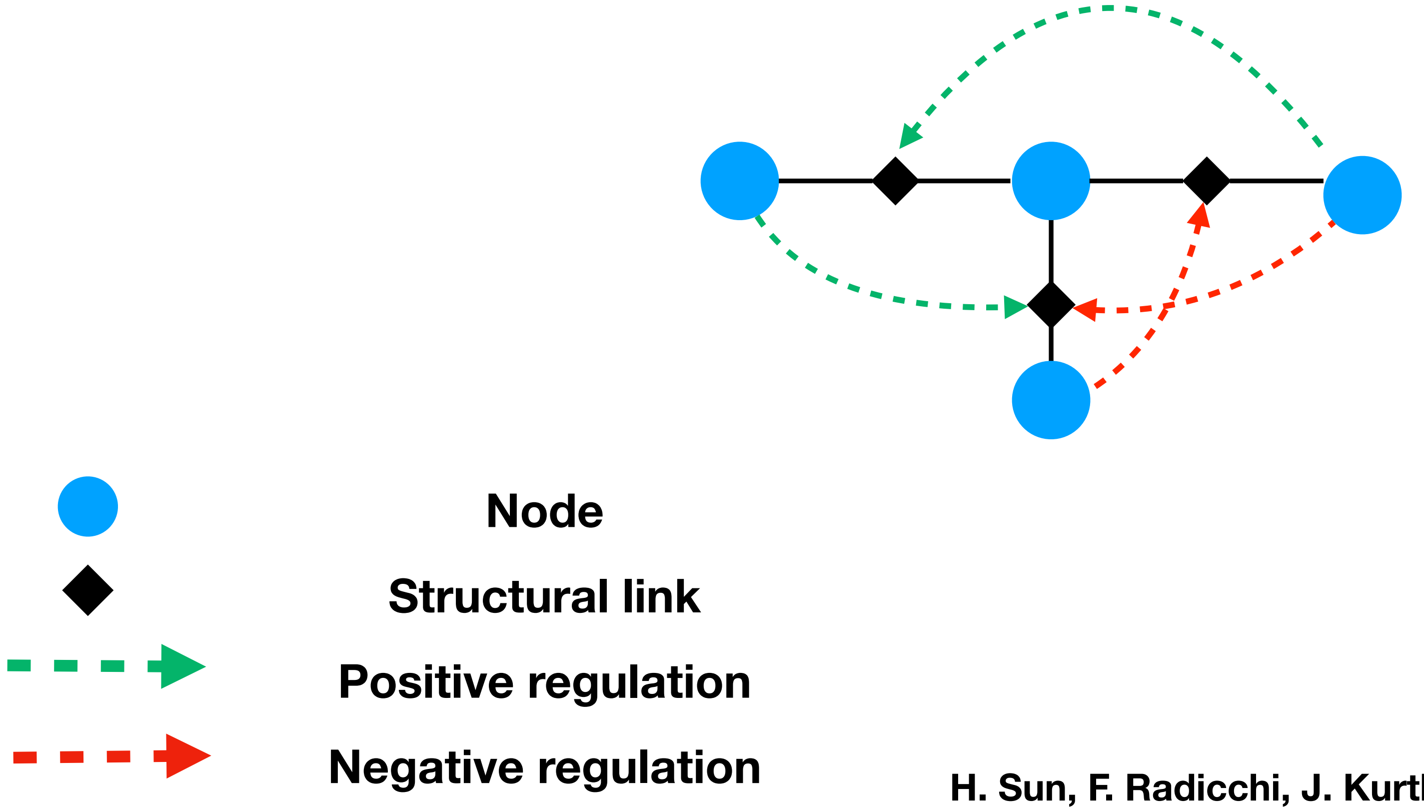
does not reach a steady state and is dynamical

Can percolation be turned

into a fully fledged dynamical process?

H. Sun, F. Radicchi, J. Kurths, G. Bianconi Nature Communications (2023)

Higher-order network with signed triadic interactions



H. Sun, F. Radicchi, J. Kurths, and G. Bianconi (2023)

Activity of nodes and structural links

Regulatory interactions determine which links are active.

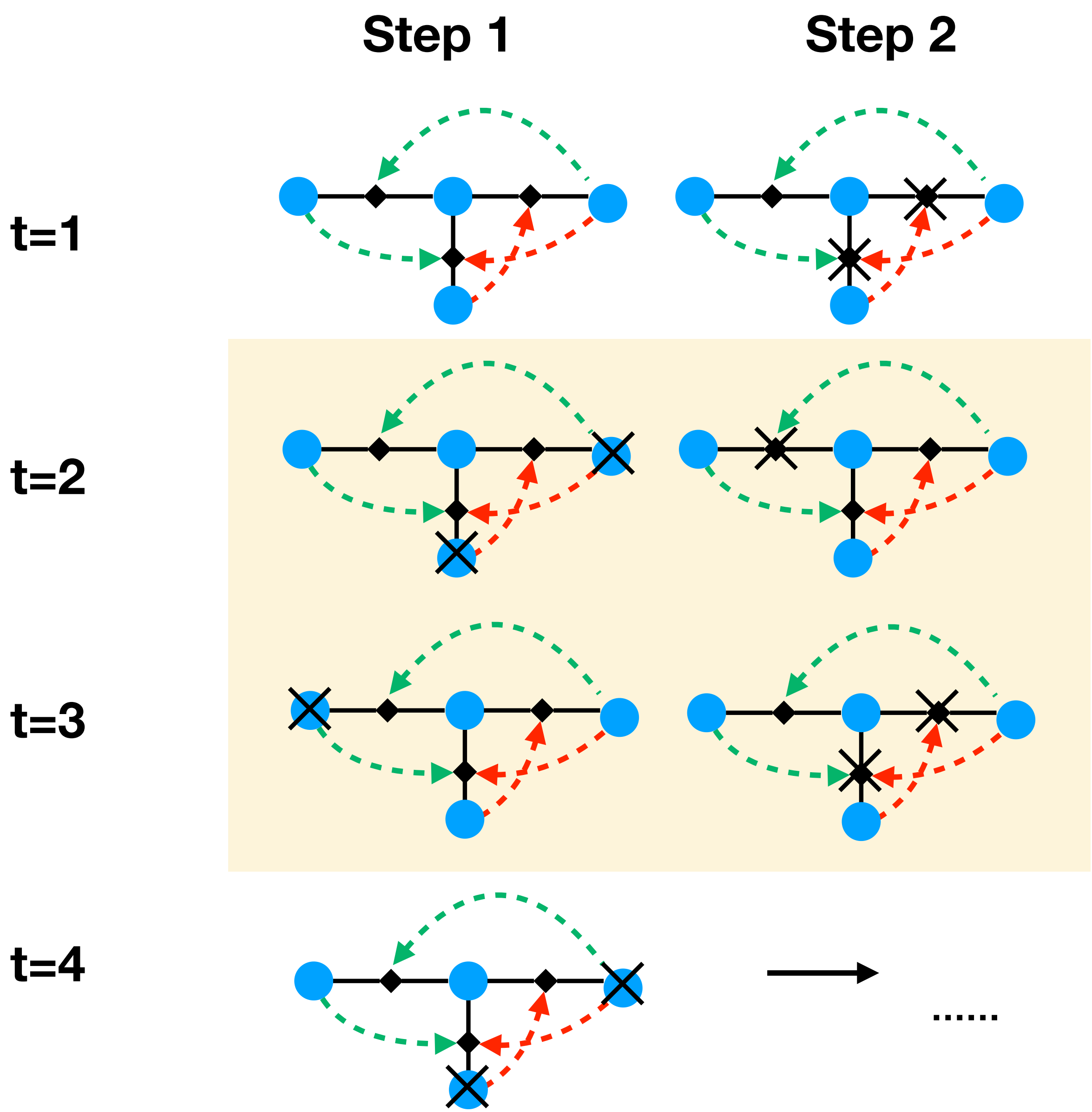
Structural links are active if they are connected to a least a active positive regulator node and they are not connected to any active negative regulator node

Structural interactions determine which nodes are active.

A node is active if it belongs to the giant component of the structural network

Dynamic nature of percolation

- Agorithm:
- Step 1: Evaluate the nodes in the giant component of the structural network. Nodes are active if and only if they belong to the giant component of the network
- Step 2: Deactivate the links that are connected to at least one active negative regulator node or that are not connected to any active positive regulator node. All the other links are damaged with probability q=1-p.
- Repeat from Step 1



Step 1
Step 2
t=1
t=2
t=3
t=4
......

Theory

Step 1

$$S^{(t)} = 1 - G_1\left(1 - p_L^{(t-1)} S^{(t)}\right)$$
$$R^{(t)} = G_0(1 - p_L^{(t-1)} S^{(t)})$$

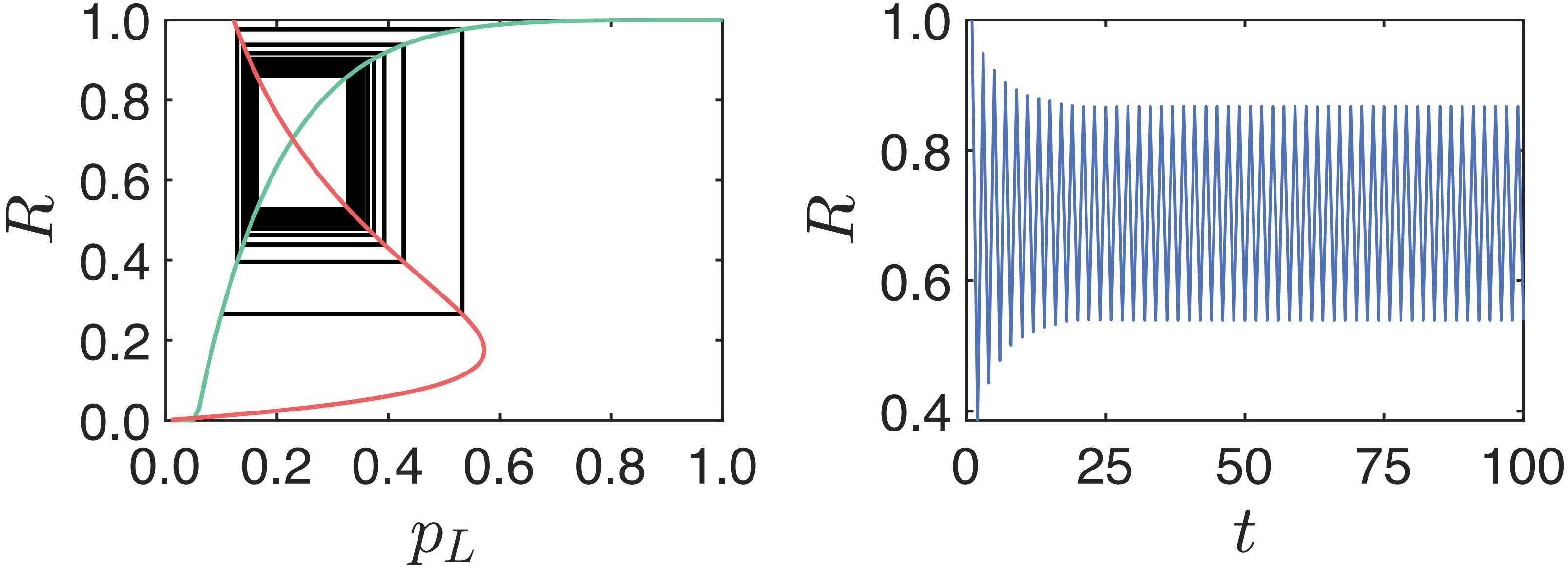
Step 2

$$p_L^{(t)} = pG_0^{[-]}\left(1 - R^{(t)}\right)\left[1 - G_0^{[+]}\left(1 - R^{(t)}\right)\right]$$

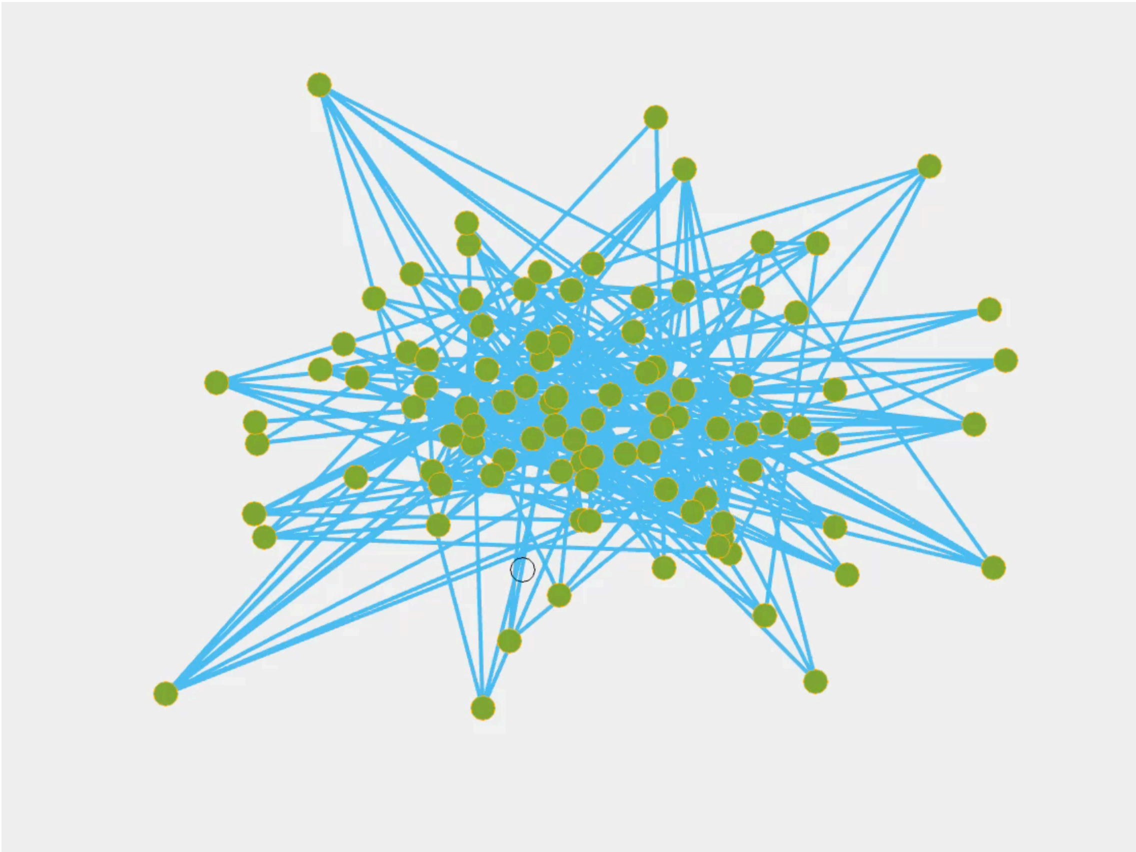
$$R^{(t)} = f(p_L^{(t-1)})$$

$$p_L^{(t)} = g(R^{(t)})$$

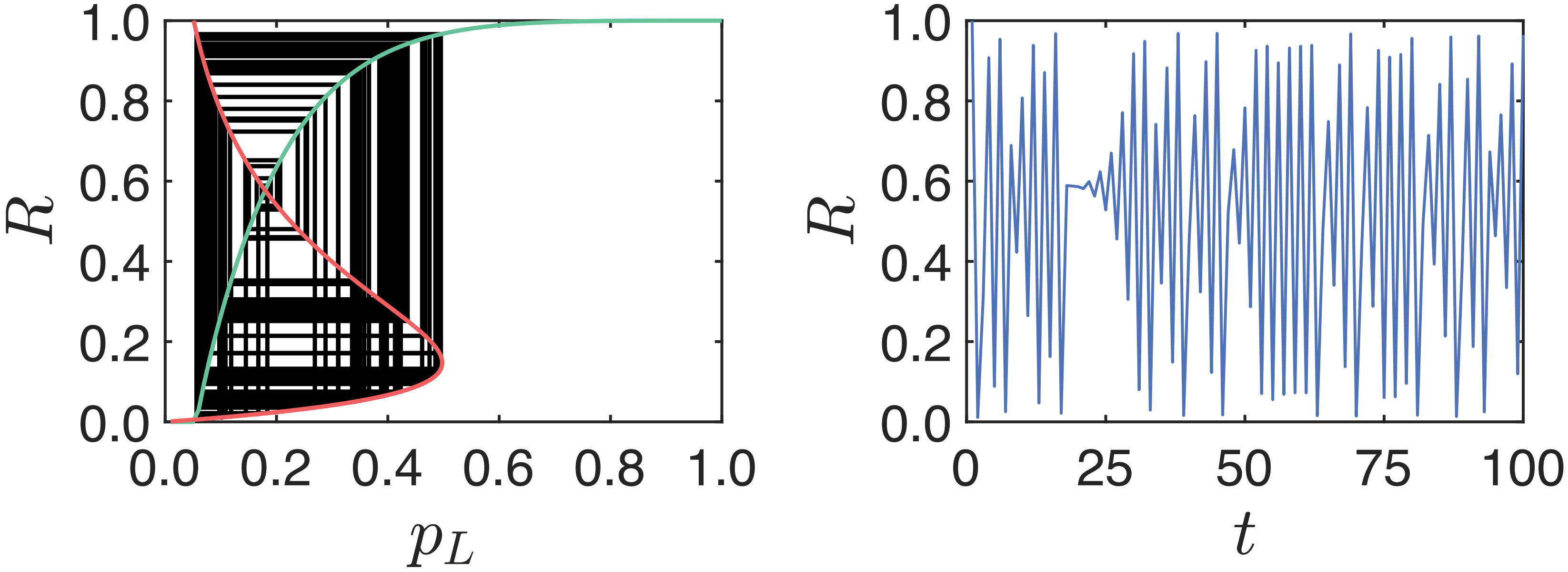
Blinking of the network



Blinking



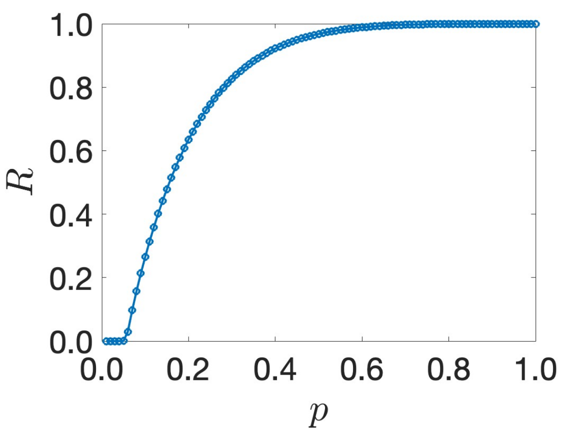
Chaotic pattern of the order parameter of percolation



Chaos in connectivity of the network

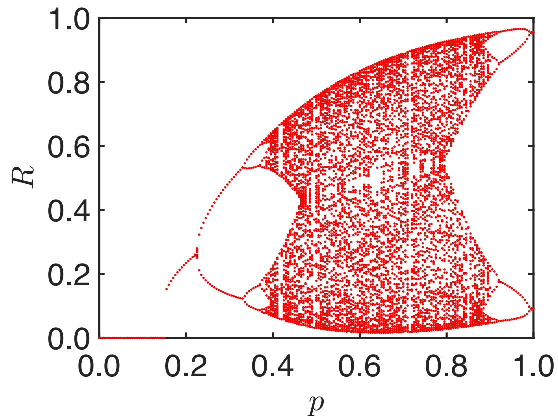
Route to chaos in scale-free networks

Absence of triadic interactions

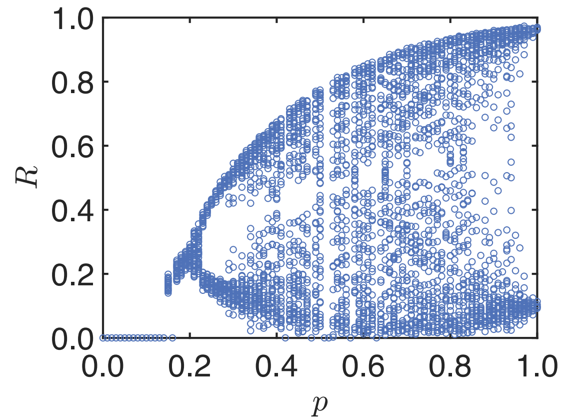


In presence of triadic interactions

Theoretical prediction



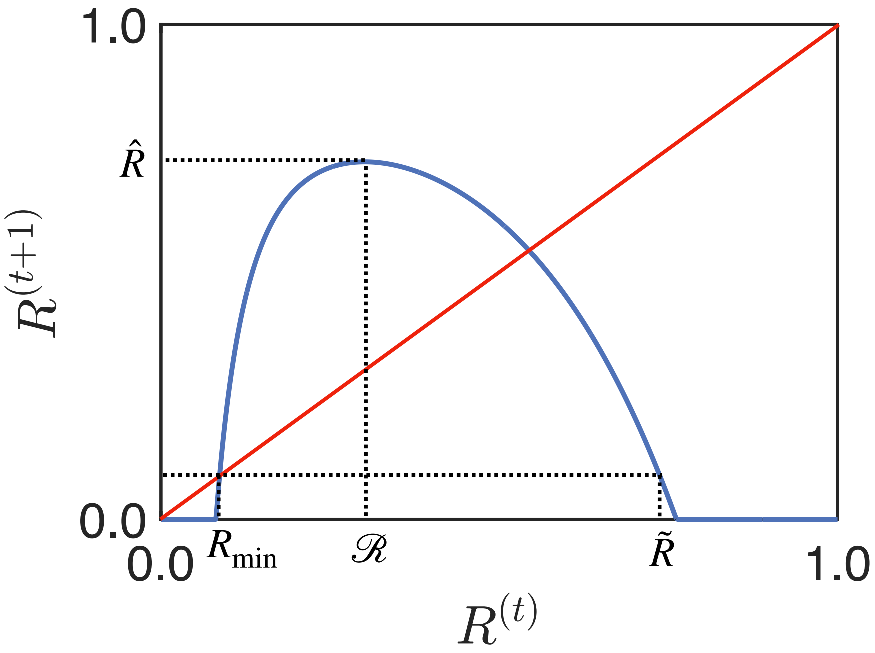
Monte Carlo simulations

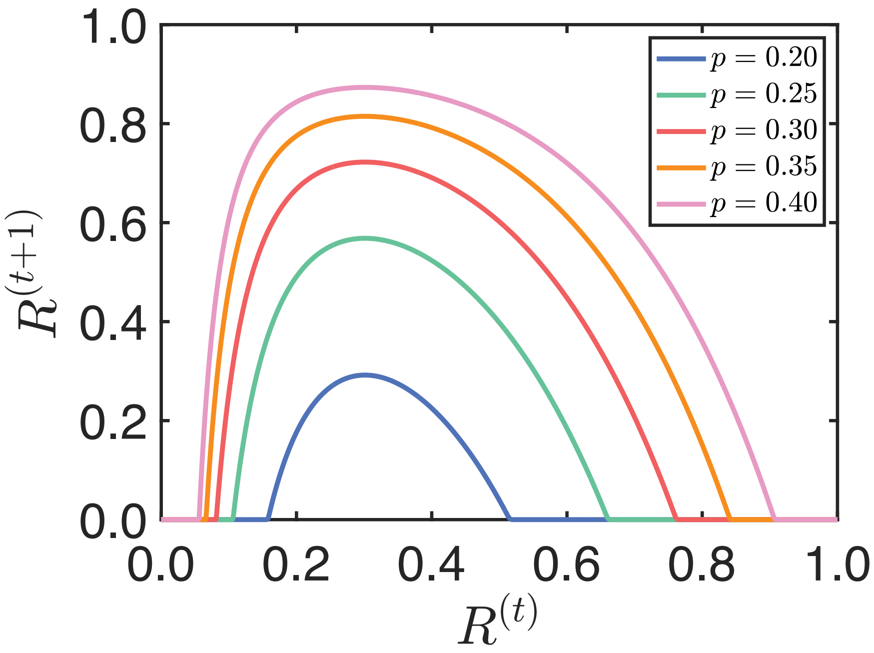


The map of triadic percolation

The map $R^{t+1} = h(R^t)$

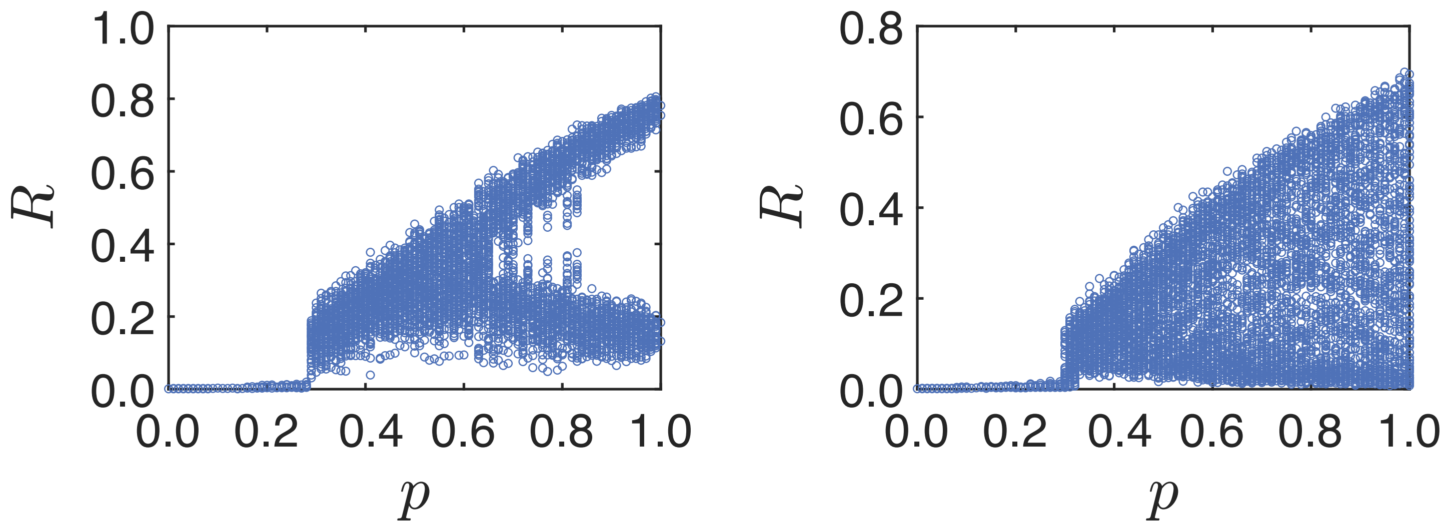
is in the universality class of the logistic map





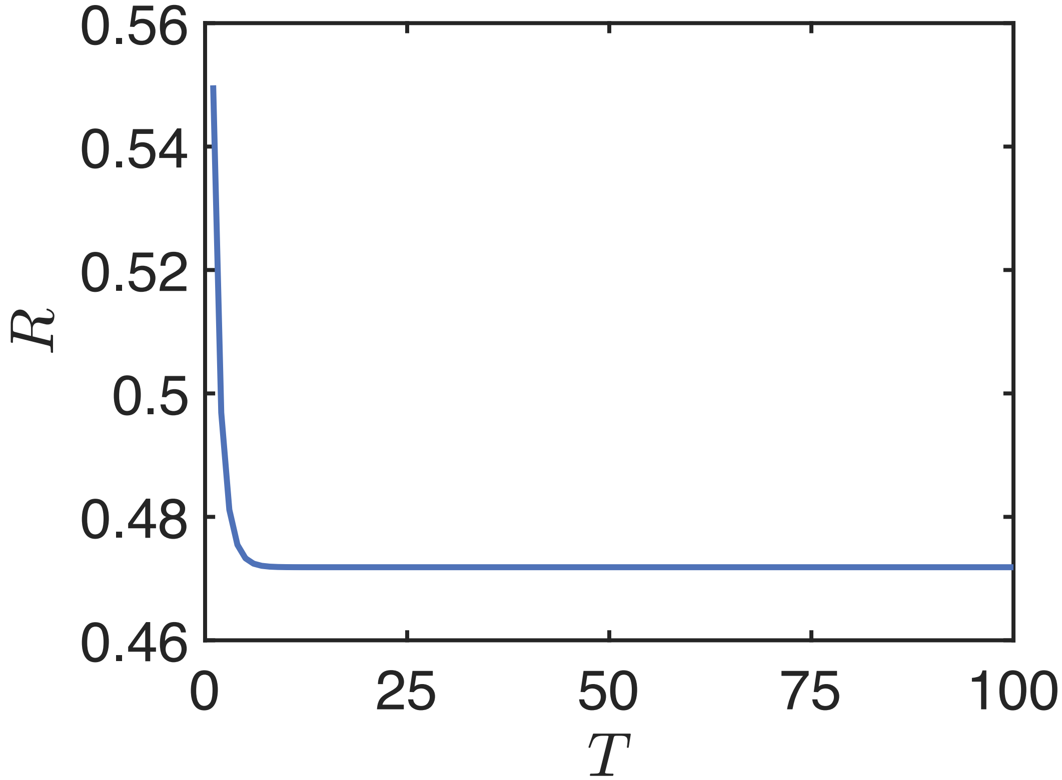
Blinking and chaos in mouse brain network

Mouse brain network+ random regulatory interactions

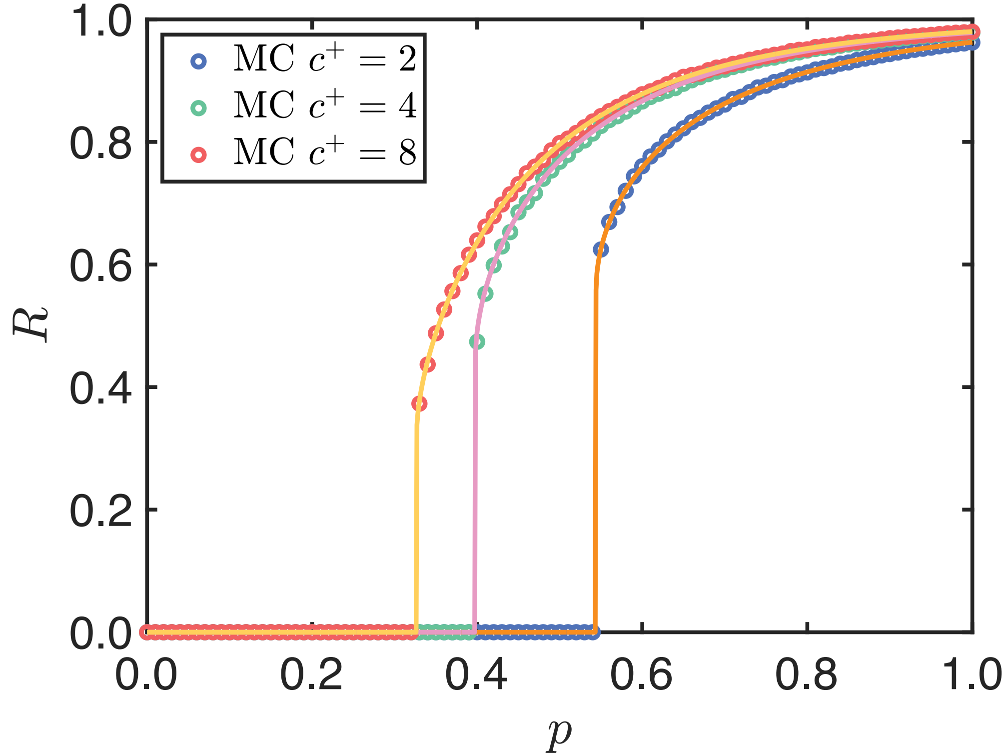


Only positive regulations

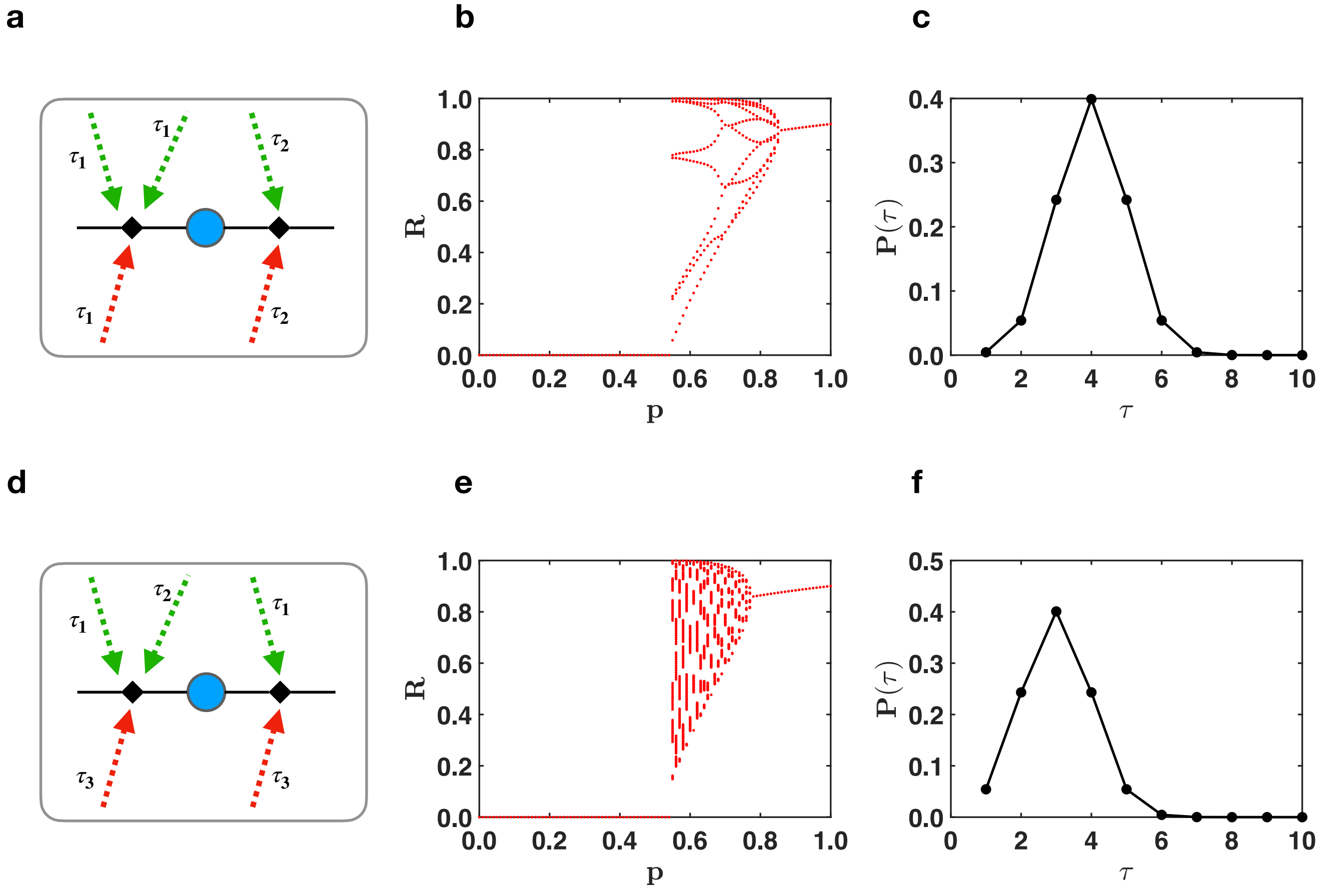
The dynamics always reaches a steady state



The percolation transition is discontinuous and hybrid

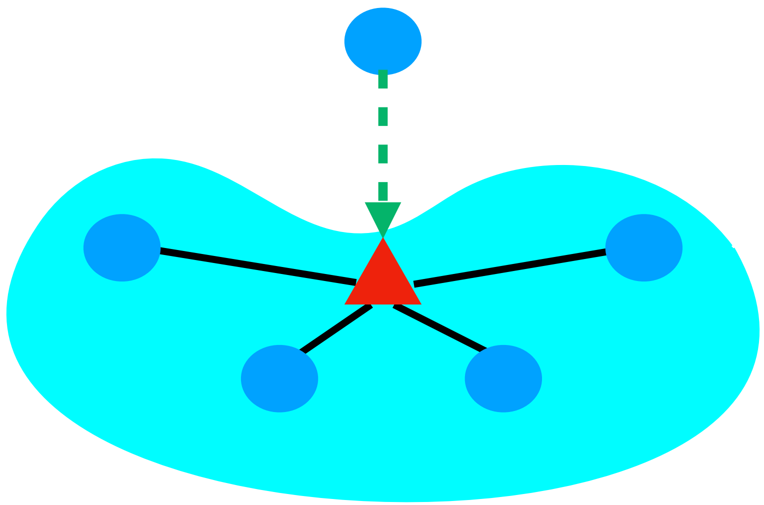


Triadic percolation with time delays

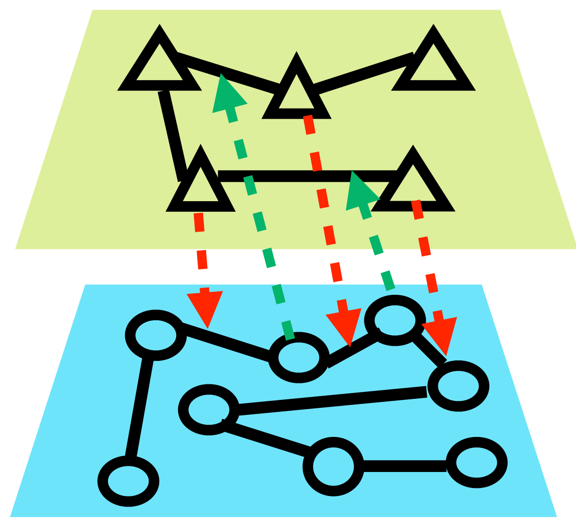


Triadic interactions in more complex settings

Hypergraphs



Multiplex networks



Lesson IV:
Dirac synchronization, Global Topological Synchronisation and more

- Dirac synchronisation
    - Phenomenology and Theory

- Global topological synchronisation and Master Stability Function

- Global synchronisation on graphs
- Global synchronisation on simplicial and cell complexes

- Turing patterns coupled by the Dirac operator

References

Global synchronisation

Carletti, T., Giambagli, L. and Bianconi, G., 2022. Global topological synchronization on simplicial and cell complexes. *Physical Review Letter (in press)*.

Triadic percolation

Sun, H., Radicchi, F., Kurths, J. and Bianconi, G., 2023. The dynamic nature of percolation on networks with triadic interactions. *Nature Communications*, *14*(1), p.1308.

References Dirac operators: Applications

The Dirac operator

G. Bianconi, Topological Dirac equation on networks and simplicial complexes JPhys Complexity (2021)

Bianconi, G., 2022. Dirac gauge theory for topological spinors in 3+ 1 dimensional networks. *arXiv preprint arXiv:2212.05621 (2022)*.

Dirac operator: Classical Dynamics and Signal processing

L. Calmon, M. Schaub and G. Bianconi *Dirac signal processing of higher-order topological signals* arXiv:2301.10137 (2023).

L. Calmon, J. G. Restrepo, J. J. Torres, G. Bianconi Dirac Synchronization is rhythmic and explosive Communication Physics 5, 253 (2022).

L. Calmon, G. Bianconi Local Dirac Synchronization on networks arxiv:2210.16124 (2022).

L. Giambagli, L. Calmon, R. Muolo,T. Carletti, G. Bianconi, Diffusion-driven instability of topological signals coupled by the Dirac operator PRE (2022).

The Dirac operator on simplicial complexes

The Dirac operator allows to study interacting topological signals of different dimensions coexisting in the same network topology

Dirac operator

$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{B}_1 & 0 \\ \mathbf{B}_1^\top & 0 & \mathbf{B}_2 \\ 0 & \mathbf{B}_2^\top & 0 \end{pmatrix},$$

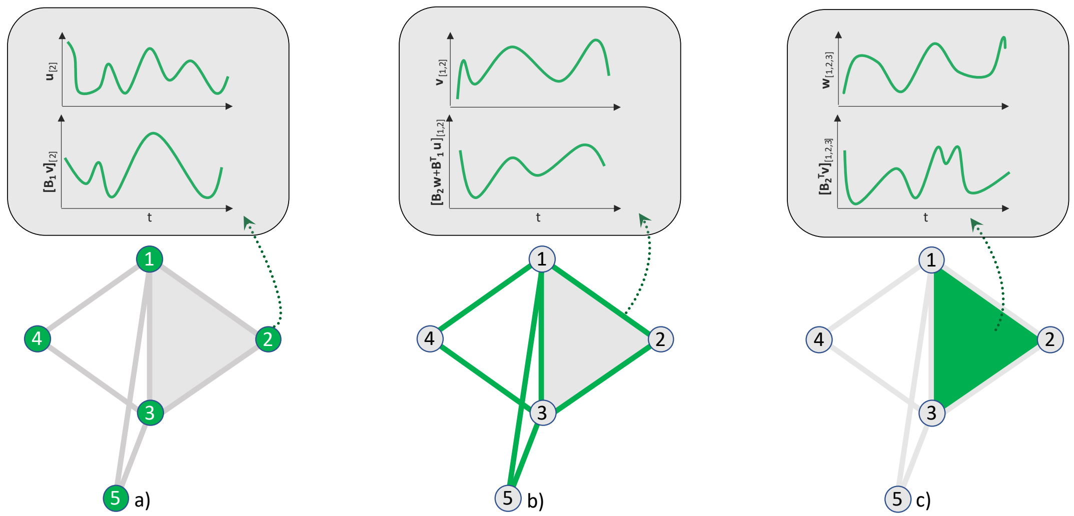
Topological signal "spinor"

$$\mathbf{s} = \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix}$$

$\mathbf{s}_0$	**Node signal**
$\mathbf{s}_1$	**Link signal**
$\mathbf{s}_2$	**Triangle signal**

The action of the Dirac operator

The Dirac operator allows cross-talking between signals of different dimension



$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{B}_1 & 0 \\ \mathbf{B}_1^\top & 0 & \mathbf{B}_2 \\ 0 & \mathbf{B}_2^\top & 0 \end{pmatrix}, \text{ acts on } \mathbf{s} = \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix} \quad \rightarrow \quad \mathbf{D}\mathbf{s} = \begin{pmatrix} \mathbf{B}_1\mathbf{s}_1 \\ \mathbf{B}_1^\top\mathbf{s}_0 + \mathbf{B}_1\mathbf{s}_2 \\ \mathbf{B}_2^\top\mathbf{s}_1 \end{pmatrix}$$

The Dirac as the square-root of the Laplacian

The Dirac operator
can be interpreted as the
“square-root” of the Laplacian

$$\boldsymbol{D} = \begin{pmatrix} 0 & \mathbf{B}_1 & 0 \\ \mathbf{B}_1^\top & 0 & \mathbf{B}_2 \\ 0 & \mathbf{B}_2^\top & 0 \end{pmatrix}, \text{ acts on } \mathbf{s} = \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix} \quad \rightarrow \quad \mathbf{D}^2 = \mathscr{L} = \begin{pmatrix} \mathbf{L}_{[0]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{[1]} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L}_{[2]} \end{pmatrix}$$

$\rightarrow$