

Beyond Truth-Telling: Preference Estimation with Centralized School Choice*

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Abstract

We propose novel approaches and tests for estimating student preferences with data from school choice mechanisms, e.g., the Gale-Shapley Deferred Acceptance. Without requiring truth-telling to be the unique equilibrium, we show that the matching is (asymptotically) stable, or justified-envy-free, implying that every student is assigned to her favorite school among those she is qualified for *ex post*. Having validated the methods in simulations, we apply them to data from Paris and reject truth-telling but not stability. Our estimates are then used to compare the sorting and welfare effects of alternative admission criteria prescribing how schools rank students in centralized mechanisms.

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Centralized mechanisms are common in the placement of students to public schools. Over the past decade, the Gale-Shapley Deferred Acceptance (DA) algorithm has become the leading mechanism for school choice reforms and has been adopted by many school districts around the world, including Amsterdam, Boston, New York, and Paris.

One of the reasons for the growing popularity of DA is the strategy-proofness of the mechanism (Abdulkadiroğlu and Sönmez, 2003). When applying for admission, students are asked to submit rank-order lists (ROLs) of schools, and it is in their best interest to rank schools truthfully. The mechanism therefore releases students and their parents from strategic considerations. As a consequence, it provides school districts “*with more credible data about school choices, or parent ‘demand’ for particular schools,*” as argued by the former Boston Public Schools superintendent Thomas Payzant when recommending DA in 2005. Indeed, such rank-ordered school choice data contain rich information on preferences over schools, and have been used extensively in the empirical literature, e.g., Ajayi (2013), Akyol and Krishna (2014), Burgess et al. (2014), Pathak and Shi (2014), and Abdulkadiroğlu, Agarwal and Pathak (2015), among many others.

Due to the strategy-proofness of DA, one may be tempted to assume that the submitted ROLs of schools reveal students’ true preferences. Strategy-proofness, however, means that truth-telling is only a weakly dominant strategy, which raises the potential issue of multiple equilibria because some students might achieve the same payoff by opting for non-truth-telling strategies in a given equilibrium. Making truth-telling even less likely, many applications of DA restrict the length of submittable ROLs, which destroys strategy-proofness (Haeringer and Klijn, 2009; Calsamiglia, Haeringer and Klijn, 2010).

The first contribution of our paper is to show how to estimate student preferences from school choice data under DA without assuming truth-telling. We derive identifying conditions based on a theoretical framework in which schools strictly rank applicants by some priority indices.¹ Deviating from the literature, we introduce an application cost that students have to pay when submitting ROLs, and the model therefore has the common real-life applications of DA as special cases. Assuming that both preferences

¹School/university admission based on priority indices, e.g., academic grades, is common in many countries (e.g., Australia, China, France, Korea, Romania, Taiwan, and Turkey) as well as in the U.S. (e.g., “selective,” “exam,” or “magnet” schools in Boston, Chicago, and NYC). The website, www.matching-in-practice.eu, summarizes many other examples under the section “Matching Practices in Europe.” Additional examples are provided in Table E5 in Appendix E. Our approaches therefore have great potential in analyzing the data from these systems.

and priorities are private information, we show that truth-telling in equilibrium is rather unlikely. For example, students would often omit a popular school if they expect a low chance of being accepted.

As an alternative, we consider a weaker assumption implied by truth-telling: stability, or justified-envy-freeness, of the matching outcome (Abdulkadiroğlu and Sönmez, 2003), which means that every student is assigned to her favorite school among all feasible ones. A school is feasible for a student if its *ex post* cutoff is lower than the student’s priority index. These cutoffs are well-defined and often observable: given an outcome, each school’s cutoff is the lowest priority index of the students accepted there. Conditional on the cutoffs, stability therefore defines a discrete model with personalized choice sets.

We show that stability is a plausible assumption: any equilibrium outcome of the game is asymptotically stable under certain conditions. When school capacities and the number of students are increased proportionally while the number of schools is fixed, the fraction of students not assigned to their favorite feasible school tends to zero in probability. Although stability, as an *ex post* optimality condition, is not guaranteed in such an incomplete-information game if the market size is arbitrary, we provide evidence suggesting that typical real-life markets are sufficiently large for this assumption to hold.

Based on the theoretical results, we propose a menu of approaches for preference estimation. Both truth-telling and stability lead to maximum likelihood estimation under the usual parametric assumptions. We also provide a solution if neither assumption is satisfied: as long as students do not play dominated strategies, the submitted ROLs reveal true partial preference orders of schools (Haeringer and Klijn, 2009),² which allows us to derive probability bounds for one school being preferred to another. The corresponding moment inequalities can be used for inference using the related methods (for a survey, see Tamer, 2010). When stability is satisfied, these inequalities provide over-identifying information that can improve estimation efficiency (Moon and Schorfheide, 2009).

To guide the choice between these identifying assumptions, we consider several tests. Truth-telling—consistent and efficient under the null hypothesis—can be tested against stability—consistent but inefficient under the null—using the Hausman test (Hausman, 1978). Moreover, stability can be tested against undominated strategies: if the outcome is unstable, the stability conditions are incompatible with the moment inequalities implied

²A ROL is a true partial preference order if the listed schools are ranked according to true preferences.

by undominated strategies, allowing us to use tests such as Bugni, Canay and Shi (2014).

Applying the methods to school choice data from Paris, our paper makes a second contribution by empirically studying the design of the admission criteria that determine how schools rank students. Despite the fact that admission criteria impact student sorting and student welfare by prescribing which students choose first, they have been relatively under-studied in the literature (see the survey by Pathak, 2011).³ Our empirical evaluation of the commonly-used criteria thus provides new insights into the design of student assignment systems.

The data contain 1,590 middle school students competing for admission to 11 academic-track high schools in the Southern District of Paris. As dictated by the admission criterion, schools rank students by their academic grades but give priority to low-income students. The emphasis on grades induces a high degree of stratification in the average peer quality of schools, which is essential for identifying student preferences.

To consistently estimate the preferences of the Parisian students, we apply our proposed approaches and tests, after validating them in Monte Carlo simulations. Truth-telling is strongly rejected in the data, but not stability. Incorrectly imposing truth-telling leads to a serious under-estimation of preferences for popular or small schools.

We use our preferred estimates to perform counterfactual analyses of the admission criteria. By assuming that students form preferences based on the characteristics of those who are already attending the school, our static model incorporates “dynamics”. We therefore simulate equilibrium outcomes under each admission criterion in both the short run and the long run. Holding constant how student preferences are determined, the short-run outcomes measure what happens in the first year after we replace the current criterion with an alternative; the long-run outcomes are to be observed in steady state.

The admission criteria that we consider differ in their weighting of two factors: academic grades and random priorities from lotteries. The results show that random priorities would substantially lower sorting by ability, compared with the current policy, but would slightly raise sorting by socioeconomic status (SES). By contrast, a grades-only policy would substantially increase sorting by SES. This is largely because of the low-income priority in the current admission criterion. We also consider mixed priorities

³The theoretical literature focuses on affirmative action (e.g., Kojima, 2012), the diversity of student body (Erdil and Kumano, 2014; Echenique and Yenmez, 2015), and neighborhood priority (Kominers and Sönmez, 2012; Dur, Kominers, Pathak and Sönmez, 2013). The empirical literature, however, is mostly about decentralized systems (e.g., Arcidiacono, 2005; Hickman, 2013).

where the “top two” schools select students based on grades, while admission to other schools is based on random priorities. This policy would lower sorting by ability but increase sorting by SES. In general, results are more pronounced in the long run, because school attributes evolve over time. The welfare analysis highlights the trade-off between the welfare of low- and high-ability students, in both the short run and the long run, under the grades-only or random-priorities policies. However, under the mixed-priorities criterion, this trade-off is mitigated in the long run. Overall, mixed priorities appear as a promising candidate to enhance overall welfare with modest effects on sorting, while balancing the welfare of low- and high-ability students.

Other Related Literature. Our results on asymptotic stability in Bayesian Nash equilibrium are in line with Haeringer and Klijn (2009), Romero-Medina (1998) and Ergin and Sönmez (2006), who prove the stability of Nash equilibrium outcomes under the constrained DA or the Boston mechanism. Although stability is a rather common identifying assumption in the two-sided matching literature (see the surveys by Fox, 2009; Chiappori and Salanié, forthcoming), it has not, to our knowledge, been used in empirical studies of school choice except in Akyol and Krishna (2014). Observing the matching outcome and the cutoffs of high school admissions in Turkey, the authors estimate preferences based on the assumption that every student is assigned to her favorite feasible school. This assumption is formalized and justified in our paper. Moreover, beyond stability, we propose approaches that incorporate the information from ROLs and that fully endogenize the cutoffs.

Large markets are commonly considered in theoretical studies on the properties of mechanisms (see the survey by Kojima, 2015). Closely related is Azevedo and Leshno (forthcoming), who show the asymptotics of the cutoffs of stable matchings. Our paper extends their results to Bayesian Nash equilibrium.

There is also a growing literature on preference estimation under other school choice mechanisms, e.g., the Boston mechanism (Agarwal and Somaini, 2014; Calsamiglia, Fu and Güell, 2014; He, 2015). Since this mechanism is not strategy-proof (Abdulkadiroğlu and Sönmez, 2003), observed ROLs are sometimes considered as maximizing expected utility. Taking estimated admission probabilities as students’ beliefs, one could apply the same approach to our setting, i.e., a discrete choice problem defined on the set of possible ROLs. Agarwal and Somaini (2014) show how preferences can be non-parametrically

identified when admission probabilities are non-degenerate.⁴

Organization of the Paper. The paper proceeds as follows. Section 1 presents the model that provides our theoretical foundation for preference estimation. Section 2 discusses the corresponding empirical approaches and tests, which are illustrated in Monte Carlo simulations. School choice in Paris, and our results on estimation and testing with the Parisian data, are shown in Section 3. The counterfactual analyses of commonly-observed admission criteria are described in Section 4. Section 5 concludes.

1 The Model

A (finite) school choice problem is denoted by $F \equiv \{[u_{i,s}, e_{i,s}]_{i \in \mathcal{I}, s \in \mathcal{S}}, \{q_s\}_{s \in \mathcal{S}}, C(\cdot)\}$, where $\mathcal{I} \equiv \{1, \dots, I\}$ is the set of students, and $\mathcal{S} \equiv \{1, \dots, S\}$ is the set of schools. Student i has a von Neumann-Morgenstern (vNM) utility $u_{i,s} \in [0, 1]$ of being assigned to s , and, as required by the admission criterion, school s ranks students by priority indices $e_{i,s} \in [0, 1]$, i.e., s “prefers” i over j if and only if $e_{i,s} > e_{j,s}$. To simplify notation, we assume that there are no indifferences in either vNM utilities or priority indices, and that all schools and students are acceptable. Each school has a positive capacity q_s .

Schools first announce their capacities, and every student then submits a rank-order list (ROL) of $K_i \leq S$ schools, denoted by $L_i = (l_i^1, \dots, l_i^{K_i})$, where $l_i^k \in \mathcal{S}$ is i ’s k th choice. L_i also represents the set of schools being ranked in L_i . We define \succ_{L_i} such that $s \succ_{L_i} s'$ if and only if s is ranked above s' in L_i . The set of all possible ROLs is \mathcal{L} , which includes all ROLs ranking at least one school. Student i ’s true ordinal preference is $R_i = (r_i^1, \dots, r_i^S) \in \mathcal{L}$, which ranks all schools according to cardinal preferences $[u_{i,s}]_{s \in \mathcal{S}}$.

When submitting a ROL, a student incurs a cost $C(|L|)$, which depends on the number of schools being ranked in L , $|L|$. Furthermore, $C(|L|) \in [0, +\infty]$ for all L and is weakly increasing in $|L|$. To simplify students’ participation decision, we set $C(1) = 0$.

Such a cost function flexibly captures many common applications of school choice mechanisms. If $C(|L|) = 0$ for all L , we are in the traditional setting without costs (e.g., Abdulkadiroğlu and Sönmez, 2003); if $C(|L|) = \infty$ for $|L|$ greater than a constant \bar{K} , it corresponds to the constrained school choice where one cannot rank more than \bar{K} schools (e.g., Haeringer and Klijn, 2009); when $C(|L|) = c(|L| - \underline{K})$, one has to pay a unit cost c

⁴One could also consider the algorithm proposed by Chade and Smith (2006) for portfolio choice, as in Ajayi and Sidibe (2015). However, fairly strong assumptions are needed, as discussed in Appendix A.2.

for each choice beyond the first \underline{K} choices (e.g., Biró, 2012); the monotonic cost function can simply reflect that it is cognitively burdensome to rank too many choices.

The student-school match is solved by a mechanism that takes into account students' ROLs and schools' rankings over students. Our main analysis focuses on the student-proposing Gale-Shapley Deferred Acceptance (DA), leaving the discussion of other mechanisms to Section 1.5. DA, as a computerized algorithm, works as follows:

Round 1. Every student applies to her first choice. Each school rejects the lowest-ranked students in excess of its capacity and temporarily holds the other students.

Generally, in:

Round k . Every student who is rejected in Round $(k - 1)$ applies to the next choice on her list. Each school, pooling together new applicants and those who are held from Round $(k - 1)$, rejects the lowest-ranked students in excess of its capacity. Those who are not rejected are temporarily held by the schools.

The process terminates after any Round k when no rejections are issued. Each school is then matched with the students it is currently holding.

We introduce the following definition of many-to-one matching.

Definition 1. A matching μ is a function from the set $\mathcal{I} \cup \mathcal{S}$ into the set of unordered families of elements of $\mathcal{I} \cup \mathcal{S}$ such that: (i) $|\mu(i)| = 1$ for every student i ; (ii) $|\mu(s)| = q_s$ for every school, and if the number of students in $\mu(s)$, say n_s , is less than q_s , then $\mu(s)$ contains $q_s - n_s$ copies of s itself; and (iii) $\mu(i) = s$ if and only if $i \in \mu(s)$.

1.1 Information Structure and Decision-Making

We assume that every student's preferences and indices are private information, and are i.i.d. draws from a joint distribution $G(v|e) \times H(e)$, which is common knowledge.⁵

Given others' indices and submitted ROLs (L_{-i}, e_{-i}) , i 's probability of being assigned to s is a function of her priority index vector e_i and submitted ROL L_i :

$$a_s(L_i, e_i; L_{-i}, e_{-i}) \equiv \begin{cases} \Pr(i \text{ is rejected by } l_i^1, \dots, l_i^k \text{ and accepted by } l_i^{k+1} = s \mid L_i, e_i; L_{-i}, e_{-i}) & \text{if } s \in L_i \\ 0 & \text{if } s \notin L_i \end{cases}$$

⁵The analysis can be extended to allow priority indices to be common knowledge, after conditioning on a realization of $[e_{i,s}]_{i \in \mathcal{I}, s \in \mathcal{S}}$.

Clearly, given the algorithm, $a_s(L_i, e_i; L_{-i}, e_{-i})$ is either zero or one for all s .

Student i 's strategy is $\sigma(u_i, e_i) : [0, 1]^S \times [0, 1]^S \rightarrow \Delta(\mathcal{L})$. We consider a symmetric equilibrium σ^* such that σ^* solves the following maximization problem for every student:⁶

$$\sigma^*(u_i, e_i) \in \arg \max_{\sigma} \left\{ \sum_{s \in \mathcal{S}} u_{i,s} \int \int a_s(\sigma, e_i; \sigma^*(u_{-i}, e_{-i}), e_{-i}) dG(u_{-i}|e_{-i}) dH(e_{-i}) - C(|\sigma|) \right\},$$

when $\sigma^*(u_i, e_i)$ is a pure strategy.⁷ The existence of pure-strategy Bayesian Nash equilibrium can be established by applying Theorem 4 (Purification Theorem) in Milgrom and Weber (1985), although there might be multiple equilibria.

Given an equilibrium σ^* and a realization of the economy F , we observe one matching, $\mu_{(F, \sigma^*)}$, such that the *ex post* cutoff (or shadow price) of each school is:

$$p_s(\mu_{(F, \sigma^*)}) = \begin{cases} \min \{e_{i,s} \mid i \in \mu_{(F, \sigma^*)}(s)\} & \text{if } s \notin \mu_{(F, \sigma^*)}(s) \\ 0 & \text{if } s \in \mu_{(F, \sigma^*)}(s) \end{cases}$$

That is, $p_s(\mu_{(F, \sigma^*)})$ is zero if s does not meet its capacity; otherwise, it is the lowest index among all accepted students. The vector of cutoffs is denoted by $P(\mu_{(F, \sigma^*)})$. In what follows, we sometimes shorten $p_s(\mu_{(F, \sigma^*)})$ to p_s when there is no confusion. With the cutoff, we can redefine the admission probabilities as:

$$a_s(L_i, e_i; L_{-i}, e_{-i}) \equiv \begin{cases} \Pr(p_{s'} > e_{i,s'} \text{ for } s' = l_i^1, \dots, l_i^k \text{ and } p_s \leq e_{i,s} \text{ for } s = l_i^{k+1} \mid L, L_{-i}, e_i) & \text{if } s \in L_i \\ 0 & \text{if } s \notin L_i \end{cases}$$

1.2 Truth-Telling Behavior in Equilibrium

To assess how plausible the truth-telling assumption is in empirical studies, we begin by investigating students' truth-telling behavior in equilibrium.

Definition 2. *Student i is **weakly truth-telling** if her ROL L_i ranks truthfully her top $|L_i|$ choices, i.e., $u_{i,l_i^k} > u_{i,l_i^{k+1}}$ for all $l_i^k, l_i^{k+1} \in L_i$, and $u_{i,s} > u_{i,s'}$ for all $s \in L_i$ and $s' \notin L_i$. If a weakly truth-telling L_i is a full list and thus $L_i = R_i$, i is **strictly truth-telling**.*

⁶It is innocuous to focus on symmetric equilibrium, because it does not restrict the strategies of any pair of students given that they all have different priority indices (almost surely).

⁷If $\sigma^*(u_i, e_i)$ is a mixed strategy, we then need every pure strategy being played with a positive probability to satisfy the above condition.

It is well known that DA is strategy-proof when there is no application cost.

Theorem 1 (Dubins and Freedman, 1981; Roth, 1982). *When $C(|L|) = 0$ for all $L \in \mathcal{L}$, the student-proposing DA is strategy-proof: strict truth-telling is a weakly dominant strategy for all students.*

The above theorem, however, highlights the possibility of multiple equilibria: there might exist strategies that are payoff-equivalent to truth-telling in some equilibrium. If we assume that the equilibrium where everyone is truth-telling is always selected, we implicitly impose a selection rule that may not be reasonable in real life. It is therefore useful to clarify the conditions under which strict truth-telling is a strictly dominant strategy and thus the unique equilibrium, for which we need the following definition.

Definition 3. *Fix any i and two ROLs, L_i and L'_i , such that the only difference between them is two neighboring choices: $(l_i^{k^*}, l_i^{k^*+1}) = (s, s')$, $(l_i^{k^*}, l_i^{k^*+1}) = (s', s)$, and $l_i^k = l_i^k$ for all $k \neq k^*, k^* + 1$. A mechanism satisfies **swap monotonicity** if for all (L_{-i}, e_{-i}) :*

$$a_s(L_i, e_i; L_{-i}, e_{-i}) \geq a_s(L'_i, e_i; L_{-i}, e_{-i}); a_{s'}(L_i, e_i; L_{-i}, e_{-i}) \leq a_{s'}(L'_i, e_i; L_{-i}, e_{-i}).$$

*These two conditions are either both strict or both equalities. If they are strict for any pair of (s, s') given any $(L_i, e_i; L_{-i}, e_{-i})$, the mechanism is **strictly swap monotonic**.*

DA is swap monotonic, but not strictly swap monotonic (Mennle and Seuken, 2014).⁸

Theorem 2. *Strict truth-telling is a strictly dominant strategy under DA if and only if*

- (i) there is no application cost: $C(|L|) = 0, \forall L \in \mathcal{L}$; and*
- (ii) the mechanism is strictly swap monotonic.*

All proofs can be found in Appendix A. The first condition is violated if students cannot rank as many schools as they wish, or if they suffer a cognitive burden when ranking too many schools. More importantly, DA does not satisfy the second condition.

Taking one step back, one might be interested in a (Bayesian) Nash equilibrium where truth-telling is a strict, and thus unique, best response given that others are truth-telling.

Proposition 1. *When others are truth-telling, $\sigma_{-i} = R_{-i}$, it is a strict best response for i to report true preferences, $\sigma_i = R_i$, if and only if:*

⁸Strict swap monotonicity requires that, given any ROLs of others, i 's admission probability at s strictly increases whenever s is moved up one position in i 's ROL. It is certainly violated under DA when, for example, everyone else ranks only s in their ROLs.

- (i) there is no application cost: $C(|L|) = 0, \forall L \in \mathcal{L}$; and
- (ii) the mechanism is strictly swap monotonic given $\sigma_{-i} = R_{-i}$.

Again, the second condition, although being relaxed, is still restrictive, as one may not want to restrict R_{-i} in empirical studies.⁹

We call $L_i, |L_i| \leq S$, a **true partial preference order** of schools if L_i respects the true preference ordering among those ranked in L_i . That is, if s is ranked before s' in L_i , then s is also ranked before s' according to i 's true preference R_i ; when s is not ranked in L_i , it is not possible to determine how s is ranked relative to any other school.

Theorem 3. *Under DA with cost $C(|L|)$, it is a weakly dominated strategy to submit a ROL that is not a true partial preference order. If the mechanism is strictly swap monotonic, such strategies are strictly dominated.*

Theorem 3 implies that under the truncated DA, untrue partial order is a dominated strategy (Haeringer and Klijn, 2009).

1.3 Matching Outcome: Stability

The above results show that the truth-telling assumption is rather restrictive in empirical studies. We now turn instead to the properties of equilibrium matching outcomes.

Definition 4. *Given a matching μ , (i, s) form a **blocking pair** if (i) i prefers s over her matched school $\mu(i)$ while s has an empty seat ($s \in \mu(s)$), or if (ii) i prefers s over $\mu(i)$ while s has no empty seats ($s \notin \mu(s)$) but i 's priority index is higher than its cutoff, $e_{i,s} > \min_{\{j \in \mu(s)\}}(e_{j,s})$. μ is **stable** if there is no blocking pair.*

Stability is a concept borrowed from two-sided matching and is also known as **elimination of justified envy** in school choice (Abdulkadiroğlu and Sönmez, 2003). In our setting, stability can be conveniently linked to schools' cutoffs. Given a matching μ , school s is **feasible** for i if $p_s(\mu) \leq e_{i,s}$, and we denote the set of feasible schools for i by $\mathcal{S}(e_i, P(\mu))$. We then have the following lemma, whose straightforward proof is omitted.

Lemma 1. *μ is stable if and only if $\mu(i) = \arg \max_{s \in \mathcal{S}(e_i, P(\mu))} u_{i,s}$ for all $i \in \mathcal{I}$; i.e., all students are assigned to their favorite feasible school.*

⁹When lotteries are used to rank students, as in Pathak and Shi (2014) and Abdulkadiroğlu et al. (2015), admission probabilities are non-degenerate if everyone submits a ROL of length S . In this case, strict swap monotonicity is satisfied given R_{-i} , implying that truth-telling is a Bayesian Nash equilibrium when there is no application cost.

It is well known that DA always produces a stable matching when students are strictly truth-telling (Gale and Shapley, 1962), but not when they are only weakly truth-telling. We have, however, the following result linking weak truth-telling and stability.

Proposition 2. *Suppose everyone is weakly truth-telling under DA. Given a matching:*

- (i) *every assigned student is assigned to her favorite feasible school; and*
- (ii) *if everyone is assigned to a school, the matching is stable.*

We are also interested in implementing stable matchings in dominant strategies, which would free us from specifying the information structure and from imposing additional equilibrium conditions. The following theorem provides the necessary and sufficient conditions for such dominant-strategy implementations, which are again rather restrictive.

Proposition 3. *Under DA, stable matching can be implemented in dominant strategies if and only if $C(|L|) = 0$ for all L . If additionally the mechanism is strictly swap monotonic, stable matching can be implemented in strictly dominant strategies.*

1.4 Asymptotic Stability in Bayesian Nash Equilibrium

So far, we have shown that neither truth-telling nor stability is satisfied without a set of restrictive assumptions. Following the literature on large markets, we study whether stability of the equilibrium outcome can be asymptotically satisfied.

1.4.1 Randomly Generated Finite Economies and the Continuum Economy

We consider a sequence of randomly generated finite economies $\{F^{(I)}\}_{I \in \mathbb{N}}$, such that

$$F^{(I)} = \left\{ [u_{i,s}, e_{i,s}]_{i \in \mathcal{I}^{(I)}, s \in \mathcal{S}}, \{q_s^{(I)}\}_{s \in \mathcal{S}}, C(|L|) \right\};$$

- (i) There are I students in $F^{(I)}$, whose types are i.i.d. draws from $G \times H$;
- (ii) Each school's capacity relative to I remains constant, i.e., $q_s^{(I)}/I = \bar{q}_s$ for all s , where \bar{q}_s is a positive constant.¹⁰

Each finite economy naturally leads to an empirical (joint) distribution of types, which converges to $G \times H$.¹¹ We further define a continuum economy E , where:

¹⁰To simplify notation, we ignore the fact that capacities in finite economies are integers.

¹¹Here the convergence notion is the weak convergence of measures, which is defined as $\int X d\hat{G}^{(I)} \hat{H}^{(I)} \rightarrow \int X dG dH$ for every bounded continuous function $X : [0, 1]^S \times [0, 1]^S \rightarrow \mathbb{R}$. This is also known as narrow convergence or weak-* convergence.

- (i) A mass one of students, \mathcal{I} , have types in space $[0, 1]^S \times [0, 1]^S$ associated with a (probability) measure $G \times H$;
- (ii) School s has a positive capacity \bar{q}_s .

The definitions of DA and stability can be naturally extended to continuum economies as in Azevedo and Leshno (forthcoming), who also establish the existence of stable matching in such settings. In a stable matching of E , the demand for school s is the measure of students whose priority index is above s 's cutoff and whose favorite feasible school is s :

$$D_s(P) = \int \int \mathbf{1}(s = \arg \max_{s' \in \mathcal{S}(e_i, P)} \{u_{s'}\}) dG(u|e) dH(e),$$

which is differentiable with respect to $[p_s]_{s \in \mathcal{S}}$ in usual applications (see Appendix A.4).

1.4.2 Results

We first present results on stable matchings in both finite and continuum economies, and then discuss whether stable matching can be achieved in equilibrium.

It is known that generically, there exists a unique stable matching in the continuum economy (Azevedo and Leshno, forthcoming), and we denote this matching in E as μ^∞ . Although μ^∞ is unique, there could exist some Nash equilibrium that leads to an unstable matching. We discuss this in Appendix A.4, and results are summarized in Proposition A1. In the following, we assume that all Nash equilibria of the continuum economy E result in the stable matching, which in practice can be checked using Proposition A1.

Linking finite and continuum economies, the next result shows that, as the market size grows, the only strategies that can survive in finite economies are those ranking μ^∞ .

Proposition 4. *If a strategy $\sigma: [0, 1]^S \times [0, 1]^S \rightarrow \Delta(\mathcal{L})$ results in a matching in the continuum economy $\mu_{(E, \sigma)}$ such that $G \times H(\{i \in \mathcal{I} \mid \mu_{(E, \sigma)}(i) \neq \mu^\infty(i)\}) > 0$ in E , then there must exist N such that σ is not an equilibrium in $F^{(I)}$ for all $I > N$.*

When $C(2) > 0$, we can obtain even sharper results:

Lemma 2. *If $C(2) > 0$ and $\sigma(u_i, e_i) = \mu^\infty(i)$ for all i almost surely, then there exists N such that σ is an equilibrium in $F^{(I)}$ for all $I > N$.*

The above results lead us to focus on a sequence of Bayesian Nash equilibria $[\sigma^{(I)}]_{I \in \mathbb{N}}$ such that each $\sigma^{(I)}$ results in μ^∞ in E almost surely. When the market size is large

enough, such Bayesian Nash equilibria are the only possible equilibria. It also implies that every i includes $\mu^\infty(i)$ almost surely in $\sigma^{(I)}$.¹²

Given the finite economies, when $\sigma^{(J)}$ is in pure strategy, it creates a sequence of ordinal economies, $F_{\sigma^{(J)}}^{(I)} = \left\{ [\sigma^{(J)}(u_i, e_i), e_i]_{i \in \mathcal{I}^{(I)}}, [q_s^{(I)}]_{s \in \mathcal{S}}, C(|L|) \right\}$. The original cardinal preferences $[u_{i,s}]_{i \in \mathcal{I}^{(I)}, s \in \mathcal{S}}$ are replaced by ordinal “preferences” $[\sigma^{(J)}(u_i, e_i)]_{i \in \mathcal{I}^{(I)}}$. Correspondingly, the continuum ordinal economy is $E_{\sigma^{(J)}}$ such that $F_{\sigma^{(J)}}^{(I)} \rightarrow E_{\sigma^{(J)}}$ almost surely. If $\sigma^{(J)}$ is in mixed strategies, a distribution of economies can be similarly constructed.

Given $E_{\sigma^{(J)}}$, assuming that everyone reports true ordinal preferences leads to a matching that is stable with respect to $[\sigma^{(J)}(u_i, e_i)]_{i \in \mathcal{I}}$. In this matching, the demand for each school in $E_{\sigma^{(J)}}$ as a function of the cutoffs is:

$$D_s(P, \sigma^{(J)}) = \int \int \mathbf{1}(u_s = \max_{s' \in \mathcal{S}(e_i, P) \cap \sigma^{(J)}(u_i, e_i)} \{u_{s'}\}) dG(u_i | e_i) dH(e_i),$$

where $\sigma^{(J)}(u_i, e_i)$ also denotes the set of schools ranked by i . Let $D(P, \sigma^{(J)}) = [D_s(P, \sigma^{(J)})]_{s \in \mathcal{S}}$.

Proposition 5. *Fix $\sigma^{(J)} \in \{\sigma^{(I)}\}_{I \in \mathbb{N}}$, where $\sigma^{(J)}$ is a Bayesian Nash equilibrium of $F^{(I)}$, and apply it to the sequence of finite economies $\{F^{(I)}\}_{I \in \mathbb{N}}$. We then have:*

- (i) $\sup_{J \in \mathbb{N}} \|P(\mu_{(F^{(I)}, \sigma^{(J)})}) - P^\infty\| \xrightarrow{P} 0$, and, therefore, $P(\mu_{(F^{(J)}, \sigma^{(J)})}) \xrightarrow{P} P^\infty$.
- (ii) *Fraction(students in any blocking pair in $\mu_{(F^{(J)}, \sigma^{(J)})}) \xrightarrow{P} 0$.*
- (iii) *If $E_{\sigma^{(J)}}$ has a C^1 demand function and $\partial D(P^\infty, \sigma^{(J)}) / \partial P^\infty$ is nonsingular, the asymptotic distribution of cutoffs in $F^{(I)}$ is:*

$$\sqrt{I} (P(\mu_{(F^{(I)}, \sigma^{(J)})}) - P^\infty) \xrightarrow{d} N(0, V(\sigma^{(J)}))$$

where P^∞ is the cutoff vector in E , $V(\sigma^{(J)}) = \partial D(P^\infty, \sigma^{(J)})^{-1} \Sigma (\partial D(P^\infty, \sigma^{(J)})^{-1})'$, and

$$\Sigma = \begin{pmatrix} \bar{q}_1(1 - \bar{q}_1) & -\bar{q}_1\bar{q}_2 & \cdots & -\bar{q}_1\bar{q}_S \\ -\bar{q}_2\bar{q}_1 & \bar{q}_2(1 - \bar{q}_2) & \cdots & \vdots \\ \vdots & \vdots & \ddots & -\bar{q}_{S-1}\bar{q}_S \\ -\bar{q}_S\bar{q}_1 & \cdots & -\bar{q}_S\bar{q}_{S-1} & \bar{q}_S(1 - \bar{q}_S) \end{pmatrix}.$$

Proposition 5 shows that the matching is asymptotically stable. It also sheds light on the convergence rate.

¹²Such an equilibrium in $F^{(I)}$ may not exist for small I but does exist when I is large enough.

1.4.3 Comparative Statics: Probability of Observing a Blocking Pair

Based on the above results, we can discuss “comparative statics” to evaluate how market size, the cost of submitting a list, and other factors affect the probability of observing an *ex post* blocking pair in equilibrium.

Let us consider a finite economy $F^{(I)}$ where the Bayesian Nash equilibrium being played is $\sigma^{(I)}$. Without loss of generality, it is a pure strategy, i.e., $\sigma^{(I)}(u_i, e_i) = L_i$.

Proposition 6. *In a Bayesian Nash equilibrium outcome, where L_i represents a true partial order of i 's ordinal preferences, ex post i can form a blocking pair only with a school that is not ranked in L_i . The probability that i is in a blocking pair:*

- (i) *is bounded above by a term that is increasing in the cost of including an additional school, $C(|L_i| + 1) - C(|L_i|)$, and decreasing in the cardinal utilities of omitted schools relative to the less preferable ones in L_i ; and*
- (ii) *decreases to zero as market size, I , goes to infinity.*

Remark 1. *Proposition 6 has implications for empirical studies. Stability is more plausibly satisfied when the cost of ranking more schools is lower, and/or the market is large. Moreover, in the case of constrained/truncated DA where there is a limit on the length of ROLs, the more schools are allowed to be ranked, the more likely stability is to be satisfied.*

1.5 Discussion and Extensions

Non-Equilibrium Strategies. We have thus far focused on the case in which everyone plays an equilibrium strategy with a common prior, which is rather restrictive. More realistically, some students could have different information and make mistakes when strategizing.¹³

When introducing the empirical approaches, we take this possibility into account. If students do not play equilibrium strategies, the matching is less likely to be stable. Therefore, allowing students to play non-equilibrium strategies amounts to having unstable matching outcomes. In Section 2.7, we propose a test for stability, which is then also a test for non-equilibrium strategies. If stability is rejected, one can obtain identifying information by imposing a “minimal” rationality assumption that everyone plays

¹³In the literature, both Calsamiglia et al. (2014) and He (2015) consider the possibility that students make mistakes when submitting ROLs.

undominated strategies. Theorem 3 provides the theoretical foundation for the approach to be introduced in Section 2.6.

Uncertainty in Priority Indices. In reality, it is common that there is some *ex ante* uncertainty in schools’ rankings over students. For example, students in some Chinese provinces do not know their exact test scores when applying to universities. In extreme cases, school choice in places such as Beijing and NYC uses an *ex ante* unknown lottery to rank students.

Our main analysis of finite economies allows a certain degree of uncertainty in priority indices, in that every student knows her own indices but not her precise ranking among all students; importantly, this uncertainty degenerates with market size. In the case of non-degenerate uncertainty, the fraction of students who can form at least one blocking pair with some school is small if the uncertainty is limited and the market size is large.¹⁴

Beyond School Choice. Although the analysis has focused on school choice, or college admission, our results apply to other assignment/matching based on DA or similar centralized mechanisms. The key requirement is that researchers have information on the “preferences” of the agents on one side, i.e., how they rank the agents on the other side.¹⁵ Examples include teacher assignment to public schools in France and the Scottish Foundation Allocation Scheme matching medical school graduates to training programs, which are both centralized.¹⁶ The estimation approaches discussed in Section 2 could be implemented in these settings.

Other Mechanisms. Our main theoretical results can be applied to another two popular mechanisms, the school-proposing DA and the Boston mechanism (see definition in Appendix E). In the school-proposing DA, schools propose to students following the order of student priority indices. When considering either of these mechanisms, Theorem 3 no longer holds; that is, students might have incentives not to report a true partial preference

¹⁴This is shown in Monte Carlo exercises, the results of which are available upon request. In the case where schools rank students by lotteries, *ex post* optimality (i.e., stability) is less likely to be satisfied in Bayesian Nash equilibrium. For estimation in school choice with such non-degenerate uncertainties, one can use the approaches in Agarwal and Somaini (2014), Calsamiglia et al. (2014), and He (2015).

¹⁵When researchers have no information on how either side ranks the other, we are in the classic setting of two-sided matching, where additional assumptions are often needed for identification and estimation.

¹⁶Details can be found on the website of the “Matching in Practice” network at www.matching-in-practice.eu/matching-practices-of-teachers-to-schools-france, and www.matching-in-practice.eu/the-scottish-foundation-allocation-scheme-sfas, respectively.

order (Abdulkadiroğlu and Sönmez, 2003; Haeringer and Klijn, 2009). Nonetheless, the asymptotic stability result (Proposition 5) still holds, as its proof does not rely on Theorem 3. Indeed, it is known that the matching outcome can be stable in Nash equilibrium for both mechanisms (Ergin and Sönmez, 2006; Haeringer and Klijn, 2009). It should be noted that when schools rank students strictly, the Boston mechanism is approximated by the DA where everyone can rank only one school, which imposes an infinite cost on ranking more than one school; therefore, one would need a larger market to ensure stability (Proposition 6).

Obviously, (asymptotic) stability does not hold under unstable mechanisms such as the Top-Trading Cycles (Abdulkadiroğlu and Sönmez, 2003).

2 Empirical Approaches

Building on the theoretical results from the previous section, we explain how to estimate student preferences under different sets of assumptions and propose a series of tests to select the appropriate approach. To be more concrete, we consider a logit-type random utility model, although our approaches can be extended to other specifications.

2.1 Model Setting

Throughout this section, we consider a market in which I students compete for admission into S distinct schools. Each school s has a positive capacity q_s , and students are assigned through the student-proposing DA.

Student i 's utility from attending schools s is defined as:

$$u_{i,s} = \sigma V_{i,s} + \sigma \epsilon_{i,s} = \alpha_s - d_{i,s} + \mathbf{Z}'_{i,s} \boldsymbol{\beta} + \sigma \epsilon_{i,s},$$

where $\sigma V_{i,s} = \alpha_s - d_{i,s} + \mathbf{Z}'_{i,s} \boldsymbol{\beta}$ denotes the deterministic component of utility and $\sigma \epsilon_{i,s} \in \mathbb{R}$ denotes unobserved heterogeneity; σ is a scaling parameter; α_s is school s 's fixed effect; $\mathbf{Z}_{i,s} \in \mathbb{R}^K$ are student-school specific attributes, e.g., interactions of student characteristics and school attributes; $d_{i,s}$ is the distance from i 's home to school s . It is convenient to normalize the effect of distance to be -1 , so the magnitude of other coefficients can be easily interpreted in terms of willingness to travel.

We further define $\mathbf{Z}_i \equiv \{\mathbf{Z}_{i,s}\}_{s=1}^S$, $\boldsymbol{\epsilon}_i \equiv \{\epsilon_{i,s}\}_{s=1}^S$, and $\boldsymbol{\theta}$ as the set of coefficients to be

estimated, $(\{\alpha_s\}_{s \in \mathcal{S}}, \boldsymbol{\beta}, \sigma)$. We normalize the utility functions by setting $\alpha_1 = 0$. Such a formulation rules out outside options, although this assumption can be relaxed. Finally, we assume that $\boldsymbol{\epsilon}_i \perp \mathbf{Z}_i$, and that $\epsilon_{i,s}$ is i.i.d. over i and s with the type-I extreme value (Gumbel) distribution.

2.2 Truth-Telling

Despite its likely implausibility, we start with formalizing the estimation under the truth-telling assumption. If each student i is weakly truth-telling and submits $K_i \equiv |L_i| (\leq S)$ choices, then $L_i = (l_i^1, \dots, l_i^{K_i})$ ranks truthfully i 's top K_i choices. The probability of observing student i submitting L_i is:

$$\begin{aligned} & \Pr(i \text{ submits } L_i \mid \mathbf{Z}_i; \boldsymbol{\theta}) \\ &= \Pr(L_i = (l_i^1, \dots, l_i^{K_i}) \mid \mathbf{Z}_i; \boldsymbol{\theta}; K_i) \times \Pr(i \text{ submits a ROL of length } K_i \mid \mathbf{Z}_i; \boldsymbol{\theta}). \end{aligned}$$

We can follow the literature in assuming that K_i is orthogonal to $u_{i,s}$, for all s (Hastings, Kane and Staiger, 2008; Abdulkadiroğlu et al., 2015), which allows to ignore the first term and focus instead on the following conditional choice probability:¹⁷

$$\begin{aligned} \Pr(L_i = (l_i^1, \dots, l_i^{K_i}) \mid \mathbf{Z}_i; \boldsymbol{\theta}; K_i) &= \Pr(u_{i,l_i^1} > \dots > u_{i,l_i^{K_i}} > u_{i,s'} \forall s' \in \mathcal{S} \setminus L_i \mid \mathbf{Z}_i; \boldsymbol{\theta}; K_i) \\ &= \prod_{s \in L_i} \left(\frac{\exp(V_{i,s})}{\sum_{s' \not\prec_{L_i} s} \exp(V_{i,s'})} \right) \end{aligned}$$

where $s' \not\prec_{L_i} s$ indicates that s' is not ranked before s in L_i , which includes s itself and the schools not ranked in L_i . This rank-ordered (or “exploded”) logit model can be seen as a series of conditional logit models: one for the top-ranked school (l_i^1) being the most preferred; another for the second-ranked school (l_i^2) being preferred to all schools except the one ranked first, and so on.

The model is point identified under the usual assumptions and can be estimated by maximum likelihood estimation (MLE) with the log-likelihood function:

$$\ln L_{TT}(\boldsymbol{\theta} \mid \mathbf{Z}, |\mathbf{L}|) = \sum_{i=1}^I \sum_{s \in L_i} V_{i,s} - \sum_{i=1}^I \sum_{s \in L_i} \ln \left(\sum_{s' \not\prec_{L_i} s} \exp(V_{i,s'}) \right),$$

¹⁷This assumption is justified when the length of a ROL is determined by institutional arrangements. Alternatively, one may consider that the length of ROLs depends on the number of schools that are preferred to the outside option, which could violate the above assumption.

where $|\mathbf{L}|$ is the length of all ROLs. The estimate is denoted by $\hat{\boldsymbol{\theta}}_{TT}$.

2.3 Stability

Stability of the matching outcome implies that every student is assigned to her favorite school among those she is qualified for *ex post*. We are interested in the probability of the observed matching μ being realized. Given μ , we also observe the vector of cutoffs, $P(\mu) = [p_s]_{s \in \mathcal{S}}$, which defines each student's set of feasible schools, $\mathcal{S}(e_i, P)$. This set includes every school s whose admission cutoff p_s is below the student's index $e_{i,s}$ at that school. The probability of observing μ conditional on the full matrix of observables (\mathbf{Z}), and the parameters $\boldsymbol{\theta}$ is then:

$$\begin{aligned} & \Pr \left(\text{a stable matching } \mu \text{ being realized} \mid \mathbf{Z}; \boldsymbol{\theta} \right) \\ &= \Pr \left(\text{cutoffs are } P(\mu); \mu(i) = \arg \max_{s' \in \mathcal{S}(e_i, P)} (u_{i,s'}), \forall i \in \mathcal{I} \mid \mathbf{Z}; \boldsymbol{\theta} \right) \\ &= \Pr \left(\text{cutoffs are } P(\mu) \mid \mathbf{Z}; \boldsymbol{\theta} \right) \times \prod_{i \in \mathcal{I}} \Pr \left(\mu(i) = \arg \max_{s' \in \mathcal{S}(e_i, P)} (u_{i,s'}) \mid \mathbf{Z}_i, P(\mu); \boldsymbol{\theta} \right). \end{aligned}$$

The first equation reflects the fact that a stable matching can be fully characterized by the cutoffs $P(\mu)$ and by students being matched with their favorite school given $P(\mu)$; the last equation is implied by the i.i.d. assumption on student preferences and by the assumption that the individual student's $\boldsymbol{\epsilon}_i$ has no impact on $P(\mu)$.

Note that the probability, $\Pr \left(\mu(i) = \arg \max_{s' \in \mathcal{S}(e_i, P)} (u_{i,s'}), \forall i \in \mathcal{I} \mid \mathbf{Z}, P(\mu); \boldsymbol{\theta} \right)$, explicitly conditions on the cutoffs $P(\mu)$, highlighting the fact that we use the cutoffs to specify each student's set of feasible schools $\mathcal{S}(e_i, P)$. However, $P(\mu)$ does not affect the probability beyond that, as preferences do not depend on cutoffs. Therefore,

$$\begin{aligned} \Pr \left(\mu(i) = \arg \max_{s' \in \mathcal{S}(e_i, P)} (u_{i,s'}), \forall i \in \mathcal{I} \mid \mathbf{Z}, P(\mu); \boldsymbol{\theta} \right) &= \Pr \left(\mu(i) = \arg \max_{s' \in \mathcal{S}(e_i, P)} (u_{i,s'}), \forall i \in \mathcal{I} \mid \mathbf{Z}; \boldsymbol{\theta} \right) \\ &\geq \Pr \left(\text{a stable matching } \mu \text{ is realized} \mid \mathbf{Z}; \boldsymbol{\theta} \right), \end{aligned}$$

where the last inequality becomes an equality only when there is a unique stable matching conditional on $(\mathbf{Z}, \boldsymbol{\theta})$, i.e., $\Pr(\text{cutoffs are } P(\mu) \mid \mathbf{Z}; \boldsymbol{\theta}) = 1$. The multiplicity of stable matchings and thus of cutoffs conditional on $(\mathbf{Z}, \boldsymbol{\theta})$ comes from the randomness in utility shocks, $\boldsymbol{\epsilon}$, as well as from the potential multiplicity of stable matchings given $\boldsymbol{\epsilon}$.

Given the parametric assumptions on utility functions, the corresponding (condi-

tional) log-likelihood function is:

$$\ln L_{ST}(\boldsymbol{\theta} \mid \mathbf{Z}, \mu) = \sum_{i=1}^I V_{i,\mu(i)} - \sum_{i=1}^I \ln \left(\sum_{s' \in \mathcal{S}_i} \exp(V_{i,s'}) \right) + \ln \left(\Pr(\text{cutoff is } P(\mu) \mid \mathbf{Z}; \boldsymbol{\theta}) \right). \quad (1)$$

The last term above deserves some careful discussion, because we do not have an analytic form of this probability. In a finite economy, the distribution of cutoffs is approximated by the asymptotic normal distribution derived in Proposition 5, which provides a solution:

$$\ln \left(\Pr(\text{cutoff is } P(\mu) \mid \mathbf{Z}; \boldsymbol{\theta}) \right) \approx \ln \phi \left(P(\mu) - P^\infty(\mathbf{Z}, \boldsymbol{\theta}), V(\mathbf{Z}, \boldsymbol{\theta})/I \right),$$

where $\phi(\cdot)$ is the density function of the $|\mathcal{S}|$ -dimensional normal distribution; $P^\infty(\mathbf{Z}, \boldsymbol{\theta})$ is the vector of cutoffs for the unique stable matching in the continuum economy given $(\mathbf{Z}, \boldsymbol{\theta})$; and $V(\mathbf{Z}, \boldsymbol{\theta})$ is the asymptotic variance calculated based on the formula in Proposition 5. Both terms can be calculated by the simulation methods set out in Appendix C.¹⁸

With this approximation, the model can be estimated by MLE as well, and the estimator is denoted by $\hat{\boldsymbol{\theta}}_{ST}$. Note that we can omit the term, $\Pr(\text{cutoff is } P(\mu) \mid \mathbf{Z}; \boldsymbol{\theta})$, when there is a unique stable matching, e.g., in large enough economies. In our applications, we report results both with and without the cutoff term in the likelihood function.

2.4 Testing Truth-Telling against Stability

Having two distinct estimates $\hat{\boldsymbol{\theta}}_{TT}$ and $\hat{\boldsymbol{\theta}}_{ST}$ for the parameters of the school choice model provides an opportunity to test the truth-telling assumption against the stability assumption by carrying out a Hausman-type specification test.

As summarized in Proposition 2, if every student is weakly truth-telling and is assigned to a school, the matching outcome is stable. Stability, however, does not imply that students are weakly truth-telling and is therefore a less restrictive assumption. Under the null hypothesis that students are weakly truth-telling, both estimators are consistent but only $\hat{\boldsymbol{\theta}}_{TT}$ is asymptotically efficient. Under the alternative that the matching outcome is stable but students are not weakly truth-telling, only $\hat{\boldsymbol{\theta}}_{ST}$ is consistent.

In this setting, the general specification test developed by Hausman (1978) can be

¹⁸ In the application, we approximate the joint probability density function at the true variances but zero pairwise correlations. This is justified, because these correlations are small. In the Monte Carlo samples discussed in Appendix C, the true covariance matrix shows that the average of all the off-diagonal terms is only 3.5 percent of the average of all the diagonal terms.

applied by computing the following test statistic:

$$T_H = (\hat{\boldsymbol{\theta}}_{ST} - \hat{\boldsymbol{\theta}}_{TT})'(\hat{\mathbf{V}}_{ST} - \hat{\mathbf{V}}_{TT})^{-1}(\hat{\boldsymbol{\theta}}_{ST} - \hat{\boldsymbol{\theta}}_{TT}),$$

where $(\hat{\mathbf{V}}_{ST} - \hat{\mathbf{V}}_{TT})^{-1}$ is the inverse of the difference between the asymptotic covariance matrices of $\hat{\boldsymbol{\theta}}_{ST}$ and $\hat{\boldsymbol{\theta}}_{TT}$. Under the null hypothesis, $T_H \sim \chi^2(d_\theta)$, where d_θ is the dimension of $\boldsymbol{\theta}$. If the model is correctly specified and the matching is stable, the rejection of the null hypothesis implies that (weak) truth-telling is violated in the data.

2.5 Stability and Undominated Strategies

An important advantage of the stability assumption is that it only requires data on the assignment outcomes. However, as submitted ROLs are often observed, one might prefer to use the identifying information contained in such data as well.

Under the rationality assumption that students play undominated strategies, observed ROLs are students' true partial preference orders in the context of the student-proposing DA. That is, every L_i respects student i 's true preference ordering among the schools ranked in L_i . These partial orders provide over-identifying information that can be used in combination with the stability assumption to estimate student preferences.

The potential benefits from this approach can be illustrated through a simple example. Consider a constrained/truncated DA where students are only allowed to rank up to three schools out of four. With personalized sets of feasible schools under the stability assumption, the preferences over two schools, say s_1 and s_2 , are estimated mainly from the sub-sample of students who are assigned to either of these schools while having priority indices above the cutoffs of both. Yet it is possible that all students include s_1 and s_2 in their ROLs, even if these schools are not *ex post* feasible for some students. In such a situation, all students could be used to estimate the preference ranking of s_1 and s_2 , rather than just a sub-sample. As shown below, this argument can be extended to the case where two or more schools are observed being ranked by a subset of students.

Moment inequalities. Students' ROLs can be used to form over-identifying *conditional moment inequalities*. Without loss of generality, consider two schools s_1 and s_2 .

Since not everyone ranks both schools, the probability of i ranking s_1 before s_2 is:

$$\Pr(s_1 \succ_{L_i} s_2 | \mathbf{Z}_i; \boldsymbol{\theta}) = \Pr(u_{i,s_1} > u_{i,s_2} \text{ and } s_1, s_2 \in L_i | \mathbf{Z}_i; \boldsymbol{\theta}) \leq \Pr(u_{i,s_1} > u_{i,s_2} | \mathbf{Z}_i; \boldsymbol{\theta}) \quad (2)$$

The first equality is because of undominated strategy, and the inequality defines a lower bound for the probability of $u_{i,s_1} > u_{i,s_2}$. Similarly, one can derive an upper bound:

$$\Pr(u_{i,s_1} > u_{i,s_2} | \mathbf{Z}_i; \boldsymbol{\theta}) \leq 1 - \Pr(s_2 \succ_{L_i} s_1 | \mathbf{Z}_i; \boldsymbol{\theta}) \quad (3)$$

Inequalities (2) and (3) yield the following conditional moment inequalities:

$$\begin{aligned} \Pr(u_{i,s_1} > u_{i,s_2} | \mathbf{Z}_i; \boldsymbol{\theta}) - E[\mathbb{1}(s_1 \succ_{L_i} s_2) | \mathbf{Z}_i; \boldsymbol{\theta}] &\geq 0 \\ 1 - E[\mathbb{1}(s_2 \succ_{L_i} s_1) | \mathbf{Z}_i; \boldsymbol{\theta}] - \Pr(u_{i,s_1} > u_{i,s_2} | \mathbf{Z}_i; \boldsymbol{\theta}) &\geq 0 \end{aligned}$$

Similar moment inequalities can be computed for any pair of schools, and the above formulas can be generalized to any n schools in \mathcal{S} , where $2 \leq n < S$. In the application, we focus on inequalities derived with two schools. The bounds become uninformative if $n \geq 3$, because not many schools are simultaneously ranked by the majority of students.

We interact \mathbf{Z}_i with the above conditional inequalities and thus obtain unconditional ones.¹⁹ This results in M_1 moment inequalities, (m_1, \dots, m_{M_1}) .

Moment equalities. To combine the above over-identifying information in ROLs with that from stability, we must reformulate the likelihood function described in equation (1) into *moment equalities*. The “choice” probability of the matched school can be rewritten as a moment condition by equating theoretical and empirical probabilities:

$$\sum_{i \in \mathcal{I}} \Pr\left(u_{i,s} = \max_{s' \in \mathcal{S}(e_i, P)} (u_{i,s'}) \mid \mathbf{Z}_i, P(\mu); \boldsymbol{\theta}\right) - E\left(\sum_{i \in \mathcal{I}} \mathbb{1}_{(\mu(i)=s)}\right) = 0, \forall s \in \mathcal{S};$$

where $\mathbb{1}_{(\mu(i)=s)}$ is an indicator function taking the value of one if and only if $\mu(i) = s$. We again interact the variables in \mathbf{Z} with the above conditions, leading to more moment equalities.

The delicate task is to incorporate $\ln(\Pr(\text{cutoff is } P(\mu) \mid \mathbf{Z}; \boldsymbol{\theta}))$ into the moment conditions, because this probability has no sample analog. Based on the asymptotic

¹⁹Such variables in \mathbf{Z}_i are known as instruments in the methods of moments literature.

distribution of cutoffs in Proposition 5, we focus on the following two moment equalities:

$$P(\mu) - P^\infty(\mathbf{Z}, \boldsymbol{\theta}) = 0; \quad \text{Diagonal}(V(\mathbf{Z}, \boldsymbol{\theta})) = 0.$$

$\text{Diagonal}(\cdot)$ returns the diagonal terms of a matrix. In other words, we let the observed cutoffs $P(\mu)$ be as close as possible to their means ($P^\infty(\mathbf{Z}, \boldsymbol{\theta})$), while minimizing the variance of each cutoff. Together, we have M_2 moment equalities, $(m_{M_1+1}, \dots, m_{M_1+M_2})$.

Estimation with Moment (In)equalities. To obtain consistent point estimates with both equality and inequality moments (henceforth, moment (in)equalities), we follow the approach of Andrews and Shi (2013), which is valid for both point and partial identifications. The objective function is a test statistic, $T_{MI}(\boldsymbol{\theta})$, of the Cramer-von Mises type with the modified method of moments (or sum function). This test statistic is constructed as follows from the previously defined unconditional moment equalities and inequalities:

$$T_{MI}(\boldsymbol{\theta}) = \sum_{j=1}^{M_1} \left[\frac{\bar{m}_j(\boldsymbol{\theta})}{\hat{\sigma}_j(\boldsymbol{\theta})} \right]_-^2 + \sum_{j=M_1+1}^{M_1+M_2} \left[\frac{\bar{m}_j(\boldsymbol{\theta})}{\hat{\sigma}_j(\boldsymbol{\theta})} \right]^2 \quad (4)$$

where $\bar{m}_j(\boldsymbol{\theta})$ and $\hat{\sigma}_j(\boldsymbol{\theta})$ are the sample mean and standard deviation of $m_j(\boldsymbol{\theta})$, respectively; and the operator $[]_-$ is such that $[a]_- = \min\{0, a\}$. We denote the point estimate $\hat{\boldsymbol{\theta}}_{MI}$, which minimizes $T_{MI}(\boldsymbol{\theta})$, and, to construct the *marginal confidence intervals*, we use the method in Bugni et al. (2014). For a given coordinate θ_k of $\boldsymbol{\theta}$, the authors provide a test for the null hypothesis $H_0 : \theta_k = \gamma$, for any given $\gamma \in \mathfrak{R}$. The confidence interval for the true value of θ_k is the set of all γ 's for which H_0 is not rejected.

2.6 Undominated Strategies without Assuming Stability

The estimation methods described in Sections 2.3 and 2.5 are only valid when the matching outcome is stable. However, as we have shown theoretically, stability can fail. Without stability, the undominated-strategy assumption leads to partial identification. Using equation (4) but without the moment equalities, we can take the same approach as in 2.5 to construct marginal confidence intervals for $\boldsymbol{\theta}$.

2.7 Testing Stability against Undominated Strategies

The moment inequalities add over-identifying information to the moment equalities implied by stability, which constitutes a test of stability, provided that students do not play dominated strategies. More precisely, if both assumptions are satisfied, the moment (in)equality model in Section 2.5 should yield a point estimate that fits the data relatively well; otherwise, there should not exist a point θ that satisfies all moment (in)equalities. Formally, we follow the specification test in Bugni, Canay and Shi (2015).

It should be noted that, for the above test, we maintain the undominated-strategies assumption, which might raise concerns, because students could make mistakes; moreover, untrue partial preference ordering is not dominated under other mechanisms (Section 1.5). The discussion in Section 2.6 provides another test of the undominated-strategies assumption, which also relies on the non-emptiness of the identified set under the null hypothesis (Bugni et al., 2015).

2.8 Results from Monte Carlo Simulations

To validate the estimation approaches and tests, we carry out Monte Carlo (MC) simulations, the details of which are consigned to Appendix C.

The Bayesian Nash equilibrium of the school choice game is simulated in two distinct settings where 500 students compete for admission into 6 schools. The first is the constrained/truncated DA where students are allowed to rank up to 4 or 5 schools. The second setting, called the DA with cost, allows students to rank as many schools as they wish but imposes a constant marginal cost per additional school in the list.

Several lessons can be drawn from these simulations. The first key result is that in both settings, the distribution of school cutoffs is close to jointly normal and degenerates as the number of seats and the number of students increase while holding constant the number of schools; the matching outcome is therefore “almost” stable (i.e., almost every student is assigned to her favorite feasible school) even in moderately-sized markets and with uncertainty in priority indices. By contrast, truth-telling is often violated by the majority of the students, even when they can rank 5 out of 6 schools (constrained DA) or when the cost of including an extra school is negligibly small (DA with cost). When the cost of ranking more schools becomes larger, the Bayesian Nash equilibrium of the game

can result in all students submitting fewer than 6 schools even when they are allowed to rank all of them. Based on these results, observing that only a few students make full use of their ranking opportunities cannot be viewed as a compelling argument in favor of truth-telling when cost is a legitimate concern.

A second important insight from the MC simulations is that the truth-telling estimates ($\hat{\theta}_{TT}$) are severely biased. In particular, we note that students' valuation of the most popular schools tends to be underestimated, especially for students with low priority indices, because such schools are more likely to be omitted from their ROLs due to their low chance of admission. This bias is also present among small schools, which are often left out of ROLs because their admission cutoffs tend to be higher than those of equally desirable but larger schools.

By contrast, the stability estimates ($\hat{\theta}_{ST}$) are reasonably close to the true parameter values, whether cutoffs are endogenized or not, but their standard errors are larger than those obtained under truth-telling. This efficiency loss is a direct consequence of restricting the choice sets to feasible schools and ignoring the information content of ROLs. Under the assumption that the matching outcome is stable, the Hausman-test presented in Section 2.4 strongly rejects truth-telling in our simulations.

The estimates from the moment (in)equalities approach ($\hat{\theta}_{MI}$), which incorporates the over-identifying information contained in students' ROLs, are also consistent. Compared with using stability alone, the inclusion of moment inequalities is informative to the extent that these inequalities define sufficiently tight bounds for the probability of a preference ordering over some pairs of schools. This is more likely when the constraint on the length of ROLs is mild and/or when the cost of ranking an extra school is low, since these situations increase the chances of observing subsets of schools being ranked by a large fraction of students. A limitation of this approach, however, is that the currently available methods for conducting inference based on moment (in)equality models are typically conservative. As a result, the marginal confidence intervals based on moment (in)equalities tend to be wider than those obtained using moment equalities alone, although the point estimates are closer to the true parameter values.

3 School Choice in Paris

Since 2008, the Paris Education Authority assigns students to public high schools based on a version of the school-proposing DA called AFFELNET (Hiller and Tercieux, 2014).

Towards the end of the Spring term, final-year middle school students who are admitted to the upper secondary academic track (*Seconde Générale et Technologique*)²⁰ are requested to submit a ROL of up to 8 public high schools to the Paris Education Authority. Students' priority indices are determined as follows:

- (i) Students' academic performance during the last year of middle school is graded on a scale of 400 to 600 points.
- (ii) Paris is divided into four districts. Students receive a "district" bonus of 600 points for each school in their list that is located in their home district, and no bonus for the others. Therefore, students applying to a school in their home district have full priority over out-of-district applicants to the same school.
- (iii) Low-income students are awarded an additional bonus of 300 points. As a result, these students are given full priority over all other students from the same district.²¹

The DA algorithm is run at the end of the academic year to determine school assignment for the following academic year. Unassigned students can participate in a secondary round of admissions by submitting a new ROL of schools among those with remaining seats, the assignment mechanism being the same as for the main round.

3.1 Data

For our empirical analysis, we use data from Paris' Southern District (*Sud*) and study the choices of 1,590 within-district middle school students who applied for admission to the district's 11 public high schools for the academic year starting in 2013. Owing to the 600-point "district" bonus, this district is essentially an independent market.²² Moreover, within the district, student priority indices are not school-specific, since all schools rank

²⁰In the French educational system, students are tracked at the end of the final year of *collège* (equivalent to middle school), at the age of 15, into vocational or academic upper secondary education.

²¹The low-income status is conditional on a student applying for and being granted the means-tested low-income financial aid in the last year of middle school. A family with two children would be eligible for this financial aid in 2013 if its taxable income was below 17,155 euros. The aid ranges from 135 to 665 euros per year.

²²Out-of-district applicants could affect the availability of school seats in the secondary round, but this is of little concern since in the district, only 22 students out of 1,590 are unassigned at the end of the main round (for the comparison of assigned and unassigned students, see Appendix Table E6).

all students in the same way. The school-proposing DA is therefore equivalent to the student-proposing DA.

Along with information on socio-demographic characteristics and home addresses, our data contain all the relevant variables to replicate the matching outcome, including the schools' capacities, the students' ROLs of schools and their priority indices (converted into percentile ranks between 0 and 1). Individual examination results for the *Diplôme national du brevet* (DNB)—a national exam that all students take at the end of middle school—are used to construct different measures of academic ability (math, French, and composite score), which are normalized as percentile ranks between 0 and 1.²³

Table 1 reports students' characteristics, choices, and admission outcomes. Almost half of the applicants are high socioeconomic status (SES), while 15 percent receive the low-income bonus. 99 percent are assigned to a within-district school in the main admission round, but only half obtain their first choice. Compared to their assigned schools, applicants' first-choice schools tend to have higher ability and more socially privileged students.

More detailed summary statistics for the 11 academic-track high schools in the district are presented in Table 2. Columns 1–4 show that there is a high degree of stratification among schools, both in terms of the average ability of students enrolled in 2012 and of their social background (measured by the fraction of high SES students). Columns 5–8 report a number of outcomes from the 2013 round of assignment. The district's total capacity (1,692 seats) is unevenly distributed across schools: the smallest school has 62 seats while the largest has 251. Admission cutoffs in 2013 are strongly correlated with the different measures of school quality, albeit not perfectly. The last column shows the fraction of ROLs in which each school appears. The least popular three schools are ranked by less than 24 percent of applicants, and two of them remain under-subscribed (Schools 1 and 3) and thus have cutoffs equal to zero. Consistent with our Monte Carlo results, we note that small schools are omitted by many students, even if they are of high quality (e.g., School 8). Likewise, a sizeable fraction of students (20 percent) do not rank the highest-achieving school (School 11) in their lists.

²³See Appendix B for a detailed description of the data sources. A map of the district is provided in Appendix Figure E4.

Table 1: High School Applicants in the Southern District of Paris: Summary Statistics

Variable	Mean	SD	Min	Max	N
<i>Panel A. Student characteristics</i>					
Age	15.0	0.4	13	17	1,590
Female	0.51	0.50	0	1	1,590
French score	0.56	0.25	0.00	1.00	1,590
Math score	0.54	0.24	0.01	1.00	1,590
Composite score	0.55	0.21	0.02	0.99	1,590
high SES	0.48	0.50	0	1	1,590
With low-income bonus	0.15	0.36	0	1	1,590
<i>Panel B. Choices and outcomes</i>					
Number of choices within district	6.6	1.3	1	8	1,590
Assigned to a within-district school	0.99	0.12	0	1	1,590
Assigned to first choice school	0.56	0.50	0	1	1,590
<i>Panel C. Attributes of first choice school</i>					
Distance (km)	1.52	0.93	0.01	6.94	1,590
Mean student French score	0.62	0.11	0.32	0.75	1,590
Mean student math score	0.61	0.13	0.27	0.78	1,590
Mean student composite score	0.61	0.12	0.31	0.77	1,590
Fraction high SES in school	0.53	0.15	0.15	0.71	1,590
<i>Panel D. Attributes of assigned school</i>					
Distance (km)	1.55	0.89	0.06	6.94	1,577
Mean student French score	0.56	0.12	0.32	0.75	1,577
Mean student math score	0.54	0.14	0.27	0.78	1,577
Mean student composite score	0.55	0.13	0.31	0.77	1,577
Fraction high SES in school	0.48	0.15	0.15	0.71	1,577

Notes: This table provides summary statistics on the choices of middle school students from the Southern District of Paris who applied for admission to the district's 11 public high schools for the academic year starting in 2013, based on administrative data from the Paris Education Authority (*Rectorat de Paris*). All scores are from the exams of the *Diplôme national du brevet* (DNB) in middle school and are measured in percentiles and normalized to be in $[0, 1]$. The composite score is the average of the scores in French and math. The correlation coefficient between French and math scores is 0.50. School attributes, except distance, are measured by the average characteristics of students enrolled in each school in the previous year (2012).

Table 2: High Schools in the Southern District of Paris: Summary Statistics

School	School attributes (2012)				Assignment outcomes (2013)			
	Mean math score (1)	Mean French score (2)	Mean composite score (3)	Fraction high SES students (4)	Capacity (5)	Count (6)	Admission cutoffs (7)	Fraction ROLs ranking it (8)
School 1	0.32	0.31	0.31	0.15	72	19	0.000	0.22
School 2	0.36	0.27	0.32	0.17	62	62	0.015	0.23
School 3	0.37	0.34	0.35	0.16	67	36	0.000	0.14
School 4	0.44	0.35	0.39	0.46	140	140	0.001	0.59
School 5	0.47	0.44	0.46	0.47	240	240	0.042	0.83
School 6	0.47	0.46	0.46	0.32	171	171	0.069	0.71
School 7	0.58	0.54	0.56	0.56	251	251	0.373	0.91
School 8	0.58	0.66	0.62	0.30	91	91	0.239	0.39
School 9	0.68	0.62	0.63	0.66	148	148	0.563	0.83
School 10	0.68	0.66	0.67	0.49	237	237	0.505	0.92
School 11	0.75	0.78	0.77	0.71	173	173	0.705	0.80

Notes: This tables provides summary statistics on the attributes of high schools in the Southern District of Paris and on the outcomes of the 2013 assignment round, based on administrative data from the Paris Education Authority (*Rectorat de Paris*). School attributes in 2012 are measured by the average characteristics of the schools' enrolled students in 2012–2013. All scores are from the exams of the *Diplôme national du brevet* (DNB) in middle school and are measured in percentiles and normalized to be in $[0, 1]$. The composite score is the average of the scores in French and math. The correlation coefficient between school-average math and French scores is 0.97.

3.2 Estimation and Test Results

Similar to Section 2, we assume that student i 's utility from attending school s can be represented by the following random utility model:

$$u_{i,s} = \theta_s - d_{i,s} + \mathbf{X}'_{i,s}\boldsymbol{\beta} + \sigma\epsilon_{i,s}, \quad s = 1, \dots, 11; \quad (5)$$

where θ_s is the school fixed effect, $d_{i,s}$ is the distance to school s from i 's place of residence, and $\mathbf{X}_{i,s}$ is a vector of student-school-specific observables. As observed heterogeneity, $\mathbf{X}_{i,s}$ includes two variables that capture potential non-linearities in the disutility of distance and control for potential behavioral biases towards certain schools: “closest school” is a dummy variable equal to one if s is the closest to student i among all 11 schools; “high school co-located with middle school” is another dummy that equals one if high school s and the student's middle school are co-located at the same address.²⁴ To account for students' heterogeneous valuations of school quality, interactions between student scores and school scores are introduced separately for French and math, as well as an interaction between own SES and the fraction of high SES students in the school. We normalize the variables in $\mathbf{X}_{i,s}$ so that each school's fixed effect can be interpreted as the mean valuation, relative to School 1, of a non-high-SES student who has median scores in both French and math, whose middle school is not co-located with that high school, and for whom the high school is not the closest to her residence.

The idiosyncratic error term $\epsilon_{i,s}$ is assumed to be an i.i.d. type-I extreme value, and the variance of unobserved heterogeneity is σ^2 multiplied by the variance of $\epsilon_{i,s}$. The effect of distance is normalized to -1 , and, therefore, the fixed effects and $\boldsymbol{\beta}$ are all measured in terms of willingness to travel. As a usual position normalization, $\theta_1 = 0$. We do not consider an outside option because almost all students enroll in one of the 11 schools.²⁵

Using the same procedures as in the Monte Carlo simulations (described in Appendix C), we obtain the results summarized in Table 3, where each column shows estimates under a given identifying assumption: (i) truth-telling (column 1); (ii) stability with and without endogenizing the cutoffs (columns 2 and 3, respectively); and (iii) stability with undominated strategies (column 4).²⁶ Endogenizing the cutoffs in the last

²⁴There are five such high schools in the district.

²⁵Among students still living in Paris at the start of the academic year following the admission procedure, 97 percent eventually enroll in one of the 11 high schools, while only 3 percent attend a private school instead (see Appendix Table E6).

²⁶For the estimates reported in column 4, we rely on the method of moment (in)equalities where

case does not make any difference, so we report only the estimates without endogenizing the cutoffs to save space.²⁷

The results make evident that the truth-telling estimates (column 1) are rather different from the others. More specifically, the “small-school” downward bias is apparent: 69 percent of students do not include School 8 in their ROLs due to the school’s small capacity (91 seats); the truth-telling assumption dictates that School 8 is less desirable than all the schools included in one’s ROL, which leads to a low estimation of its fixed effect. This under-estimation disappears when the model is estimated under the other two sets of assumptions. Similarly, there is a noticeable difference in the quality estimate of School 11, which is one of the most popular schools.

The Hausman test rejects truth-telling in favor of stability (p-value < 0.01); the test based on moment (in)equalities does not reject the null hypothesis that stability is consistent with undominated strategies at a 5 percent significance level. The results in columns 2 and 3 are very similar, indicating that the market may be large enough for us to treat the cutoffs as exogenous. In the following, we focus on the results from columns 3 and 4.

The results show that “closest school” has no significant effect, but students significantly prefer co-located schools. Compared with low-score students, those with high French (math) scores prefer more schools with higher French (math) scores. Moreover, high SES students prefer schools that have admitted a larger fraction of high SES students.

It is worth noting that the truth-telling estimates of the covariates (Panel B) are not noticeably different from our preferred ones. However, one cannot conclude that the truth-telling assumption produces reasonable results, as the estimates of fixed effects have shown. Moreover, the results of goodness of fit measures reported in Appendix D show that our preferred estimates fit the data well, as opposed to those based on truth-telling, whose predictions are far off the observed outcomes.

inequalities are constructed as described in Section 2.5. Determined by our selection of $\mathbf{X}_{i,s}$, we interact French score, math score, and distances to Schools 1 and 2 with the conditional moments. Although one could use more variables, e.g., SES status and distance to other schools, they provide only little additional variation.

²⁷In principle, the assumption of undominated strategies alone implies partial identification (see Section 2.6). Because stability is not rejected by our test, we do not present results based on this approach (they are available upon request).

Table 3: Estimation Results under Different Identifying Assumptions

	Identifying assumptions			
	Truth-telling	Stability of matching outcome		Stability and undominated strategies
	(1)	(2)	(3)	(4)
<i>Panel A. School fixed effects</i>				
School 2	-0.71 [-1.17; -0.24]	1.46 [0.636; 2.276]	1.45 [0.56; 2.35]	1.21 [0.14; 2.29]
School 3	-2.12 [-2.66; -1.58]	1.03 [0.19; 1.86]	1.04 [0.13; 1.95]	0.84 [-0.56; 2.01]
School 4	3.31 [2.75; 3.86]	2.91 [2.07; 3.76]	2.91 [2.00; 3.82]	2.90 [2.36; 3.39]
School 5	5.13 [4.41; 5.84]	4.16 [3.22; 5.10]	4.15 [3.14; 5.16]	4.16 [3.71; 4.49]
School 6	4.87 [4.21; 5.54]	4.24 [3.29; 5.18]	4.27 [3.26; 5.28]	4.30 [3.73; 4.82]
School 7	7.32 [6.47; 8.18]	6.81 [5.65; 7.98]	6.81 [5.58; 8.04]	6.24 [5.76; 7.28]
School 8	1.59 [1.10; 2.08]	4.46 [3.46; 5.47]	4.47 [3.40; 5.55]	4.27 [2.98; 5.26]
School 9	6.84 [6.07; 7.61]	7.77 [6.55; 8.99]	7.76 [6.46; 9.05]	6.57 [5.84; 7.26]
School 10	7.84 [6.94; 8.75]	7.25 [6.01; 8.49]	7.24 [5.94; 8.54]	6.44 [5.87; 7.05]
School 11	5.35 [4.62; 6.08]	7.28 [6.06; 8.51]	7.33 [6.06; 8.60]	5.61 [4.98; 7.33]
<i>Panel B. Covariates</i>				
Closest school	-0.37 [-0.63; -0.11]	-0.19 [-0.47; 0.10]	-0.19 [-0.47; 0.09]	-0.15 [-0.75; 0.57]
High school co-located with middle school	2.54 [2.02; 3.07]	1.76 [1.19; 2.32]	1.77 [1.22; 2.31]	1.54 [0.17; 3.12]
Student French score [$\times 10$] \times school French score [$\times 10$]	0.30 [0.26; 0.34]	0.27 [0.21; 0.32]	0.27 [0.21; 0.32]	0.30 [0.18; 0.40]
Student math score [$\times 10$] \times school math score [$\times 10$]	0.20 [0.16; 0.23]	0.18 [0.13; 0.24]	0.18 [0.12; 0.24]	0.23 [0.10; 0.35]
high SES \times fraction high SES in school	6.79 [5.62; 7.97]	4.92 [3.31; 6.54]	4.92 [3.24; 6.59]	8.12 [4.18; 12.55]
Scaling parameter	3.09 [2.79; 3.38]	1.33 [1.16; 1.50]	1.32 [1.15; 1.49]	1.50 [1.20; 1.64]
Endogenizing cutoffs	No	No	Yes	No
Number of students	1,590	1,568	1,590	1,590

Notes: This table reports the estimates of model (5) for the Southern District of Paris, with the coefficient on distance being normalized to -1 . All estimates are based on maximum likelihood except those reported in column 4, which are based on moment equalities and inequalities. In brackets, we report the 95 percent confidence interval. “Endogenizing cutoffs” refers to the inclusion of the likelihood terms or moment conditions that are derived based on the asymptotic distribution of cutoffs. An alternative approach to column 4, which also endogenizes cutoffs, yields exactly the same results as in column 4.

Model selection tests: Hausman tests, testing (1) against (2) or (1) against (3), reject the null hypothesis that the truth-telling assumption is satisfied in favor of stability (p-value < 0.01); a test based on moment equalities and inequalities does not reject the null hypothesis that stability is consistent with undominated strategies at the 95 percent level.

4 Admission Criteria in School Choice

This section uses our preference estimates to evaluate the performance of alternative admission criteria. In the absence of monetary transfers, these criteria determine the priority indices and thus dictate which students are allowed to choose before others. The choice of criteria can therefore affect student sorting across schools as well as student welfare. We study the effects of four commonly observed admission criteria:

- (i) *Grades with affirmative action.* In the current Paris high school choice plan, student priority indices are based on academic grades and low-income status. Similar practices are observed in China’s college admission (Chen and Kesten, 2013; Zhu, 2014) and in many other school choice programs.
- (ii) *Grades only.* Some school districts use only academic grades to rank students, for example in Turkey (Akyol and Krishna, 2014).
- (iii) *Random priorities.* Priorities are random *ex ante*, if they are determined by an *ex post* lottery. This is observed, for instance, in Amsterdam (De Haan et. al, 2015) and Beijing (He, 2015). In the school choice programs of Boston and NYC, lottery-determined random priorities are also used in addition to some other coarse priorities such as those given to sibling or neighborhood applicants.
- (iv) *Mixed priorities.* Some places have a mixed system. Within a school district, a few academically high-achieving schools are labeled as “selective” (or “exam,” “magnet”) schools and are allowed to select students by academic grades, while other schools use random priorities to rank students. For example, unlike the other schools that (partially) rely on random priorities, the eight specialized high schools in NYC use the Specialized High Schools Admissions Test to rank applicants.

To evaluate the sorting and welfare effects of these admission criteria, we simulate assignment outcomes by “implementing” the criteria in Paris. For mixed priorities, we assume that admission to the “top two” schools, 10 and 11, is based on grades, while others use random priorities. This gives 410 (or 25 percent) of the applicants access to the selective schools. We also assume that academic grades used in the admission criteria are measured in the same way as the current system’s academic performance component.

4.1 Short-Run and Long-Run Effects in Equilibrium

In the counterfactual exercises, we focus on the stable outcome with respect to students' preferences and schools' rankings over students.²⁸ Students have the observed characteristics as in the data, while utility shocks are randomly drawn in each simulation sample; school attributes, however, depend on the equilibrium concept being considered. Note that our static model incorporates “dynamics,” because students form preferences based on school attributes measured by the characteristics of those already attending the school. We maintain these assumptions when performing the counterfactual analyses.

The short-run equilibrium is simulated by keeping school attributes as in the data. It can be interpreted as the equilibrium that would arise in the first year following adoption of the counterfactual policy. This short-run view can be misleading, however, because school attributes evolve over time due to the policy shift and can therefore be drastically different in the long run. To approximate the long-run steady state under a given policy, we let students play the school choice game given the observed school attributes and obtain a set of new school attributes from the matching outcome; we then iterate until we obtain a fixed point of school attributes.²⁹

We use 300 simulation samples with different random utility shocks for every student. When an admission criterion uses random priorities, we use 1,000 sets of lottery realizations for each simulation sample. In the main text, we present counterfactual analyses based on estimates under the stability assumption with endogenized cutoffs (Table 3, column 3), and additional results are collected in Appendix E. Results based on the stability-and-undominated-strategies assumption are quantitatively similar, while those based on truth-telling are noticeably different, especially in the long run.

4.2 Effects on Student Sorting across Schools

The first set of outcomes under investigation is student sorting across schools, both by ability and by SES. Sorting by ability, which is between 0 and 1, is defined as the ratio of the between-school variance of the student composite score to its total variance. It

²⁸With random or mixed priorities, stability is defined with respect to *ex post* lottery realizations. This is justified if students can rank all schools, which makes strict truth-telling a dominant strategy.

²⁹We also update school fixed effects in this process, based on the results from an OLS regression of the estimated fixed effects on school attributes (fraction of high SES students and mean scores in French and math), which are reported in Appendix Table E7. When attributes are updated, fixed effects are also updated, whereas the coefficients on attributes and the estimated disturbances are held constant.

Table 4: Short-Run and Long-Run Effects of Counterfactual Admission Criteria on Student Sorting and Welfare: Estimates from Stability

	Baseline values		Impact of switching admission criterion to					
	Current criterion		Grades only		Random priorities		Mixed priorities	
	Short run (1)	Long run (2)	Short run (3)	Long run (4)	Short run (5)	Long run (6)	Short run (7)	Long run (8)
<i>Panel A. Students sorting across schools</i>								
By ability	0.409 (0.014)	0.429 (0.014)	0.112 (0.013)	0.113 (0.015)	-0.273 (0.019)	-0.284 (0.019)	-0.007 (0.016)	-0.042 (0.016)
By SES	0.067 (0.010)	0.048 (0.007)	0.039 (0.008)	0.064 (0.010)	0.002 (0.012)	0.028 (0.012)	0.026 (0.011)	0.062 (0.011)
<i>Panel B. Welfare: fraction of winners and losers</i>								
Winners	-	-	0.150 (0.011)	0.608 (0.008)	0.285 (0.009)	0.299 (0.008)	0.253 (0.009)	0.472 (0.008)
Losers	-	-	0.092 (0.006)	0.392 (0.008)	0.291 (0.009)	0.589 (0.008)	0.250 (0.009)	0.528 (0.008)
Indifferent	-	-	0.758 (0.014)	0.000 (0.000)	0.424 (0.014)	0.000 (0.000)	0.497 (0.014)	0.000 (0.000)
<i>Panel C. Welfare measured by willingness to travel (in kilometers)</i>								
Average	1.345 (0.028)	1.161 (0.024)	0.164 (0.022)	0.375 (0.024)	-0.316 (0.041)	-0.299 (0.034)	-0.038 (0.032)	0.187 (0.031)
Ability Q1	0.267 (0.028)	0.144 (0.044)	-0.018 (0.040)	-0.051 (0.042)	0.377 (0.063)	0.419 (0.053)	0.167 (0.059)	0.073 (0.053)
Ability Q2	0.109 (0.049)	-0.306 (0.043)	0.060 (0.043)	-0.038 (0.051)	0.616 (0.096)	0.928 (0.069)	0.173 (0.059)	0.353 (0.053)
Ability Q3	1.343 (0.062)	1.196 (0.055)	0.232 (0.053)	0.312 (0.062)	-0.287 (0.132)	-0.332 (0.082)	-0.274 (0.097)	-0.165 (0.076)
Ability Q4	3.664 (0.072)	3.613 (0.056)	0.382 (0.055)	1.279 (0.064)	-1.972 (0.176)	-2.214 (0.106)	-0.216 (0.097)	0.490 (0.086)
Low SES	1.042 (0.035)	0.834 (0.031)	-0.013 (0.028)	0.053 (0.032)	-0.208 (0.063)	-0.148 (0.044)	-0.180 (0.046)	-0.039 (0.040)
high SES	1.667 (0.048)	1.509 (0.039)	0.352 (0.038)	0.718 (0.045)	-0.431 (0.093)	-0.460 (0.061)	0.114 (0.065)	0.428 (0.057)

Notes: This table reports the effects of different admission criteria on student sorting across high schools (by ability and by SES) and on student welfare in the Southern District of Paris. The effects are measured relative to the current system, in which priority indices are based on students' grades and low-income status. The three admission criteria considered in this table are: (i) Grades only; (ii) Random priorities; and (iii) Mixed priorities (the "top two" schools rank students by grades while the other schools use random priorities). Short-run effects measure what would happen in the first year of the alternative policy compared with the current admission criterion; long-run effects represent those in steady state. Sorting by ability (SES) is the fraction of total variance of ability (SES) explained by the between-school variance. Welfare is measured in terms of willingness to travel (in kilometers). The quartiles of ability are computed using students' composite score on the DNB exam. All effects are based on estimates from stability with endogenized cutoffs (Table 3, column 3).

has an intuitive interpretation: it measures the fraction of the variance in scores that is “explained” by between-school differences and can be obtained as the R-squared of the regression of students’ composite scores on school dummies.³⁰ Sorting by SES is similarly defined, with the difference that SES status is binary (high vs. low).³¹

The first row in Panel A of Table 4 reports the effects of the counterfactual admission criteria on sorting by ability. Compared with the current Paris admission criterion, sorting is worsened when moving to the grades-only criterion and is mitigated when moving to either completely random priorities or—to a lesser extent—mixed priorities. Intuitively, compared with the current criterion or the grades-only criterion, introducing random priorities gives more access to applicants with lower academic grades, which yields a higher degree of mixing of students with heterogeneous abilities. Switching to random priorities decreases ability sorting in the district by 0.273 in the short run, and by 0.284 in the long run, which represents a two-third reduction from the baseline levels.

The second row of Panel A in Table 4 reports the effects of the counterfactual admission criteria on sorting by SES. Compared with the current admission criterion, sorting is worsened when moving to any of the alternatives, especially to the grades-only policy. This is because of the low-income bonus in the current system, which is positively correlated with SES (the correlation coefficient being 0.3).

4.3 Effects on Student Welfare

We now compare the welfare effects of the admission criteria. Panel B of Table 4 shows the fractions of winners and losers, without making inter-student welfare comparison. In the short run, only the grades-only criterion has significantly more winners than losers (by 6 percentage points) compared with the current policy, although the majority (79 percent) are indifferent; the other two criteria produce similar shares of winners and losers. In the long run, however, the outcomes are evidently different. First, no student is indifferent between any pair of criteria, precisely because no school stays the same across criteria. Second, the effects are more pronounced: the grades-only criterion brings more

³⁰In the literature, this index is commonly used to measure income or residential segregation (e.g., Farley, 1977; Yang and Jargowsky, 2006).

³¹It is also known as the Normalized Exposure Index or the Variance Ratio Index. In the case of two groups, it can be interpreted as the exposure of one group to another, normalized by the maximum possible exposure. It has been widely used in the racial school segregation literature and its properties are discussed in Frankel and Volij (2011).

winners (by 20 percentage points); random priorities produce more losers (by 30 percentage points); and mixed priorities have slightly more losers (by 5 percentage points).

Aggregating across students, Panel C shows the average welfare effects for the entire population and for different ability/SES subgroups. The baseline outcomes under the current criterion are normalized so that the welfare is measured in comparison to the welfare that would be achieved by purely random assignment in the long-run steady state.³² While switching to random priorities decreases average welfare in both the short run (by -23 percent) and the long run (by -26 percent), the grades-only policy is welfare-enhancing for both time horizons (by 12 percent and 32 percent, respectively). Mixed priorities represent an intermediate situation, with a small average welfare loss in the short run (of -3 percent) but an average welfare gain (of 16 percent) in the long run.

On the distribution of welfare, Figure 1 provides a visual comparison of the four policies in both the short run (left panel) and the long run (right panel). To plot each distribution, we again measure each student's welfare relative to that achieved by purely random assignment. Unsurprisingly, random priorities make the welfare distribution more concentrated. In the short run, the current admission criterion leads to a welfare distribution similar to that obtained under the grades-only policy, but in the long run the grades-only policy is more beneficial to students in the upper tail.

A closer examination of the welfare effects by ability (rows 2–5 of Panel C in Table 4) reveals that low-ability students benefit from random priorities at the expense of high achievers (columns 5 and 6), whereas removing the low-income bonus from the current admission criterion would harm low-ability students (columns 3 and 4). Figure 2 provides a more detailed overview on these heterogeneous welfare impacts by ability. Compared with the current policy, admission criteria that incorporate random priorities tend to harm students with above-median ability. However, for the students in the upper quartile of the ability distribution, having two selective schools mitigates these negative welfare consequences, especially in the long run where high-ability students enjoy the highest welfare level under the mixed-priorities criterion. Again, as shown in Figure 2, removing

³²Under the purely random assignment, students' preferences are ignored and all applicants have the same probability of being assigned to a given school. In the steady state, observable school attributes (average French and math scores, fraction of high SES) are therefore invariant across schools and are equal to the average characteristics of the student population. By contrast, the unobservable component of the school fixed effects (which is approximated by the vector of residuals from the OLS regressions in Appendix Table E7) is assumed to persist in the long run.

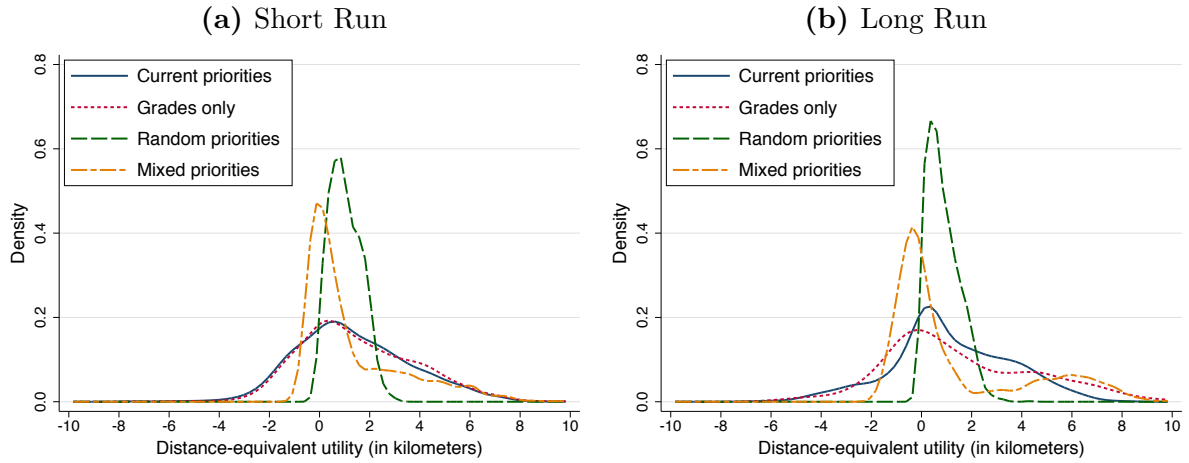


Figure 1: Distribution of Student Welfare under Alternative Admission Criteria: Estimates from Stability

Notes: The two graphs show the short-run (left panel) and long-run (right panel) distributions of student welfare in the Southern District of Paris, under each of the four school priority structures described in Section 4: (i) Current admission criterion (based on academic grades and low-income status); (ii) academic grades only; (iii) Random priorities; (iv) Mixed priorities (based on grades for the “top two” schools and random priorities for other schools). The welfare of each student is measured in terms of willingness to travel (in kilometers) and is normalized by subtracting the distance-equivalent utility that she would experience under a purely random assignment in the long-run steady state. The welfare distributions are computed by averaging over 300 simulation samples with different vectors of random utility shocks for each student. Preferences over the 11 within-district schools are simulated using estimates based on the stability assumption with endogenized cutoffs (Table 4, column 3). The line fits are from a Gaussian kernel with optimal bandwidth using MATLAB’s `ksdensity` command. See Section 4.1 for details on the simulations.

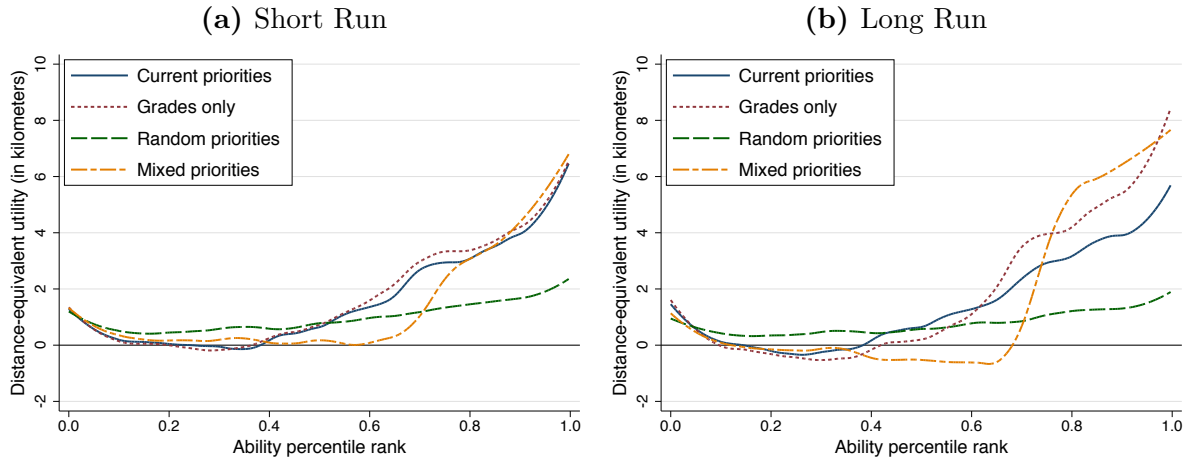


Figure 2: Student Welfare by Ability under Alternative Admission Criteria: Estimates from Stability

Notes: See notes in Figure 1. For each of the four school priority structures described in Section 4, the graphs show the short-run (left panel) and long-run (right panel) distributions of average student welfare in the Southern District of Paris, for students belonging to the different quartiles of academic ability. Student ability is proxied by the percentile rank on the DNB exam composite score (normalized to be between 0 and 1) among all applicants in the data. Preferences over the 11 within-district schools are simulated using estimates based on the stability assumption with endogenized cutoffs (Table 4, column 3). The lines are fitted using MATLAB’s `smooth/rloess` command with a span of 20 percent.

the low-income bonus does not change welfare much for any student in the short run, but in the long run, the top 35 percent of students benefit substantially, while the around-median students are slightly worse off.

Finally, from the bottom two rows of Panel C in Table 4, the grades-only admission criterion helps high SES students in the short run, while harming the low SES ones slightly; surprisingly, it benefits both types of students in the long run. Random priorities reduce the welfare of both low and high SES students, with larger losses for the latter. Mixed priorities benefits high SES students at the expense of low SES ones.

5 Conclusion

We present novel approaches to estimating student preferences with school choice data under the popular Deferred Acceptance mechanism. Our approaches are applicable to many educational systems around the world, as summarized in Table E5, and also to two-sided matching where the preferences of one side are observed. We provide theoretical and empirical evidence showing that it is rather restrictive to assume that students truthfully rank schools when applying for admission. Instead, stability (or justified-envy-freeness) of the matching outcome provides rich identifying information, while being a much weaker assumption. Assuming that students do not play dominated strategies, we also discuss methods with moment inequalities, which can be useful whether stability is satisfied or not. A series of tests are proposed to guide the selection of the appropriate approach.

The estimation and testing procedures are illustrated with Monte Carlo simulations, and our results confirm the theoretical predictions. When applied to school choice data from Paris, our tests strongly reject the truth-telling hypothesis but not stability. Compared with our preferred estimates based on stability (with or without moment inequalities), the incorrect imposition of truth-telling leads to a serious under-estimation of preferences for popular or small schools.

Our preferred estimates are then used to study the sorting and welfare effects of the commonly observed admission criteria, which differ in their use of academic grades and random priorities from lotteries. The results show that finding the optimal admission criterion requires a delicate trade-off between sorting by ability and overall welfare, and between the welfare of low-ability students and that of high-ability students.

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Appendix for

Beyond Truth-Telling: Preference Estimation with
Centralized School Choice

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Appendix A Definition, Additional Results, and Proofs

A.1 Definition of the Boston Mechanism

The **Boston mechanism**, also known as the Immediate Acceptance mechanism, solicits rank-ordered lists of schools from students, uses pre-defined rules to determine schools' strict ranking over students, and has multiple rounds:

Round 1. Each school considers all the students who rank it first and assigns its seats in order of their ranking at that school until either no seats remain or no student remains who has listed it as first choice.

Generally, in:

Round k ($k > 1$). The k th choice of the students who have not yet been assigned is considered. Each school that still has available seats assigns the remaining seats to students who rank it as k th choice in order of their ranking at that school until either no seats remain or no student remains who has listed it as k th choice.

The process terminates after any Round k when every student is assigned a seat at some school, or if the only students who remain unassigned listed no more than k choices.

A.2 School Choice versus Portfolio Choice

Our model of school choice with costs is closely related to the portfolio choice model in Chade and Smith (2006), where an individual simultaneously chooses among ranked stochastic options, only one of which may be exercised from those that succeed. The main contribution of Chade and Smith (2006) is the proposed Marginal Improvement Algorithm for finding an optimal portfolio quickly, which would otherwise be computationally difficult due to the problem's combinatorial nature.

Although the algorithm can be a promising candidate for school choice, there are several concerns. First, as mentioned in their working paper version, Chade and Smith's portfolio choice problem requires that the success probability of each option is independent or correlated in a rather restrictive way, which is in general not satisfied in school choice (see Proposition 5). This issue is not mitigated even if the pairwise correlations are small, because the optimal decision requires the probability of admission to one school conditional on being rejected by multiple schools. Second, portfolio choice implicitly ranks the chosen options according to one's own preference order, which is not true in

school choice equilibrium except under the student-proposing DA. Ranking of the schools would not matter only if admission probabilities are independent. Third and most importantly, even if a particular school choice problem could be considered as a portfolio choice problem, one would need precise data on, or a precise estimation of, the admission probabilities to implement the marginal improvement algorithm, which is usually not feasible when schools rank students by priority indices.³³

A.3 Additional Results and Proofs

This subsection collects the additional results and the proofs for a given finite economy, while those related to asymptotics and results in the continuum economy are presented in Section A.4.

Before proving the main results in the paper, we introduce the following definitions in addition to the swap monotonicity in Definition 3.

Definition A1. Fix any i and two ROLs, L_i and L'_i , such that the only difference between them is two neighboring choices: $(l_i^{k^*}, l_i^{k^*+1}) = (s, s')$, $(l_i'^{k^*}, l_i'^{k^*+1}) = (s', s)$, and $l_i^k = l_i'^k$ for all $k \neq k^*, k^* + 1$.

(i) A mechanism is **upper invariant** if for all (L_{-i}, e_{-i}) :

$$a_{s^*}(L_i, e_i; L_{-i}, e_{-i}) = a_{s^*}(L'_i, e_i; L_{-i}, e_{-i}),$$

where s^* is the k th choice in both L_i and L'_i for all $k < k^*$.

(ii) A mechanism is **lower invariant** if for all (L_{-i}, e_{-i}) :

$$a_{s^*}(L_i, e_i; L_{-i}, e_{-i}) = a_{s^*}(L'_i, e_i; L_{-i}, e_{-i}),$$

where s^* is the k th choice in both L_i and L'_i for all $k > k^* + 1$.

The following theorem is useful for proving Theorem 2.

Theorem A1 (Mennle and Seuken, 2014). A mechanism is strategy-proof if and only if it is swap monotonic, upper invariant, and lower invariant. Therefore, DA satisfies these three properties.

³³When lotteries are used to rank students, approaches to estimate such probabilities are considered in Agarwal and Somaini (2014), Calsamiglia et al. (2014) and He (2015). In essence, these assumptions rely on the assumption that the empirical distribution of strategies approximates the theoretical one in large markets. A similar approach is also used in Carvalho, Magnac and Xiong (2014).

Proof of Theorem 2.

(i) Sufficiency.

When $C(|L|) = 0$, it is known that strict truth-telling is a weakly dominant strategy, so we need to show that any other strategy is strictly dominated.

Suppose that for a given student i , there is a ROL $L \neq R_i$, which is payoff-equivalent to R_i given a profile of (L_{-i}, e_{-i}) : $\sum_{s \in S} [a_s(L, e_i; L_{-i}, e_{-i}) - a_s(R_i, e_i; L_{-i}, e_{-i})] u_{i,s} = 0$. We only consider L being a full list. Otherwise, we can add the omitted schools to the end of L_i without decreasing the payoff to i .

Recall that $R_i = (r_i^1, \dots, r_i^S)$. We can always transform L to R_i by a series of swaps as in Definition A1 (or Definition 3). We show that each step of the following procedure strictly increases i 's expected utility. Let $L_{(0)} = L$ and perform the following iteration.

Iteration t , $t \geq 1$:

If $\exists k^* \in \{1, \dots, S-1\}$ s.t. $l_{(t-1)}^{k^*} = r_i^S$, swap $l_{(t-1)}^{k^*}$ with $l_{(t-1)}^{k^*+1}$ to create a new ROL, $L_{(t)}$, such that $l_{(t)}^{k^*+1} = r_i^S$, $l_{(t)}^{k^*} = l_{(t-1)}^{k^*+1}$, and $l_{(t)}^k = l_{(t-1)}^k$ for all $k \neq k^*, k^*+1$; take $L_{(t)}$ to start the next iteration ($t+1$);

else if $\exists k^* \in \{1, \dots, S-2\}$ s.t. $l_{(t-1)}^{k^*} = r_i^{S-1}$, swap $l_{(t-1)}^{k^*}$ with $l_{(t-1)}^{k^*+1}$ to create a new ROL, $L_{(t)}$, such that $l_{(t)}^{k^*+1} = r_i^{S-1}$, $l_{(t)}^{k^*} = l_{(t-1)}^{k^*+1}$, and $l_{(t)}^k = l_{(t-1)}^k$ for all $k \neq k^*, k^*+1$; take $L_{(t)}$ to start the next iteration ($t+1$);

...

else if $k^* = 1$ s.t. $l_{(t-1)}^{k^*} = r_i^2$, swap $l_{(t-1)}^{k^*}$ with $l_{(t-1)}^{k^*+1}$ to create a new ROL, $L_{(t)}$, such that $l_{(t)}^{k^*+1} = r_i^2$, $l_{(t)}^{k^*} = l_{(t-1)}^{k^*+1}$, and $l_{(t)}^k = l_{(t-1)}^k$ for all $k \neq k^*, k^*+1$; take $L_{(t)}$ to start the next iteration ($t+1$);

else if $L_{(t-1)} = R_i$, terminate the iteration process.

Note that the above iteration must terminate in finite steps. The swap in every iteration involves moving a less preferred school ($l_{(t-1)}^{k^*}$) further down the ROL. Since the mechanism is strictly swap monotonic, each swap strictly increases the probability of being assigned to a more preferred school ($l_{(t-1)}^{k^*+1}$) and decreases that of being assigned to a less preferred school ($l_{(t-1)}^{k^*}$), while assignment probabilities of other schools are constant due to upper and lower invariance.

Therefore, $\sum_{s \in S} [a_s(L, e_i; L_{-i}, e_{-i}) - a_s(R_i, e_i; L_{-i}, e_{-i})] u_{i,s} = 0$ cannot be satisfied, and strict truth-telling is a strictly dominant strategy.

(ii) Necessity.

If $C(|L|) > 0$ for some L , strong truth-telling is not a weakly dominant strategy, as sometimes it might be optimal to rank fewer schools.

Suppose that $C(|L|) = 0$ for all L , but that strict swap monotonicity is violated. There must exist L_i and L'_i , such that the only difference between them is two neighboring choices: $(l_i^{k^*}, l_i^{k^*+1}) = (s, s')$, $(l_i^{k^*}, l_i^{k^*+1}) = (s', s)$, and $l_i^k = l_i^k$ for all $k \neq k^*, k^* + 1$. For some (L_{-i}, e_{-i}) :

$$\begin{aligned} a_s(L_i, e_i; L_{-i}, e_{-i}) &= a_s(L'_i, e_i; L_{-i}, e_{-i}); \\ a_{s'}(L_i, e_i; L_{-i}, e_{-i}) &= a_{s'}(L'_i, e_i; L_{-i}, e_{-i}). \end{aligned}$$

Therefore, due to the above two equalities and the upper and lower invariance, L_i and L'_i lead to exactly the same admission probability at every school.

Suppose that i 's true ordinal preference is $R_i = L_i$. Given (L_{-i}, e_{-i}) , i is then indifferent between submitting L_i and L'_i , which contradicts strict truth-telling being a strictly dominant strategy. This proves the necessity of the two conditions. \blacksquare

Proof of Proposition 1. The proof of Theorem 2 can be applied to show this proposition. \blacksquare

Proof of Theorem 3. If $C(|L|) = 0$ for all L , Theorem 1 implies that strict truth-telling is a weakly dominant strategy. Therefore, any ROL that is not a true partial order is weakly dominated by strict truth-telling. Furthermore, from Theorem 2, the dominance becomes strict when the mechanism is strictly swap monotonic.

If $C(|L|) \neq 0$ for some L , suppose that L_i is not a true partial order of schools. We can always find a true partial order L'_i that ranks the same set of schools as L_i . That is, $s \in L'_i$ if and only if $s \in L_i$. We show that L'_i weakly dominates L_i .

The DA satisfies swap monotonicity, and upper and lower invariance (Theorem A1). Similar to the procedure in the proof of Theorem 2, we can find a series of swaps of two adjacent choices to transform L_i and L'_i . Any such swap involves moving a less-preferred school further down the list, which weakly decreases the probability of being assigned to that school and weakly increases the probability of being assigned to a more-preferred school, while other admission probabilities are unchanged. Therefore, L'_i must weakly dominate L_i . Moreover, if the mechanism is strictly swap monotonic, each such swap

strictly increases i 's payoff, and thus the dominance becomes strict. ■

Proof of Proposition 2.

(i) Suppose that given a matching μ , there is a student-school pair (i, s) such that $\mu(i) \in \mathcal{S} \setminus \{s\}$, $u_{i,s} > u_{i,\mu(i)}$, and $e_{i,s} \geq p_s$.

Since i is weakly truth-telling, she must have ranked all schools that are more preferred to $\mu(i)$, including s . The DA algorithm implies that i must have been rejected by s at some round given that she is accepted by a lower-ranked school $\mu(i)$. As the “cutoff” of each school must increase over rounds, the final cutoff of s must be higher than $e_{i,s}$. This contradiction rules out the existence of such matchings.

(ii) Given the result in part (i), when every student is assigned, everyone must be assigned to her favorite feasible school. Therefore, the matching is stable. ■

Proof of Proposition 3. Sufficiency is implied by Theorems 1 and 2. That is, truth-telling is a dominant strategy if $C(|L|) = 0$ for all L , which always leads to stability. If the mechanism is strictly swap monotonic, truth-telling is a strictly dominant strategy.

To prove necessity, it suffices to show that there is no dominant strategy when $C(|L|) > 0$ for some $L \in \mathcal{L}$.

If $C(|L|) = +\infty$ for some L , we are in the case of the constrained/truncated DA, and it is well known that there is no dominant strategy (see, e.g., Haeringer and Klijn, 2009).

Now suppose that $0 < C(|L|) < +\infty$ for some $L \in \mathcal{L}$. If a strategy always ranks all schools, then it must be weakly dominated by strict truth-telling, which is not a dominant strategy. If a strategy ranks fewer than S schools with a positive probability, we know that it cannot be a dominant strategy for the same reason as in the constrained/truncated DA.

Therefore, there is no dominant strategy when $C(|L|) > 0$ for some $L \in \mathcal{L}$, and hence stability cannot be implemented in dominant strategy. ■

A.4 Asymptotics: Proofs and Additional Results

This subsection presents the proofs of results as well as some additional results on the asymptotics and the continuum economy.

A.4.1 Differentiability of Demand Function

Given the distributional assumption on ϵ , the demand for school s in the continuum economy can be written as a function of cutoffs. For example, when ϵ is Type-I extreme value, we have:

$$D_s(P) = \int_{e \in [0,1]^S} \frac{\mathbb{1}_{(e_s < P_s)} \exp(v_s)}{1 + \sum_{s'=1}^S \mathbb{1}_{(e_{s'} < P_{s'})} \exp(v_{s'})} dH(e)$$

As long as $H(e)$ has a density function $g(e)$, $D(P)$ is differentiable with respect to P . The same is true when ϵ is normally distributed.

A.4.2 Nash equilibrium and Stable Outcome

Example A1 (An unstable Nash equilibrium outcome in the continuum economy). Suppose that the continuum of students consists of three types, $\mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$, with each type being of the same measure, $1/3$. Let $S = \{s_1, s_2, s_3\}$ be the set of schools and $Q = (1/3, 1/3, 1/3)$ the vector of capacities. Students' preferences and the priority structure (i.e., student priority indices at each school) are given in the table below. Student priority indices are random draws from a uniform distribution on the interval therein, which allows schools to strictly rank students. It is assumed that students are only allowed to rank up to two schools.

	Student preferences			Student priority indices		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
School s_1	1	0.5	0	(1/3, 2/3]	[0, 1/3]	(2/3, 1]
School s_2	0	0	0.5	(1/3, 2/3]	[0, 1/3]	(2/3, 1]
School s_3	0.5	1	1	(2/3, 1]	(1/3, 2/3]	[0, 1/3]

	ROL by type		
	Type 1	Type 2	Type 3
1st choice	s_1	s_1	s_3
2nd choice	s_3	s_2	s_1

Students submit ROLs as shown in the table. One can verify that the outcome is that Type-1 students are matched with s_1 , Type-2 with s_2 , and Type-3 with s_3 , which is indicated by the boxes in the table. This matching is not stable as Type-2 students have justified envy for s_3 , i.e., a positive measure of Type-2 students can form a blocking

pair with School s_3 . However, there is no profitable deviation for any subset of Type-2 students.

The above example is similar to the discrete version in Haeringer and Klijn (2009), who show that in discrete and finite economies, DA with constraints implements stable matchings in Nash equilibria if and only if the student priority indices at all schools satisfy the so-called Ergin acyclicity condition (Ergin, 2002). We extend this result to the continuum economy and to a more general class of DA mechanisms where the cost function of ranking more than one school, $C(|L|)$, is flexible.

Definition A2. *In a continuum economy, we fix a vector of capacities, $\{q_s^\infty\}_{s=1}^S$, and a distribution of priority indices, H . An Ergin cycle is constituted of distinct schools (s_1, s_2) and subsets of students $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ (of equal measure $q_0 > 0$), whose elements are denoted by i_1, i_2 , and i_3 , respectively, such that the following conditions are satisfied:*

(i) *Cycle condition: $e_{i_1, s_1} > e_{i_2, s_1} > e_{i_3, s_1}$, and $e_{i_3, s_2} > e_{i_1, s_2}$, for all i_1, i_2 , and i_3 .*

(ii) *Scarcity condition: there exist (possibly empty) disjoint sets of agents $\mathcal{I}_{s_1}, \mathcal{I}_{s_2} \in \mathcal{I} \setminus \{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$ such that $e_{i, s_1} > e_{i_2, s_1}$ for all $i \in \mathcal{I}_{s_1}$, $|\mathcal{I}_{s_1}| = q_{s_1} - q_0$; $e_{i, s_2} > e_{i_1, s_2}$ for all $i \in \mathcal{I}_{s_2}$, and $|\mathcal{I}_{s_2}| = q_{s_2} - q_0$.*

A priority index distribution H is Ergin-acyclic if it allows no Ergin cycles almost surely.

Note that this acyclicity condition is satisfied if all schools rank students in the same way. Under this acyclicity condition, we can extend Theorem 6.3 in Haeringer and Klijn (2009) to the continuum economy.

Proposition A1. *In the continuum economy E :*

(i) *Every stable outcome is an outcome of some Nash equilibrium.*

(ii) *If $C(2) = 0$, every (pure-strategy) Nash equilibrium outcome is stable if and only if the economy satisfies Ergin-acyclicity (Haeringer-Klijn, 2009).*

(iii) *If $C(2) > 0$, all (pure-strategy) Nash equilibrium outcomes are stable.*

(iv) *If $C(|L|) = 0$ for all L , the unique trembling hand perfect equilibrium (THPE) is when everyone ranks all schools truthfully, and the corresponding outcome is the student-optimal stable matching.*

Proof. Part (i) can be shown by letting every student i submit a one-school list including only $\mu^\infty(i)$.

To prove parts (ii) and (iii), we can use the proof of Theorem 6.3 in Haeringer and Klijn (2009) and, therefore, that of Theorem 1 in Ergin (2002). It can be directly extended to the continuum economy under more general DA mechanisms, although it is for discrete economies. We notice the following:

- (a) The continuum economy can be “discretized” in a similar manner as in Example A1 and each subset of students can be treated as a single student. When doing so, we do not impose restrictions on the sizes of the subsets, as long as they have a positive measure. This allows us to use the derivations in the aforementioned proofs.
- (b) The flexibility in the cost function of ranking more schools does not impose additional restrictions. As we focus on Nash equilibrium, for any strategy with more than one school listed, we can find a one-school list that has the same or higher payoff. And indeed, many steps in the aforementioned proofs involve such a treatment.

To show part (iv), we notice that a perturbed game is a copy of a base school choice game, with the restriction that only totally mixed strategies are allowed to be played. It is known that the set of trembling hand perfect equilibria is the set of undominated strategies, which implies that strict truth-telling is the unique THPE. ■

Proof of Proposition 4. Suppose instead that there is a subsequence of finite economies $\{F^{(\hat{I})}\}_{\hat{I} \in \mathbb{N}}$ such that σ is an equilibrium. Note that we still have $F^{(\hat{I})} \rightarrow E$ almost surely.

If σ is in pure strategy, given the finite economies, it creates a (sub-)sequence of ordinal economies,

$$F_{\sigma}^{(\hat{I})} = \left\{ [\sigma(u_i, e_i), e_i]_{i \in \mathcal{I}(\hat{I})}, [q_s^{(\hat{I})}]_{s \in \mathcal{S}}, C(|L|) \right\}.$$

That is, the original cardinal preferences $[u_{i,s}]_{i \in \mathcal{I}(\hat{I}), s \in \mathcal{S}}$ are replaced by ordinal “preferences” $[\sigma(u_i, e_i)]_{i \in \mathcal{I}(\hat{I})}$. Correspondingly, we can define the continuum ordinal economy E_{σ} such that $F_{\sigma}^{(\hat{I})} \rightarrow E_{\sigma}$ almost surely.

Given the student-proposing DA, we focus on the student-optimal stable matching (SOSM) in the ordinal economies induced by σ , and $P_{\sigma}^{(\hat{I})}$ are the associated cutoffs in $F_{\sigma}^{(\hat{I})}$. It must be that $P_{\sigma}^{(\hat{I})} \rightarrow P_{\sigma}$ almost surely, where the latter is the cutoff of the SOSM in E_{σ} . Moreover, P_{σ} is also the cutoff in E given $\mu_{(E, \sigma)}$.

Because there is a unique Nash equilibrium outcome, which is also the unique stable matching, in E by assumption, $G \times H(\{i \in \mathcal{I} \mid \mu_{(E,\sigma)}(i) \neq \mu^\infty(i)\}) > 0$ in the continuum economy implies that P_σ is not the market-clearing cutoff in E . Thus we can find a positive-measure set of students $\mathcal{I}_{(\eta,\xi)} = \{i \in \mathcal{I} \mid u_{i,\mu_{(E,\sigma)}(i)} + \xi < \max_{s \in \mathcal{S}(e_i, \eta + P(\mu_{(E,\sigma)}))}(u_{i,s})\}$, where $\mathcal{S}(e_i, \eta + P(\mu_{(E,\sigma)}))$ is the set of schools whose cutoffs $\mu_{(E,\sigma)}$ are below i 's priority index by at least $\eta (> 0)$. $\xi > 0$ implies that students in $\mathcal{I}_{(\eta,\xi)}$ would deviate from σ in E .

We further define $\mathcal{I}_{(\eta,\xi)}^{(\hat{I})}$ the sub-set of students in $F^{(\hat{I})}$ corresponding to $\mathcal{I}_{(\eta,\xi)}$. That is, if $i \in \mathcal{I}_{(\eta,\xi)}^{(\hat{I})}$, there exists $j \in \mathcal{I}_{(\eta,\xi)}$ such that $e_i = e_j$ and $u_i = u_j$.

Since $P_\sigma^{(\hat{I})} \rightarrow P_\sigma$ almost surely, for any given η and ξ , and for $0 < \phi < \xi$, there exists \hat{N} such that in $F^{(\hat{I})}$ for all $\hat{I} > \hat{N}$:

- (i) every i in $\mathcal{I}_{(\eta,\xi)}^{(\hat{I})}$ is assigned to $\mu_{(E,\sigma)}(i)$ with probability at least $(1 - \phi)$,
- (ii) $\Pr\left(\|P_\sigma^{(\hat{I})} - P_\sigma\| > \eta\right) < \min\left\{\phi, 1 - \frac{\phi}{\xi}\right\}$.

For a small enough ϕ , everyone in $\mathcal{I}_{(\eta,\xi)}^{(\hat{I})}$ then has an incentive to deviate from σ , and therefore σ cannot be an equilibrium for $\hat{I} > \hat{N}$.

If σ is in mixed strategies, the above arguments can still be applied after taking into account that σ now transforms each finite (cardinal) economy $F^{(I)}$ into a probability distribution over a set of ordinal economies.

Since there is no subsequence of finite economies where σ is always an equilibrium, there exists N such that σ is not an equilibrium in $F^{(I)}$ for all $I > N$. ■

Proof of Lemma 2. With every student ranking the school prescribed by μ^∞ , we then transform the cardinal economies into ordinal economies where everyone has only one acceptable school. As before (e.g., part (iii) of Proposition 5), the cutoffs in finite ordinal economies converge in probability to that in the continuum ordinal economy (P^∞). Similar to the proof of Proposition 4, ranking the school prescribed by μ^∞ is a Bayesian Nash equilibrium when $C(2) > 0$ and the market is large enough. ■

Proof of Proposition 5. Recall that we assume that (a) there is a unique stable matching in the continuum economy, which is the only outcome of all Nash equilibria and (b) $\sigma^{(J)}$, a Bayesian Nash equilibrium in the economy $F^{(J)}$, includes σ^∞ almost surely.

Part (iii) is a result from Proposition G1 in Azevedo and Leshno (forthcoming). We prove (i), and (i) implies (ii).

We observe that, in the continuum economy E , there can exist a school with a cutoff equal to zero. Let \mathcal{S}^+ be the set of schools with positive cutoffs in E , and thus $\mathcal{S}^+ \subset \mathcal{S}$.

We further let σ_1 be the strategy such that everyone ranks only the school prescribed by μ^∞ , i.e., $\sigma_1(u_i, e_i) = \mu^\infty(i)$ for all i . Applying σ_1 to $F^{(I)}$, we know that the cutoff vector in that randomly generated economy satisfies:

$$\Pr(P(\mu_{(F^{(I)}, \sigma_1)}) \neq P^\infty) = \Pr\left(\begin{array}{l} \exists s \in \mathcal{S}^+, s.t., D_s(P^\infty, \sigma_1 | F^{(I)}) < I\bar{q}_s; \text{ or} \\ \exists s \in \mathcal{S} \setminus \mathcal{S}^+, s.t., D_s(P^\infty, \sigma_1 | F^{(I)}) > I\bar{q}_s \end{array}\right),$$

where $D_s(P^\infty, \sigma_1 | F^{(I)})$ is the demand for school s at price P^∞ given the induced ordinal economy $F_{\sigma_1}^{(I)}$.

By assumption, $D_s(P^\infty, \sigma_1 | F^{(I)})/I$ converges in probability to $D_s(P^\infty, \sigma_1)$ (the demand in the continuum economy). Since P^∞ is the unique cutoff associated with the stable matching μ^∞ , $D_s(P^\infty, \sigma_1) = \bar{q}_s$ for all s . It thus follows that:

$$\Pr\left(\begin{array}{l} \exists s \in \mathcal{S}^+, s.t., D_s(P^\infty, \sigma_1 | F^{(I)}) < I\bar{q}_s; \text{ or} \\ \exists s \in \mathcal{S} \setminus \mathcal{S}^+, s.t., D_s(P^\infty, \sigma_1 | F^{(I)}) > I\bar{q}_s \end{array}\right) \xrightarrow{p} 0,$$

which implies $P(\mu_{(F^{(I)}, \sigma_1)}) \xrightarrow{p} P^\infty$.

For any given $\sigma^{(J)}$ and the induced random ordinal economy $F_{\sigma^{(J)}}^{(I)}$, we also have the following:

$$\begin{aligned} & \Pr(P(\mu_{(F^{(I)}, \sigma^{(J)})}) \neq P^\infty) \\ &= \Pr\left(\begin{array}{l} \exists s \in \mathcal{S}^+, s.t., D_s(P^\infty, \sigma^{(J)} | F^{(I)}) < I\bar{q}_s; \text{ or} \\ \exists s \in \mathcal{S} \setminus \mathcal{S}^+, s.t., D_s(P^\infty, \sigma^{(J)} | F^{(I)}) > I\bar{q}_s \end{array}\right) \\ &= \Pr\left(\begin{array}{l} \exists s \in \mathcal{S}^+, s.t., D_s(P^\infty, \sigma_1 | F^{(I)}) < I\bar{q}_s; \text{ or} \\ \exists s \in \mathcal{S} \setminus \mathcal{S}^+, s.t., D_s(P^\infty, \sigma_1 | F^{(I)}) > I\bar{q}_s \end{array}\right), \end{aligned}$$

where the second equality is because at the cutoffs P^∞ , the matching μ^∞ prescribes the favorite feasible school to every student, and thus $D_s(P^\infty, \sigma^{(J)} | F^{(I)}) = D_s(P^\infty, \sigma_1 | F^{(I)})$.

Hence, $\sup_{J \in \mathbb{N}} \|P(\mu_{(F^{(I)}, \sigma^{(J)})}) - P^\infty\| \xrightarrow{p} 0$, which is equivalent to $P(\mu_{(F^{(J)}, \sigma^{(J)})}) \xrightarrow{p} P^\infty$.

To show Part (ii), we observe that a blocking pair is only possible when cutoffs deviate from P^∞ , given that $\sigma^{(J)}$ ranks the school prescribed by μ^∞ almost surely. $P(\mu_{(F^{(J)}, \sigma^{(J)})}) \xrightarrow{p} P^\infty$ thus implies that the fraction of students that can form a blocking pair goes to zero in probability. ■

Proof of Proposition 6. Suppose that i is in a blocking pair with some school s . It means that the *ex post* cutoff of s is lower than i 's priority index at s . Therefore, if $s \in L_i$,

the stability of DA implies that i must be accepted by s or by schools ranked above and thus preferred to s . Therefore, i and s cannot form a blocking pair if $s \in L_i$, which proves the first statement in the proposition.

We let $\mathcal{S}_0 = \mathcal{S} \setminus L_i$ and are interested in the following probability (implicitly conditional on $\sigma^{(I)}$):

$$P_i^{block} \equiv \Pr \left(\exists s \in \mathcal{S}_0, s.t., (i, s) \text{ is a blocking pair} \mid L_i \right),$$

which is the probability of i being in a blocking pair in equilibrium $\sigma^{(I)}$.

We use p_s to denote the cutoff of s of the match $\mu_{(F^{(I)}, \sigma^{(I)})}$. The cutoffs are random variables (*ex ante*), and can be used to bound P_i^{block} .

$$P_i^{block} \leq \sum_{s \in \mathcal{S}_0} \Pr \left(p_t > e_{i,t}, \forall t \in L_i, s.t., t \succ_{R_i} s; p_s < e_{i,s} \mid L_i \right) \equiv \sum_{s \in \mathcal{S}_0} P_{i,s}^{block},$$

where $\Pr \left(p_t > e_{i,t}, \forall t \in L_i, s.t., t \succ_{R_i} s; p_s < e_{i,s} \mid L_i \right)$ is the probability that i is not assigned to any school that is preferred to s while s is feasible for i , and we thus denote it as $P_{i,s}^{block}$. $\sum_{s \in \mathcal{S}_0} P_{i,s}^{block}$ is an upper bound for P_i^{block} because there is a positive probability that i simultaneously forms blocking pairs with multiple schools in \mathcal{S}_0 .

We then show how this probability changes with primitives for a given $s \in \mathcal{S}_0$. Since $s \in \mathcal{S}_0$ and L_i is *ex ante* optimal, it implies:

$$\sum_{s \in \mathcal{S}} A_s(L_i, e_i) u_{i,s} - C(|L_i|) \geq \sum_{s \in \mathcal{S}} A_s(L'_i, e_i) u_{i,s} - C(|L_i| + 1),$$

where $L'_i = \left(l_i^1, \dots, l_i^k, s, l_i^{k+1}, \dots, l_i^{|L_i|} \right)$, and $l_i^1 \succ_{R_i} \dots \succ_{R_i} l_i^k \succ_{R_i} s \succ_{R_i} l_i^{k+1} \succ_{R_i} \dots \succ_{R_i} l_i^{|L_i|}$, i.e., adding s to the true partial preference order L_i while keeping the new list a true partial preference order. For notational convenience, we relabel the schools such that $l_i^1 = 1, \dots, l_i^k = k, l_i^{k+1} = k+1, \dots, l_i^{|L_i|} = K$.

It then follows that:

$$\begin{aligned} & C(|L_i| + 1) - C(|L_i|) \\ & \geq \sum_{t=1}^k 0 + P_{i,s}^{block} u_{i,s} + \sum_{t=k+1}^K \left(\Pr(p_t < e_{i,t}; p_s > e_{i,s}; p_\tau > e_{i,\tau}, \forall \tau \in \{1, \dots, t-1\} \mid L_i) \right. \\ & \quad \left. - \Pr(p_t < e_{i,t}; p_\tau > e_{i,s}, \forall \tau \in \{1, \dots, t-1\} \mid L_i) \right) u_{i,t}, \end{aligned}$$

where the zeros in the first term on the right come from the upper invariance of DA

(Definition A1). This leads to:

$$P_{i,s}^{block} \leq \frac{C(|L_i| + 1) - C(|L_i|)}{u_{i,s} - \sum_{t=k+1}^K \Pr \left(\begin{array}{c|c} p_t < e_{i,t}; p_\tau > e_{i,\tau}, & L_i; p_s < e_{i,s}; \\ \forall \tau \in \{k+1, \dots, t-1\} & p_\tau > e_{i,\tau}, \forall \tau \in \{1, \dots, k\} \end{array} \right) u_{i,t}},$$

where the set $\{k+1, \dots, k\}$ should be interpreted as an empty set; the denominator of the right-side term must be positive, because $u_{i,s} > u_{i,t}$ for all $t \geq k+1$ and the sum of all the probabilities is less than or equal to one. This inequality thus implies that $P_{i,s}^{block}$ is bounded by a term that is monotonically increasing in $C(|L_i| + 1) - C(|L_i|)$ and is decreasing in the cardinal utility $u_{i,s}$ relative to the less preferred schools.

Moreover, parts (i) and (ii) in Proposition 5 imply that P_i^{block} goes to zero as $I \rightarrow +\infty$. ■

Appendix B Data

B.1 Data Sources

For the empirical analysis, we use three administrative data sets on Parisian students, which are linked using an encrypted version of the French national student identifier (*Identifiant National Élève*).

- (i) **Application Data:** The first data set was provided to us by the Paris Education Authority (*Rectorat de Paris*) and contains all the information necessary to replicate the assignment of students to public upper secondary schools in the city of Paris for the 2013-2014 academic year. This includes the schools' capacities, the students' ROLs of schools, and their priority indices. Moreover, it contains information on students' socio-demographic characteristics (age, gender, parents' SES, low-income status, etc.), and their home addresses, allowing us to compute distances to each school in the district.
- (ii) **Enrollment Data:** The second data set is a comprehensive register of students enrolled in Paris' lower and upper secondary schools during the 2012–2013 and 2013–2014 academic years (*Base Elèves Académique*), which is also from the Paris Educational Authority. This data set allows to track students' enrollment status in all Parisian public and private middle and high schools.
- (iii) **DNB Exam Data:** The third data set contains all Parisian middle school students' individual examination results for a national diploma, the *Diplôme national du brevet* (or DNB), which students take at the end of middle school. We obtain this data set from the statistical office of the French Ministry of Education (*Direction de l'Évaluation, de la Prospective et de la Performance du Ministère de l'Éducation Nationale*).

B.2 Definition of Variables

Priority Indices. Students' priority indices for each school are recorded as the sum of three main factors: (i) students receive a “district” bonus of 600 points on each of the schools in their list which are located in their home district; (ii) students' academic performance during the last year of middle school is graded on a scale of 400 to 600

points; (iii) low-income students are awarded an additional bonus of 300 points. We convert these priority indices into percentile ranks between 0 and 1.

Student Scores. Based on the DNB exam data set, we compute several measures of student academic performance, which are normalized as percentile ranks between 0 and 1 among all Parisian students who took the exam in the same year. Both French and math scores are used, and we also construct the students' composite score, which is the average of the French and math scores. Note that students' DNB scores are different from the academic performance measure used to calculate student priority indices as an input into the DA mechanism. Recall that the latter is based on the grades obtained by students throughout their final year of middle school.

Socio-Economic Status. Students' socio-economic status is based on their parents' occupation. We use the French Ministry of Education's official classification of occupations to define "high SES": if the occupation of the student's legal guardian (usually one of the parents) belongs to the "very high SES" category (company managers, executives, liberal professions, engineers, academic and art professions), the student is coded as high SES, otherwise she is coded as low SES.³⁴

B.3 Construction of the Main Data Set for Analyses

For our empirical analysis, we use data from the Southern District of Paris (*District Sud*). We focus on public middle school students who are allowed to continue their studies in the academic track of upper secondary education and whose official residence is in the Southern District. We exclude those with disabilities, those who are repeating the first year of high school, and those who were admitted into specific selective tracks that are offered by certain public high schools in Paris (e.g., music majors, bilingual courses, etc.), as these students are given absolute priority in the assignment over other students. This leads to the exclusion of 350 applicants, or 18 percent of the total, the majority of whom are grade repeaters. Our data thus include 1,590 students from 57 different public middle schools, with 96 percent of applicants coming from one of the district's 24 middle schools.

³⁴There are four official categories: low SES, medium SES, high SES, and very high SES.

Appendix C Monte Carlo Simulations

This appendix provides details on the Monte Carlo simulations that we perform to assess our empirical approaches and model selection tests. Section C.1 specifies the model, Section C.2 describes the data generating processes, Section C.3 reports a number of summary statistics for the simulated data, Section C.4 presents the estimation and testing procedures, and, finally, Section C.5 discusses the main results.

C.1 Model Specification

Market Size. For the Monte Carlo results presented in this appendix, we consider a market where $I = 500$ students compete for admission to $S = 6$ schools. The vector of school capacities is specified as follows:

$$\{q_s\}_{s=1}^6 = \{50, 50, 25, 50, 150, 150\}.$$

Setting the total capacity of schools (475 seats) to be strictly smaller than the number of students (500) simplifies the analysis by ensuring that each school has a strictly positive cutoff in equilibrium.

Spatial Configuration. The school district is stylized as a disc of radius 1 (Figure C1). The schools (represented by red circles) are evenly located on a circle of radius 1/2 around the district centroid; the students (represented by blue circles) are uniformly distributed across the district area. The cartesian distance between student i and school s is denoted by $d_{i,s}$.

Student Preferences. To represent students' preferences over schools, we adopt a parsimonious version of the random utility model described in Section 2.3. Student i 's utility from attending school s is specified as follows:

$$\begin{aligned} u_{i,s} &= 10 + \theta_s - d_{i,s} + \beta(a_i \times \bar{a}_s) + \epsilon_{i,s}, \forall i; s = 2, \dots, 6; \\ u_{i,1} &= 10 + \epsilon_{i,s}; \end{aligned} \tag{6}$$

where $10 + \theta_s$ is school s 's fixed effects; $d_{i,s}$ is the walking distance from student i 's residence to school s ; a_i is student i ' ability; \bar{a}_s is school s 's quality, which is measured as

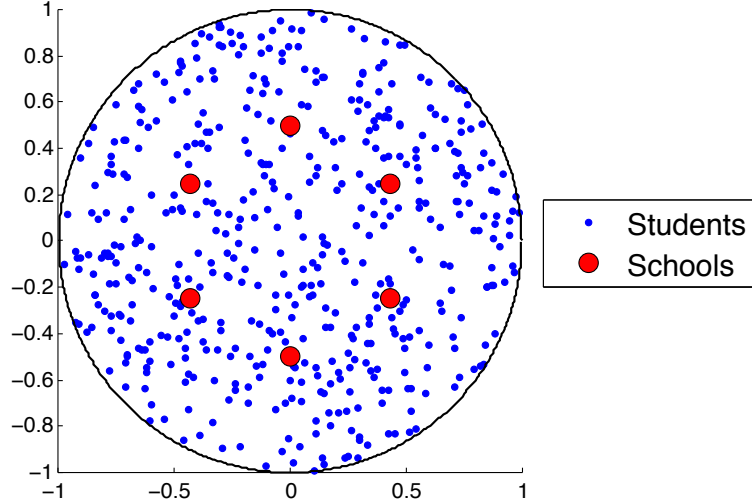


Figure C1: Monte Carlo Simulations: Spatial Distribution of Students and Schools

Notes: This figure shows the spatial configuration of the school district considered in the Monte Carlo simulations, for the case with 500 students and 6 schools. The school district is represented as a disc of radius 1. The blue and red circles show the location of students and of schools, respectively.

the average ability of currently enrolled students; and $\epsilon_{i,s}$ is an error term that is drawn from a type-I extreme value distribution. Setting the effect of distance to -1 ensures that other coefficients can be interpreted in terms of willingness to travel.

The school fixed effects beyond the common valuation are specified as follows:

$$\{\theta_s\}_{s=1}^6 = \{0, 0.5, 1.0, 1.5, 2.0, 2.5\}$$

Adding the common value of 10 for every school ensures that all schools are acceptable in the simulated samples, i.e., are preferable to the outside option whose utility $u_{i,0}$ is normalized to zero for every student.

Students' abilities $\{a_i\}_{i=1}^I$ are represented as evenly spaced values on the interval $[0, 1]$, where the distance between two consecutive values is set to $1/(I + 1)$. Values of $\{a_i\}$ can therefore be interpreted as percentile ranks. Each of these values is randomly assigned to a student. School qualities $\{\bar{a}_s\}_{s=1}^S$ are exogenous to students' idiosyncratic preferences $\epsilon_{i,s}$. The procedure followed to ascribe values to the schools' qualities is discussed at the end of this section.

The positive coefficient on the interaction term $a_i \times \bar{a}_s$ reflects the assumption that high-ability students value school quality more than low-ability students. In the simulations, we set $\beta = 3.0$.

Priority Indices. Students are ranked separately by each school based on a school-specific index $e_{i,s}$. The vector of student priority indices at a given school s , $\{e_{i,s}\}_{i=1}^I$, is constructed as a permutation of the vector of students' abilities $\{a_i\}_{i=1}^I$, such that: (i) student i 's index at each school is correlated with her ability a_i with a correlation coefficient of ρ ; (ii) i 's indices at any two schools s_1 and s_2 are also correlated with correlation coefficient ρ . When ρ is set equal to 1, a student has the same priority at all schools. When ρ is set equal to zero, her priority indices at the different schools are uncorrelated. For the simulations presented in this appendix, we choose $\rho = 0.9$.

In the benchmark simulations, student priority indices are common knowledge, which implies that students know how they are ranked by each school. This assumption can be relaxed by allowing some uncertainty in students' knowledge of their relative priorities, without changing the main findings (results are available upon request).

School Quality. To ensure that school qualities $\{\bar{a}_s\}_{s=1}^S$ are exogenous to students' idiosyncratic preferences, while being close to those observed in the Bayesian Nash equilibrium of the school choice game, we adopt the following procedure: we consider the unconstrained student-proposing DA where students rank all schools truthfully; students' preferences are constructed using random draws of errors and a common prior belief about the average quality of each school; students rank schools truthfully and are assigned through the DA mechanism; each school's quality is computed as the average ability of students assigned to that school, i.e., $\{\bar{a}_s\}_{s=1}^S = \left\{ (1/q_s) \sum_{i \in \mu(s)} a_i \right\}_{s=1}^S$; a fixed-point vector of school qualities, denoted by $\{\bar{a}_s^*\}_{s=1}^S$, is found; the value of each school's quality is set equal to mean value of \bar{a}_s^* across the samples.

The resulting vector of school qualities is:

$$\{\bar{a}_s\}_{s=1}^6 = \{0.22, 0.35, 0.78, 0.72, 0.43, 0.67\}$$

C.2 Data Generating Processes

The simulated data are constructed under two distinct data generating processes (DGPs).

DGP 1: Constrained/Truncated DA. This DGP considers a situation where the student-proposing DA is used to assign students to schools but where the number of schools that students are allowed to rank, K , is strictly smaller than the total number of

available schools, S . For expositional simplicity, students are assumed to incur no cost when ranking exactly K schools. Hence:

$$C(|L_i|) \begin{cases} = 0 & \text{if } |L_i| \leq K \\ = +\infty & \text{if } |L_i| > K \end{cases}$$

In the simulations, we set $K = 4$ (students are allowed to rank up to 4 schools out of 6).

DGP 2: Unconstrained DA with Cost. This DGP considers the case where students are not formally constrained in the number of schools they can rank but nevertheless incur a constant marginal cost, denoted by $c(> 0)$, each time they increase the length of their ROL by one, if this list contains more than one school. Hence:

$$C(|L_i|) = c \cdot (|L_i| - 1),$$

where the marginal cost c is strictly positive. In the simulations, we set $c = 1e-6$.

For each DGP, we adopt a two-step procedure to solve for a Bayesian Nash equilibrium of the school choice game.

Step 1: Distribution of Cutoffs Under Unconstrained DA. Students' prior beliefs about the distribution of school admission cutoffs are based on the distribution of cutoffs that arises when students submit unrestricted truthful rankings of schools under the standard DA. Specifically:

- (i) M samples, each of which is of size I , are generated using different draws of students' geographic coordinates, school-specific priority indices $e_{i,s}$, and idiosyncratic preferences $\epsilon_{i,s}$ over the S schools.
- (ii) Each student i in each sample m submits a complete and truthful ranking $R_{i,m}$ of the schools.
- (iii) After collecting $\{R_{i,m}\}_{i=1}^I$, the DA mechanism assigns, in each sample, students to schools based on their priority indices.
- (iv) Each realized matching μ_m in sample m determines a vector of school admission cutoffs $\mathbf{P}_m \equiv \{p_{s,m}\}_{s=1}^S$.
- (v) The realized matchings μ_m are used to derive the empirical distribution of school admission cutoffs under the unconstrained DA, which is denoted by $\hat{\mathbf{P}} = \{\mathbf{P}_m\}_{m=1}^M$.

In the simulations, we set $M = 500$.

Step 2: Bayesian Nash Equilibrium. For each DGP, the M Monte Carlo samples generated in Step 1 are used to solve the Bayesian Nash equilibrium of the school choice game. Specifically:

- (i) Each student i in each sample m determines all possible true partial preference orders $\{L_{i,m,n}\}_{n=1}^N$ over the schools, i.e., all potential ROLs of length between 1 and K that respect i 's true preference ordering $R_{i,m}$ of schools among those ranked in $L_{i,m,n}$; for each student, there are $N = \sum_{k=1}^K S!/k!(S-k)!$ such partial orders. Under the constrained/truncated DA, students consider only true partial preference orderings of length K ($< S$), i.e., 15 candidate ROLs when they rank exactly 4 schools out of 6;³⁵ under the unconstrained DA with cost, students consider all true partial orders of length up to S , i.e., 63 candidate ROLs when they can rank up to 6 schools.
- (ii) For each candidate ROL $\{L_{i,m,n}\}_{n=1}^N$, students estimate the (unconditional) probabilities of being admitted to each school by comparing their indices $e_{i,s}$ to the expected distribution of cutoffs. Initial beliefs on the cutoff distribution are based on $\hat{\mathbf{P}}$, i.e., the empirical distribution of cutoffs under unconstrained DA with truth-telling students.
- (iii) Each student selects the ROL $L_{i,m}^*$ that maximizes her expected utility, where the utilities of each school are weighted by the students' expected probabilities of admission.
- (iv) After collecting $\{L_{i,m}^*\}_{i=1}^I$, the DA mechanism assigns students to schools.
- (v) The realized matchings μ'_m jointly determine the posterior empirical distribution of school admission cutoffs;
- (vi) Students update their beliefs and steps (i) to (v) are repeated until a fixed point is found, which occurs when the posterior distribution of cutoffs is consistent with students' prior common beliefs.
- (vii) Given the equilibrium beliefs, each student in each sample submits the ROL that maximizes her expected utility. Students are matched to schools through the DA mechanism.

³⁵This is without loss of generality, because in equilibrium the admission probability is non-degenerate and it is, therefore, in students' best interest to rank 4 schools.

The simulated school choice data are constructed by collecting students' priority indices, their submitted ROLs, the student-school matching outcome, and the realized cutoffs from each MC sample.

C.3 Summary Statistics of Simulated Data

500 Monte Carlo samples of school choice data are simulated for each DGP.

Equilibrium Distribution of Cutoffs. The empirical equilibrium distribution of school admission cutoffs is displayed in Figure C2 separately for each DGP. In line with the theoretical predictions, the empirical marginal distribution of cutoffs is approximately multivariate normal. Because both DGPs involve the same profiles of preferences and produce almost identical matchings, the empirical distribution of cutoffs under the constrained/truncated DA (left panel) is very similar to that observed under the unconstrained DA with cost (right panel).

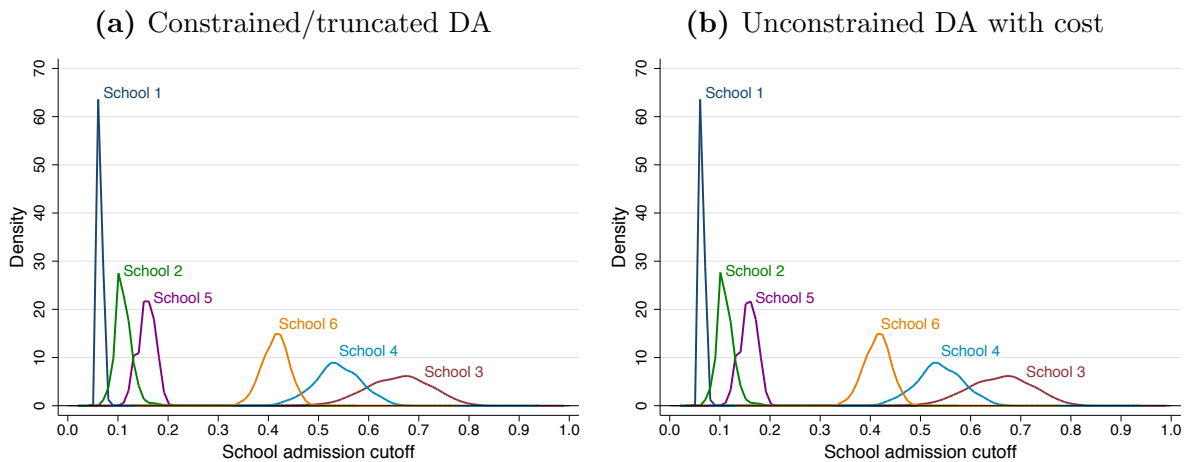


Figure C2: Monte Carlo Simulations: Empirical Distribution of School Cutoffs (6 schools, 500 students)

Notes: This figure shows the equilibrium marginal distribution of school admission cutoffs under the constrained/truncated DA (left panel) and the DA with cost (right panel) in a setting where 500 students compete for admission into 6 schools, using 500 simulated samples. The line fits are from a Gaussian kernel with optimal bandwidth using MATLAB's `ksdensity` command.

School cutoffs are not strictly aligned with the school fixed effects, since cutoffs are also influenced by the size of schools. In the simulations, small schools (e.g., School 3) tend to have higher cutoffs than larger schools (e.g., Schools 5 and 6) because, in spite

of being less popular, they can be matched only with a small number of students, which pushes their cutoffs upward.³⁶

Figure C3 reports the marginal distribution of cutoffs in the constrained/truncated DA for various market sizes. The simulations show that as the number of seats and the number of students increase while holding the number of schools constant, the distribution of school cutoffs degenerates and becomes closer to a normal distribution.

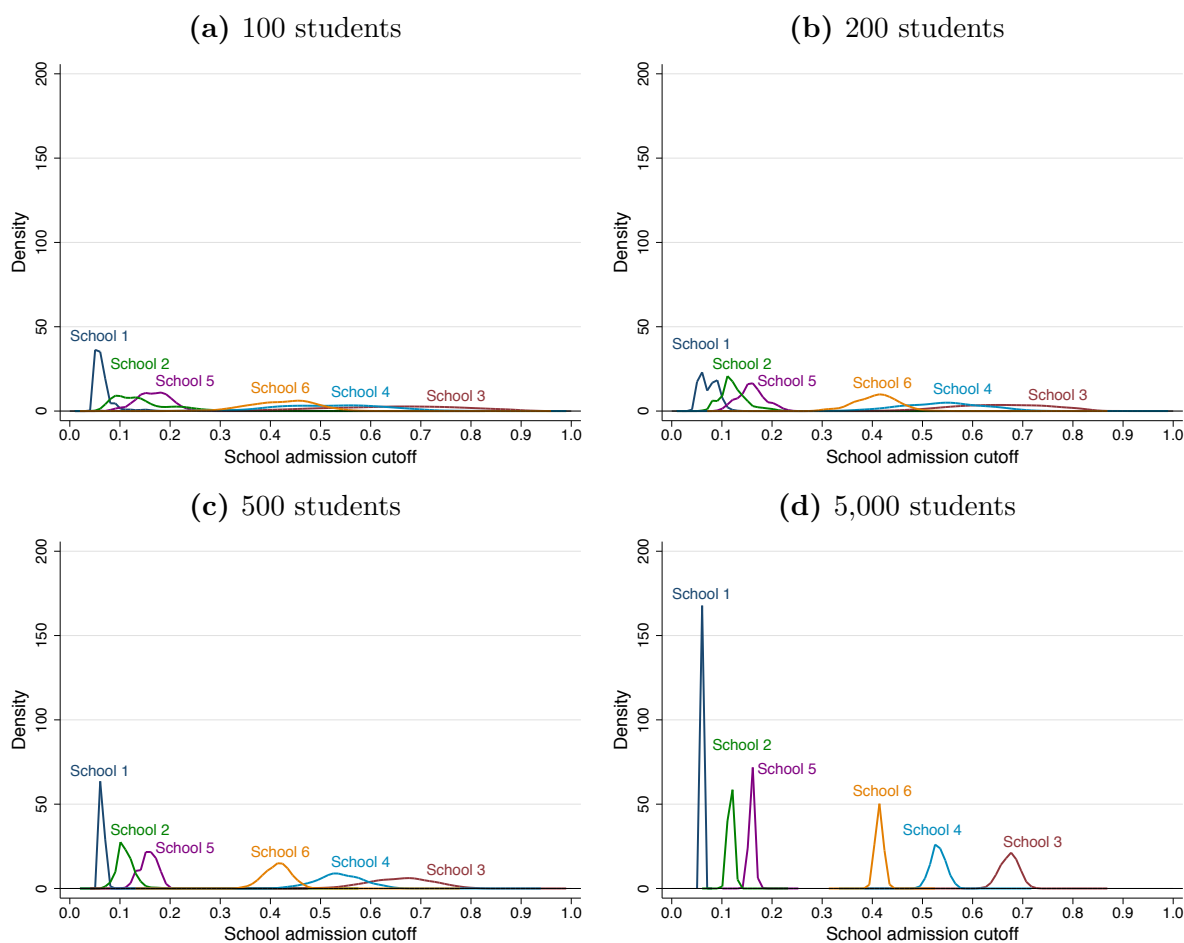


Figure C3: Monte Carlo Simulations: Impact of Market Size on the Distribution of Cutoffs (Unconstrained/Truncated DA)

Notes: This figure shows the equilibrium marginal distribution of school admission cutoffs under the constrained/truncated DA (ranking 4 out of 6 schools) when varying the number of students, I , who compete for admission into 6 schools with a total enrollment capacity of $I \times 0.95$ seats, using 500 simulated samples. The line fits are from a Gaussian kernel with optimal bandwidth using MATLAB's `ksdensity` command.

³⁶Note that this phenomenon is also observed if one sets $\beta = 0$, i.e., when students' preferences over schools do not depend on the interaction term $a_i \times \bar{a}_s$.

Summary Statistics. Table C1 shows some descriptive statistics of the simulated data from both DGPs. The reported means are averaged over the 500 Monte Carlo samples.

All students under the constrained/truncated DA submit lists of the maximum allowed length (4 schools). Under the unconstrained DA with cost, students are allowed to rank as many schools as they wish but, due to the cost of submitting longer lists, they rank 4.5 schools on average.

Under both DGPs, all school seats are assigned, and, therefore, 95 percent of students are assigned to a school. Weak truth-telling is strongly violated under the constrained/truncated DA, since less than half of submitted ROLs rank truthfully students' top K_i choices. Although less widespread, violations of truth-telling are still observed under the unconstrained DA with cost, since about 20 percent of students do not submit their top K_i choices. By contrast, almost every student is assigned to her favorite feasible school under both DGPs.

Table C1: Monte Carlo Simulations: Summary Statistics

	Data generating process	
	Constrained/truncated DA (1)	Unconstrained DA with cost (2)
<i>Panel A. Outcomes</i>		
Average size of ROL	4.00 (0.000)	4.49 (0.025)
Assigned to a school	0.950 (0.000)	0.950 (0.000)
Weakly truth-telling	0.438 (0.014)	0.804 (0.015)
Assigned to favorite feasible school	1.000 (0.000)	1.000 (0.000)
<i>Panel B. Parameters</i>		
Number of students	500	500
Number of schools	6	6
Number of simulated samples	500	500
Maximum authorized length of ROL	4	6
Cost c of ranking an extra school	0	1e-6

Notes: This table presents summary statistics of simulated data under two DGPs: (i) Constrained/Truncated DA (column 1): students are only allowed to rank 4 schools out of 6; and (ii) Unconstrained DA with cost (column 2): students can rank as many schools as they wish, but incur a constant marginal cost of $c = 1e-6$ per extra school included in their ROL. Standard deviations across the 500 simulation samples are in parentheses.

C.4 Estimation and Testing

Identifying Assumptions. With the simulated data at hand, the school choice model described by equation (6) is estimated under different identifying assumptions:

- (i) *Truth-Telling.* The choice probabilities for individual ROLs can be fully specified and the corresponding rank-ordered logit model is estimated by Maximum Likelihood Estimation (MLE), as discussed in Section 2.2.
- (ii) *Stability.* Under the assumption that students are assigned to their favorite feasible school given the *ex post* cutoffs, the model is estimated by MLE based on a conditional logit model where each student’s choice set is restricted to the *ex post* feasible schools and where the assigned school is the chosen alternative.³⁷ Two sets of estimates are reported, depending on whether we endogeneize the cutoffs by incorporating the probability of observing the realized cutoffs in the likelihood function (see Section 2.3). The procedure that we adopt to approximate the probability of observing the realized cutoffs conditional on $\boldsymbol{\theta}$ is described in the next paragraph.
- (iii) *Stability and undominated strategies.* The method of moment (in)equalities in Andrews and Shi (2013) is used to obtain point estimates, where conditional moment inequalities are derived from students’ observed orderings of all 15 possible pairs of schools (see Section 2.5). The variables that are used to interact with these conditional inequalities and thus to obtain the unconditional ones are student ability (a_i), distance to School 1 ($d_{i,1}$) and distance to School 2 ($d_{i,2}$), which brings the total number of moment inequalities to 120. The approach proposed by Bugni et al. (2014) is used to construct the marginal confidence intervals for the point estimates.

Procedure to Endogeneize School Cutoffs. We use the asymptotic distribution derived in Proposition 5, while ignoring the pairwise correlation among cutoffs.³⁸ Simulation methods are used to approximate the following terms in the likelihood function,

$$\ln \left(\Pr(\text{cutoff is } P(\mu) \mid \mathbf{Z}; \boldsymbol{\theta}) \right) \approx \ln \phi \left(P(\mu) - P^\infty(\mathbf{Z}, \boldsymbol{\theta}), V(\mathbf{Z}, \boldsymbol{\theta})/I \right),$$

³⁷The stability estimates can be equivalently obtained using a GMM estimation with moment equalities defined by the first-order conditions of the log-likelihood function.

³⁸These pairwise correlations are not zero (Proposition 5), but they are difficult to calculate precisely given the current numerical computation techniques.

where ϕ is the S -dimensional normal density function; $P^\infty(\mathbf{Z}, \boldsymbol{\theta})$ and $V(\mathbf{Z}, \boldsymbol{\theta})$ need to be simulated; all the off-diagonal terms of $V(\mathbf{Z}, \boldsymbol{\theta})$ are set to be zero.

$P^\infty(\mathbf{Z}, \boldsymbol{\theta})$ is the cutoff vector of the unique stable matching in the continuum economy given $(\mathbf{Z}, \boldsymbol{\theta})$. We adopt the following procedure to simulate this term:

- (i) We inflate the sample of all students, assigned and unassigned, by 20 times (i.e., to 10,000 students) to approximate the continuum economy. In this inflated sample, distances, student abilities and priority indices at each school are generated as described in Section C.1.³⁹
- (ii) For each student in the “continuum” economy, we then independently draw a vector of $\boldsymbol{\epsilon}$ from the type-I extreme-value distribution. With $(\mathbf{Z}, \boldsymbol{\theta})$ as well as $\boldsymbol{\epsilon}$, we calculate the preferences of every student.
- (iii) We find the student-optimal stable matching by running the student-proposing DA with students’ true ordinal preferences. The cutoffs are the simulated $P^\infty(\mathbf{Z}, \boldsymbol{\theta})$.

To further simulate $V(\mathbf{Z}, \boldsymbol{\theta})$, we need the partial derivative of demand with respect to each cutoff ($P^\infty(\mathbf{Z}, \boldsymbol{\theta})$), which is calculated by numerical differentiation (two-sided approximation). The expression in Proposition 5 is then used to calculate $V(\mathbf{Z}, \boldsymbol{\theta})$.

Model Selection Tests. Two tests are implemented:

1. *Truth-Telling vs. Stability.* This test is carried out by constructing a Hausman-type test statistic from the estimates of the truth-telling and stability approaches (with or without the cutoff terms).
2. *Stability vs. Undominated Strategies.* As shown in Section 2.5, stability implies a set of moment equalities, while undominated strategies lead to another set of moment inequalities. When undominated strategies are assumed to be satisfied, testing stability amounts to check if the identified set of the moment (in)equality model is empty. Specifically, we apply the test proposed by Bugni et al. (2015) (Test RS).

³⁹When applied to real data, the inflated sample is 10 times the size of the original sample and is randomly drawn with replacement from the original sample, with random tie-breaking of student priority indices.

C.5 Results

The results from 500 Monte Carlo samples confirm the theoretical predictions for both the constrained/truncated DA (Table C2) and the unconstrained DA with cost (Table C3). The results in Panel A ignore the endogeneity of cutoffs, whereas those in Panel B endogenize the cutoffs.

The coefficients reported in column 2 show that violation of the truth-telling assumption leads to severely biased estimates under the traditional approach (the $\hat{\theta}_{TT}$ estimator). Under both DGPs, students' valuation of popular schools tends to be underestimated. This problem is particularly acute when one considers the smaller schools (Schools 2 and 3), which often have higher admission cutoffs than the larger ones (see Figure C2), and are therefore more often left out of students' ROLs due to low chances of admission. The truth-telling estimates are more biased under the constrained/truncated DA but the biases are nevertheless substantial under the unconstrained DA with cost, considering the fact that only 20 percent of students do not submit truthful rankings under this DGP.

By contrast, estimating the school choice model under the assumption that the matching outcome is stable performs well in our simulations. The point estimates (column 5) are reasonably close to the true parameter values, although they are more dispersed than the truth-telling estimates (column 6 vs. column 3). This efficiency loss is a direct consequence of restricting the choice sets to include only feasible schools and of considering a single choice situation for each matched student.

Under the assumption that the matching outcome is stable, the Hausman-type specification test strongly rejects truth-telling in the constrained/truncated DA simulations (last row of Panel A) and rejects this assumption in 30 percent of the samples simulated under the unconstrained DA with cost (last row of Panel A), under which truth-telling is violated for only 20 percent of students.⁴⁰

The results from the moment (in)equalities approach show that in the two specific cases under study, the over-identifying information provided by students' true partial orders of schools has only a small impact on estimates (column 8 vs. column 5).⁴¹ On

⁴⁰To show the power of the two tests, especially the test for stability, we construct some simulation examples in which stability is rejected when 30 percent of students are not assigned to their favorite feasible schools. These results are available upon request.

⁴¹Larger improvements are obtained when we relax the constraint on the number of choices than students can submit or when we reduce the marginal cost of ranking an extra school (results available upon request).

average, estimates based on the method of moment (in)equalities are closer to the true values of the parameters (column 9 vs. column 6) but, unfortunately, the marginal confidence intervals obtained using the Bugni et al. (2014) approach tend to be conservative, especially relative to the stability estimates from MLE (column 10 vs. column 7).

The estimates show little sensitivity to treating the cutoffs as endogenous (Panel B vs. Panel A) when inequalities are considered, which suggests that the cutoffs can be treated as exogenous when the market is reasonably large. In the stability approach, endogenizing cutoffs does, however, contribute to reduce the dispersion of point estimates (Panel B vs. Panel A, column 6).

Table C2: Monte Carlo Results: Ranking up to 4 Schools under DA (500 Students, 6 Schools, 500 Samples)

	Identifying assumptions									
	True value (1)	Truth-telling			Stability			Stability and undominated strategies		
		Mean (2)	s.d. (3)	Coverage (4)	Mean (5)	s.d. (6)	Coverage (7)	Mean (8)	s.d. (9)	Coverage (10)
<i>Panel A. Cutoffs treated as exogenous</i>										
PARAMETERS										
School 2	0.50	-0.05	0.07	0.00	0.51	0.29	0.93	0.50	0.28	1.00
School 3	1.00	-2.79	0.14	0.00	1.05	0.79	0.95	0.97	0.77	1.00
School 4	1.50	-1.76	0.11	0.00	1.55	0.67	0.94	1.48	0.65	1.00
School 5	2.00	0.62	0.07	0.00	2.03	0.31	0.94	2.00	0.30	1.00
School 6	2.50	-0.07	0.10	0.00	2.54	0.57	0.94	2.48	0.55	1.00
Own ability \times school quality	3.00	10.19	0.44	0.00	2.97	2.13	0.95	3.19	2.17	1.00
Distance	-1.00	-0.77	0.08	0.22	-1.00	0.19	0.95	-1.00	0.19	1.00
MODEL SELECTION TESTS										
Truth-Telling (H_0) vs. Stability (H_a):	H_0 rejected in 100% of samples (at 0.05 significance level).									
Stability (H_0) vs. Undominated strategies (H_a):	H_0 rejected in 0% of samples (at 0.05 significance level).									
<i>Panel B. Cutoffs treated as endogenous</i>										
PARAMETERS										
School 2	0.50				0.47	0.25	0.96	0.50	0.28	1.00
School 3	1.00				1.04	0.70	0.96	0.97	0.77	1.00
School 4	1.50				1.54	0.60	0.96	1.49	0.65	1.00
School 5	2.00				2.00	0.27	0.96	2.00	0.30	1.00
School 6	2.50				2.53	0.51	0.96	2.48	0.55	1.00
Own ability \times school quality	3.00				2.88	1.94	0.96	3.17	2.17	1.00
Distance	-1.00				-1.02	0.20	0.94	-1.00	0.19	1.00
MODEL SELECTION TESTS										
Truth-telling (H_0) vs. Stability (H_a):	H_0 rejected in 100% of samples (at 0.05 significance level).									
Stability (H_0) vs. Undominated strategies (H_a):	H_0 rejected in 0% of samples (at 0.05 significance level).									

Notes: This table reports Monte Carlo results from estimating students' preferences under different assumptions: (i) Truth-telling; (ii) Stability; (iii) Stability and undominated strategies. 500 Monte Carlo samples of school choice data are simulated under the following data generating process for a market in which 500 students compete for admission into 6 schools: a constrained/truncated DA where students are allowed to rank up to 4 schools out of 6. Under assumptions (i) and (ii), the school choice model is fitted using maximum likelihood estimation. Under assumption (iii), the model is estimated using Andrews and Shi (2013)'s method of moment (in)equalities. Column 1 reports the true values of the parameters. The mean and standard deviation of point estimates across the Monte Carlo samples are reported in columns 2, 5 and 8, and in columns 3, 6 and 9, respectively. Columns 4, 7 and 10 report the Monte Carlo coverage probabilities for the 95 percent confidence intervals. The confidence intervals in models (i) and (ii) are the Wald-type confidence intervals obtained from the inverse of the Hessian matrix. The marginal confidence intervals in model (iii) are computed using the method proposed by Bugni et al. (2014). Truth-telling is tested against stability by constructing a Hausman-type test statistic from the estimates of both approaches. Stability is tested against undominated strategies by checking if the identified set of the moment(in)equality model is empty, using the test proposed by Bugni et al. (2015). Panel A ignores the endogeneity of cutoffs, while Panel B endogenizes them.

Table C3: Monte Carlo Results: Ranking Schools under DA with Cost (500 Students, 6 Schools, 500 Samples)

	Identifying assumptions									
	True value (1)	Truth-telling			Stability			Stability and undominated strategies		
		Mean (2)	s.d. (3)	Coverage (4)	Mean (5)	s.d. (6)	Coverage (7)	Mean (8)	s.d. (9)	Coverage (10)
<i>Panel A. Cutoffs treated as exogenous</i>										
PARAMETERS										
School 2	0.50	0.39	0.09	0.80	0.51	0.29	0.93	0.51	0.28	1.00
School 3	1.00	0.43	0.16	0.06	1.05	0.79	0.95	1.02	0.63	1.00
School 4	1.50	1.06	0.15	0.14	1.55	0.67	0.94	1.53	0.54	1.00
School 5	2.00	1.67	0.10	0.14	2.03	0.31	0.94	2.02	0.28	1.00
School 6	2.50	2.13	0.14	0.23	2.54	0.57	0.94	2.51	0.47	1.00
Own ability \times school quality	3.00	2.87	0.50	0.94	2.97	2.13	0.95	3.03	1.85	1.00
Distance	-1.00	-0.95	0.09	0.91	-1.00	0.20	0.95	-1.00	0.19	1.00
MODEL SELECTION TESTS										
Truth-telling (H_0) vs. Stability (H_a):	H_0 rejected in 30% of samples (at 0.05 significance level).									
Stability (H_0) vs. Undominated strategies (H_a):	H_0 rejected in 0% of samples (at 0.05 significance level).									
<i>Panel B. Cutoffs treated as endogenous</i>										
PARAMETERS										
School 2	0.50				0.47	0.25	0.96	0.51	0.28	1.00
School 3	1.00				1.04	0.70	0.96	1.03	0.63	1.00
School 4	1.50				1.54	0.59	0.96	1.53	0.54	1.00
School 5	2.00				2.00	0.27	0.97	2.02	0.28	1.00
School 6	2.50				2.53	0.50	0.96	2.52	0.47	1.00
Own ability \times school quality	3.00				2.88	1.94	0.96	3.02	1.84	1.00
Distance	-1.00				-1.02	0.20	0.94	-1.00	0.19	1.00
MODEL SELECTION TESTS										
Truth-telling (H_0) vs. Stability (H_a):	H_0 rejected in 27% of samples (at 0.05 significance level).									
Stability (H_0) vs. Undominated strategies (H_a):	H_0 rejected in 0% of samples (at 0.05 significance level).									

Notes: This table reports Monte Carlo results from estimating students' preferences under different assumptions: (i) Truth-telling; (ii) Stability; (iii) Stability and undominated strategies. 500 Monte Carlo samples of school choice data are simulated under the following data generating process for a market in which 500 students compete for admission into 6 schools: an unconstrained DA where students can rank as many schools as they wish, but incur a constant marginal cost $c=1e-6$ for including an extra school in their ROL. Under assumption (iii), the model is estimated using Andrews and Shi (2013)'s method of moment (in)equalities. Column 1 reports the true values of the parameters. The mean and standard deviation of point estimates across the Monte Carlo samples are reported in columns 2, 5 and 8, and in columns 3, 6 and 9, respectively. Columns 4, 7 and 10 report the Monte Carlo coverage probabilities for the 95 percent confidence intervals. The confidence intervals in models (i) and (ii) are the Wald-type confidence intervals obtained from the inverse of the Hessian matrix. The marginal confidence intervals in model (iii) are computed using the method proposed by Bugni et al. (2014). Truth-telling is tested against stability by constructing a Hausman-type test statistic from the estimates of both approaches. Stability is tested against undominated strategies by checking if the identified set of the moment(in)equality model is empty, using the test proposed by Bugni et al. (2015). Panel A ignores the endogeneity of cutoffs, while Panel B endogenizes them.

Appendix D Goodness of Fit

This appendix reports in-sample goodness of fit statistics for the estimates of model (5), which are obtained under different identifying assumptions and reported in Table 3. To measure goodness of fit, we keep fixed the estimated coefficients and $\mathbf{X}_{i,s}$, and draw utility shocks as type-I extreme values. This leads to the simulated utilities for every student in the simulation samples. When studying the truth-telling estimates, we let students submit their top 8 schools according to their simulated preferences; the matching outcome is obtained by running DA. For the other sets of estimates, because stability is assumed, we focus on the stable matching in each sample, which is calculated using students' priority indices and simulated ordinal preferences. Table D4 reports the average over the results from the 300 simulated samples, with standard deviations in parentheses.

Panel A presents the predicted cutoffs based on estimates from the three assumptions. The results make clear that estimates based on stability (with or without assuming undominated strategies) predict cutoffs very close to the observed ones. Truth-telling estimates substantially under-predict the cutoffs of School 8, the “small” school, and of School 11, the top school, which is a direct consequence of the under-estimation of the qualities of these schools.

Panel B compares the predicted levels of student sorting across schools, by ability and by SES, to the observed levels in the data. Sorting by ability is defined as the ratio of the between-school variance of the student composite score over its total variance. Sorting by SES is similarly defined, with the difference that SES status is binary (high vs. low). The stability assumption leads to predicted levels of sorting by ability that are much closer to the observed value (0.427) than those derived from truth-telling. The stability estimates also predict more accurate levels of sorting by SES.

Panel C compares students' predicted assignment with the observed one. Our preferred estimates have between 32 and 38 percent successful prediction rates, whereas truth-telling estimates accurately predict only 22 percent of assignments. We also consider an alternative measure of prediction accuracy: if the observed assignment is predicted to have the highest assignment probability, we define it as a correct prediction. According to this definition, the last line in the table shows that stability estimates correctly predict half of the students' observed assignments, whereas truth-telling estimates have a lower accurate prediction rate of 37 percent.

Table D4: Goodness-of-Fit of Measures Based on Different Identifying Assumptions

	Observed sample	Simulated samples with estimates from		
		Truth-telling	Stability	Stability and undominated strategies
	(1)	(2)	(3)	(4)
<i>Panel A. School admission cutoffs</i>				
School 1	0.000	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
School 2	0.015	0.004 (0.006)	0.024 (0.012)	0.019 (0.013)
School 3	0.000	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
School 4	0.001	0.043 (0.015)	0.017 (0.007)	0.017 (0.008)
School 5	0.042	0.064 (0.016)	0.052 (0.012)	0.040 (0.013)
School 6	0.069	0.083 (0.025)	0.084 (0.022)	0.062 (0.024)
School 7	0.373	0.254 (0.020)	0.373 (0.010)	0.320 (0.012)
School 8	0.239	0.000 (0.001)	0.241 (0.023)	0.153 (0.047)
School 9	0.563	0.371 (0.033)	0.564 (0.017)	0.505 (0.023)
School 10	0.505	0.393 (0.029)	0.506 (0.011)	0.444 (0.014)
School 11	0.705	0.409 (0.040)	0.708 (0.009)	0.663 (0.013)
<i>Panel B. Student sorting across schools</i>				
By ability	0.427	0.304 (0.018)	0.409 (0.014)	0.462 (0.015)
By SES	0.087	0.056 (0.010)	0.067 (0.010)	0.111 (0.013)
<i>Panel C. Predicted assignment</i>				
Mean predicted fraction of students assigned to observed assignment		0.220 (0.011)	0.383 (0.010)	0.326 (0.012)
Observed assignment has highest predicted assignment probability		0.374 (0.484)	0.526 (0.499)	0.463 (0.499)

Notes: This table reports reports three sets of goodness-of-fit measures that compare the observed outcomes to those simulated under various identifying assumptions as in Table 3, for the high school assignment of students in the Southern District of Paris. Estimates under the stability assumption endogenize the cutoff. 300 samples are used in the simulations, and the standard deviations across the samples are reported in parentheses. In all simulations, we vary only the utility shocks, which are kept common across columns 2-4. Panel A compares the observed school admission cutoffs with the (average) simulated cutoffs. Panel B compares the observed and simulated sorting of students across schools by ability and by SES, where sorting is measured as the fraction of the total variance of ability (SES) that is explained by the between school variance. Panel C compares students' observed and predicted assignment using two distinct measures. The first is the average predicted fraction of students who are assigned to their observed assignment school; in other words, this is the average fraction of times each student is assigned to her observed assignment in the 300 simulated samples. The second measure is the average fraction of students for whom the observed assignment school has the highest assignment probability among all 11 schools, where the assignment probability is estimated from the 300 samples.

Appendix E Supplementary Figures and Tables

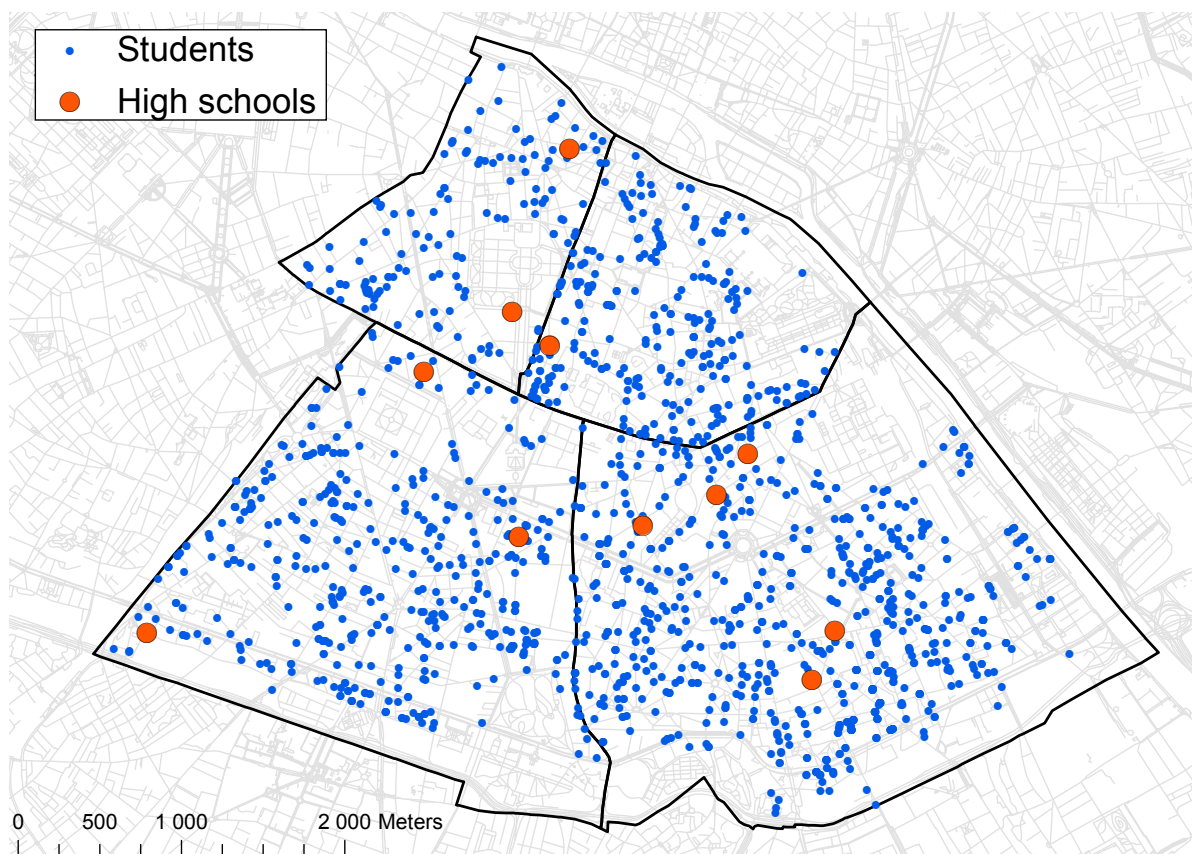


Figure E4: the Southern District of Paris for Public High School Admission

Notes: the Southern District of Paris covers four of the city's 20 *arrondissements* (administrative divisions): 5th, 6th, 13th and 14th. The red circles show the location of the district's 11 public high schools (*lycées*). The blue circles show the home addresses of the 1,590 students in the data.

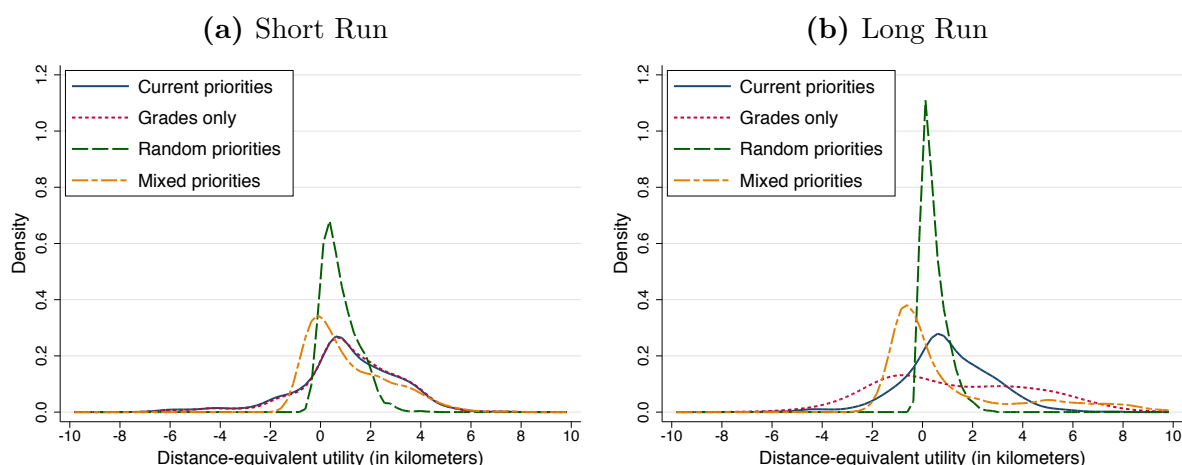


Figure E5: Distribution of Student Welfare under Alternative Admission Criteria: Estimates from Truth-Telling

Notes: The two graphs show the short-run (left panel) and long-run (right panel) distributions of student welfare in the Southern District of Paris, under each of the four school priority structures described in Section 4: (i) Current admission criterion (based on academic grades and low-income status); (ii) academic grades only; (iii) Random priorities; (iv) Mixed priorities (based on grades for the “top two” schools and random priorities for the other schools). The welfare of every student is measured in terms of willingness to travel (in kilometers) and is normalized by subtracting the distance-equivalent utility that she would experience under a purely random assignment in the long-run steady state. The welfare distributions are computed by averaging over 300 simulation samples with different vectors of random utility shocks for each student. Preferences over the 11 within-district schools are simulated using estimates based on the truth-telling assumption (Table 4, column 4). The line fits are from a Gaussian kernel with optimal bandwidth using MATLAB’s `ksdensity` command. See Section 4.1 for details on the simulations.

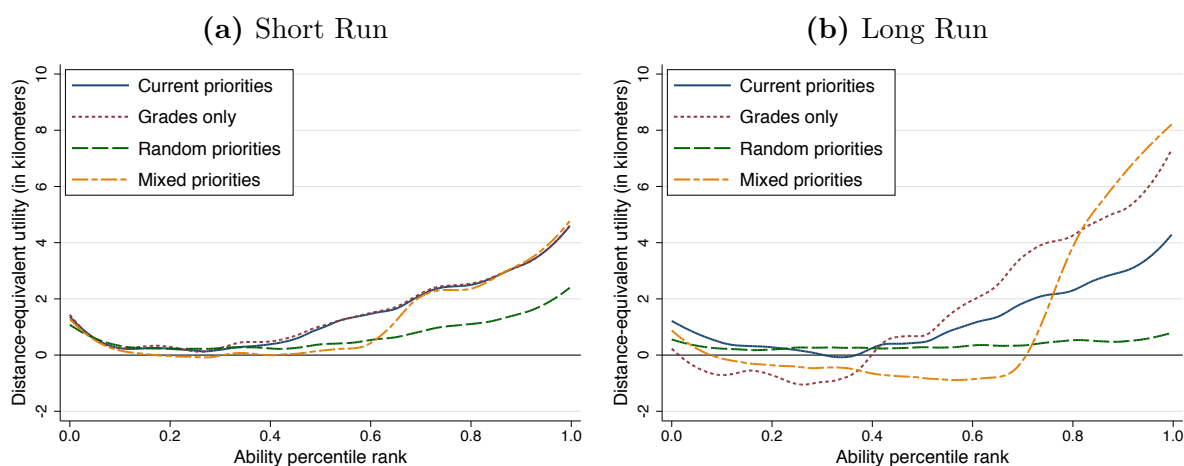


Figure E6: Student Welfare by Ability under Alternative Admission Criteria: Estimates from Truth-Telling

Notes: See notes in Figure E7. For each of the four school priority structures described in Section 4, the graphs show the short-run (left panel) and long-run (right panel) distributions of average student welfare in the Southern District of Paris, for students belonging to the different quartiles of academic ability. Student ability is proxied by the percentile rank on the DNB exam composite score (normalized to be between 0 and 1) among all applicants in the data. Preferences over the 11 within-district schools are simulated using estimates based on the truth-telling assumption (Table 4, column 4). The lines are fitted using MATLAB’s `smooth/roess` command with a span of 20 percent.

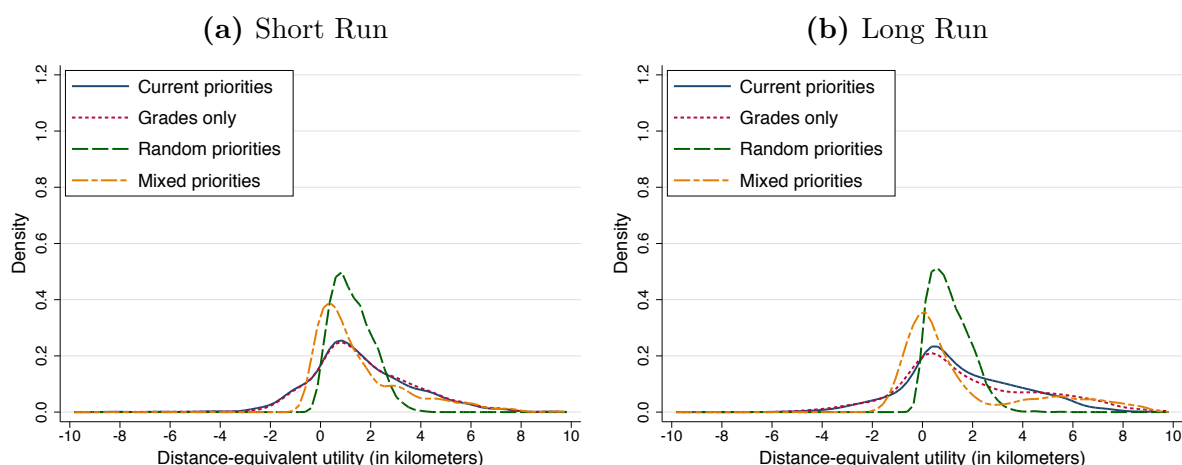


Figure E7: Distribution of Student Welfare under Alternative Admission Criteria: Estimates from Stability and Undominated Strategies

Notes: The two graphs show the short-run (left panel) and long-run (right panel) distributions of student welfare in the Southern District of Paris, under each of the four school priority structures described in Section 4: (i) Current admission criterion (based on academic grades and low-income status); (ii) academic grades only; (iii) Random priorities; (iv) Mixed priorities (based on grades for the “top two” schools and random priorities for the other schools). The welfare of every student is measured in terms of willingness to travel (in kilometers) and is normalized by subtracting the distance-equivalent utility that she would experience under a purely random assignment in the long-run steady state. The welfare distributions are computed by averaging over 300 simulation samples with different vectors of random utility shocks for each student. Preferences over the 11 within-district schools are simulated using estimates based on the stability assumption with undominated strategies (Table 4, column 4). The line fits are from a Gaussian kernel with optimal bandwidth using MATLAB’s `ksdensity` command. See Section 4.1 for details on the simulations.

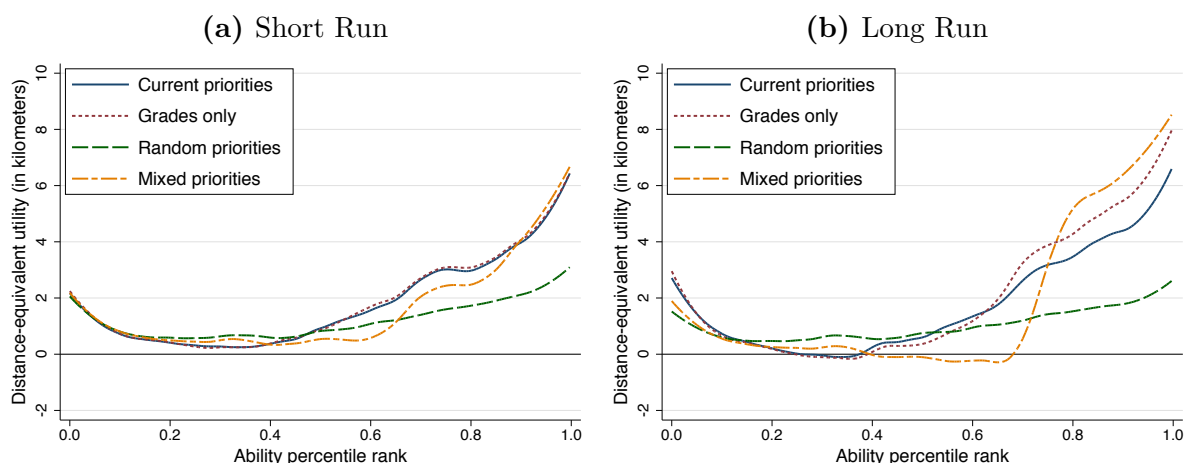


Figure E8: Student Welfare by Ability under Alternative Admission Criteria: Estimates from Stability and Undominated Strategies

Notes: See notes in Figure E7. For each of the four school priority structures described in Section 4, the graphs show the short-run (left panel) and long-run (right panel) distributions of average student welfare in the Southern District of Paris, for students belonging to the different quartiles of academic ability. Student ability is proxied by the percentile rank on the DNB exam composite score (normalized to be between 0 and 1) among all applicants in the data. Preferences over the 11 within-district schools are simulated using estimates based on the stability assumption with undominated strategies (Table 4, column 4). The lines are fitted using MATLAB’s `smooth/roess` command with a span of 20 percent.

Table E5: Centralized School Choice and College Admission based on Deferred Acceptance Mechanism with Strict Priority Indices

Country/city	Education stage	Assignment mechanism	Student priority index	Choice restrictions	Sources
Australia (Victoria)	Higher education	College-proposing DA	Mostly use a common priority indices based on grades; some have additional criteria.	Up to 12 choices	VTAC (2015a, 2015b)
Boston	Selective high schools	Student-proposing DA	School-specific priority indices based on grades and entrance exam results	Unrestricted	Abdulkadirođlu et al. (2014)
Chile	Higher education	College-proposing DA	College-specific priority indices based on grades, entrance exams and aptitude tests	Up to 8 choices	Hastings et al (2013); Ríos et al. (2014)
Chicago	Selective high schools	DA (Serial dictatorship)	Non-school-specific priority indices based on grades and entrance exam	Up to 6 choices (since 2010)	Pathak and Sömez (2013)
Finland	Secondary schools	School-proposing DA	School-specific priority indices based on grades, entrance exams and interviews	Up to 5 choices	Salonen (2014)
Ghana	Secondary schools	DA (Serial dictatorship)	Non-school-specific priority indices based on entrance exam	Up to 6 choices (since 2008)	Ajayi (2013)
Hungary	Higher education	Student-proposing DA	College-specific priority indices based on grades, exit exams and other criteria	Unrestricted but a fee for each choice beyond 3rd	Biró (2012)
Ireland	Higher education	College-proposing DA	College-specific priority indices based on exit exams and other criteria	Up to 10 choices	Chen (2012)
New York City	Specialized high schools	DA (Serial dictatorship)	Non-school-specific priority indices based on entrance exam	Unrestricted	Abdulkadirođlu et al. (2014)
Norway	Higher education	College-proposing DA	College-specific priority indices based on grades and other criteria	Up to 15 choices	Kirkebøen et al. (2015)
Paris	Upper secondary schools	School-proposing DA	Non-school-specific priority indices based on grades and other criteria	Up to 8 choices (since 2013)	Hiller and Tercieux (2014)
Romania	Secondary schools	DA (Serial dictatorship)	Non-school-specific priority indices based on grades and exit exams	Unrestricted	Pop-Eleches and Urquiola (2013)
Spain	Higher education	Student-proposing DA	College-specific priority indices based on grades and exit exams	Region-specific (e.g., 8 in Catalonia, 12 in Madrid)	Mora and Romero-Medina (2001); Calsamiglia et al. (2010)
Taiwan	Higher education	College-proposing DA	College-specific priority indices based on grades	Up to 100 choices	UAC (2014)
Turkey	Secondary schools	DA (Serial dictatorship)	Non-school-specific priority indices based on entrance exams	Up to 12 choices	Balinski and Sömez (1999); Akyol and Krishna (2014)
Turkey	Higher education	College-proposing DA	College-specific priority indices based on grades and exit exam	Up to 24 choices	Balinski and Sömez (1999); Saygin (2014)

Notes: References mentioned above are listed on the next page. When priority indices are not school-specific, i.e., schools/universities rank students in the same way, DA, whether student-proposing or school-proposing, is equivalent to the “Serial Dictatorship”, under which students, in the order of their priority indices, are allowed to choose among the remaining schools.

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Table E6: Assigned and Unassigned Applicants in the Southern District of Paris

	Sample	
	Assigned	Unassigned
<i>Panel A. Student characteristics</i>		
Age	15.0	15.0
Female	0.51	0.45
French score	0.56	0.45
Math score	0.54	0.47
Composite score	0.55	0.46
high SES	0.48	0.72
With low-income bonus	0.17	0.00
<i>Panel B. Assignment outcomes</i>		
Number of submitted choices	6.6	4.5
Enrolled in a within-district school	0.97	0.60
Enrolled in a private school	0.03	0.35
Number of students	1,568	22

Notes: The summary statistics reported in this table are based on administrative data from the Paris Education Authority (*Rectorat de Paris*), for students who applied to the 11 high schools of Paris's Southern District for the academic year starting in 2013. All scores are from the exams of the *Diplôme national du brevet* (DNB) in middle school and are measured in percentiles and normalized to be in $[0, 1]$. Enrollment shares are calculated for students who are still enrolled in the Paris school system at the beginning of the 2013-2014 academic year.

Table E7: OLS Regressions of School Fixed Effects on School Attributes

School attributes	Dependent variable: estimated school fixed effects from		
	Truth-telling	Stability	Stability and undominated strategies
Constant	-1.259 (1.339)	-3.756** (1.232)	-2.243 (1.406)
School score (French)	-1.650 (6.660)	-4.718 (6.128)	-2.247 (6.994)
School score (math)	3.331 (10.038)	16.308 (9.236)	9.740 (10.541)
Fraction high SES students	3.756 (2.692)	2.322 (2.477)	2.272 (2.827)
Adjusted R^2	0.624	0.908	0.792

Notes: Coefficients from the OLS regression of the school fixed effects reported in Table 3 on the observable characteristics of the 11 high schools of the Southern District of Paris. School attributes are measured by the average characteristics of each school's enrolled students in 2012-2013. Standard errors are in parentheses. *: $p < 0.10$; **: $p < 0.05$; ***: $p < 0.01$.

Table E8: Short-Run and Long-Run Effects of Counterfactual Admission Criteria on Student Sorting and Welfare: Estimates from Truth-Telling

	Baseline values		Impact of switching admission criterion to					
	Current criterion		Grades only		Random priorities		Mixed priorities	
	Short run (1)	Long run (2)	Short run (3)	Long run (4)	Short run (5)	Long run (6)	Short run (7)	Long run (8)
<i>Panel A. Sorting of students across schools</i>								
By ability	0.304 (0.018)	0.287 (0.016)	0.049 (0.012)	0.144 (0.018)	-0.190 (0.018)	-0.239 (0.018)	0.006 (0.014)	0.040 (0.020)
By SES	0.056 (0.010)	0.042 (0.008)	0.022 (0.006)	0.024 (0.009)	-0.008 (0.009)	-0.002 (0.009)	0.017 (0.009)	0.030 (0.012)
<i>Panel B. Welfare: fraction of winners and losers</i>								
Winners	-	-	0.070 (0.008)	0.558 (0.007)	0.163 (0.010)	0.390 (0.009)	0.161 (0.011)	0.385 (0.008)
Losers	-	-	0.051 (0.005)	0.442 (0.007)	0.190 (0.010)	0.610 (0.009)	0.144 (0.010)	0.615 (0.008)
Indifferent	-	-	0.879 (0.011)	0.000 (0.000)	0.646 (0.017)	0.000 (0.000)	0.694 (0.019)	0.000 (0.000)
<i>Panel C. Welfare measured by willingness to travel (in kilometers)</i>								
Average	1.047 (0.040)	0.983 (0.033)	0.082 (0.023)	0.315 (0.035)	-0.274 (0.046)	-0.540 (0.037)	0.032 (0.033)	0.074 (0.042)
Ability Q1	-0.015 (0.094)	0.305 (0.079)	0.076 (0.061)	-0.882 (0.088)	0.469 (0.100)	0.043 (0.085)	0.277 (0.095)	-0.348 (0.096)
Ability Q2	-0.035 (0.106)	-0.021 (0.084)	0.083 (0.060)	-0.331 (0.097)	0.418 (0.118)	0.398 (0.089)	0.134 (0.110)	0.343 (0.104)
Ability Q3	1.189 (0.107)	0.896 (0.097)	0.092 (0.058)	0.638 (0.112)	-0.471 (0.140)	-0.486 (0.106)	-0.161 (0.108)	-0.337 (0.131)
Ability Q4	3.052 (0.114)	2.755 (0.109)	0.075 (0.042)	1.837 (0.124)	-1.513 (0.179)	-2.117 (0.117)	-0.121 (0.092)	1.326 (0.156)
Low SES	0.657 (0.064)	0.727 (0.058)	-0.008 (0.033)	-0.067 (0.061)	-0.129 (0.069)	-0.347 (0.060)	-0.046 (0.056)	-0.246 (0.077)
high SES	1.462 (0.084)	1.255 (0.070)	0.176 (0.044)	0.720 (0.069)	-0.428 (0.101)	-0.745 (0.073)	0.115 (0.073)	0.414 (0.093)

Notes: This table reports the effects of different admission criteria on student sorting across high schools (by ability and by SES) and on student welfare in the Southern District of Paris. The effects are measured relative to the current system, in which priority indices are based on students' grades and low-income status. The three admission criteria considered in this table are: (i) Grades only; (ii) Random priorities; and (iii) Mixed priorities (the "top two" schools rank students by grades while the other schools use random priorities). Short-run effects measure what would happen in the first year of the alternative policy relative to the current system; long-run effects represent those in steady state. Sorting by ability (SES) is the fraction of total variance of ability (SES) explained by the between-school variance. Welfare is measured in terms of willingness to travel (in kilometers). The quartiles of ability are computed using students' composite score on the DNB exam. All effects are based on estimates under the truth-telling assumption (Table 3, column 1).

Table E9: Short-Run and Long-Run Effects of Counterfactual Admission Criteria on Student Sorting and Welfare: Estimates from Stability and Undominated Strategies

	Baseline values		Impact of switching admission criterion to					
	Current criterion		Grades only		Random priorities		Mixed priorities	
	Short run (1)	Long run (2)	Short run (3)	Long run (4)	Short run (5)	Long run (6)	Short run (7)	Long run (8)
<i>Panel A. Sorting of students across schools</i>								
By ability	0.462 (0.015)	0.485 (0.013)	0.061 (0.011)	0.066 (0.012)	-0.266 (0.020)	-0.301 (0.020)	-0.030 (0.015)	-0.076 (0.015)
By SES	0.111 (0.013)	0.103 (0.011)	0.032 (0.008)	0.039 (0.009)	-0.001 (0.014)	0.032 (0.017)	0.022 (0.013)	0.057 (0.015)
<i>Panel B. Welfare: fraction of winners and losers</i>								
Winners	-	-	0.097 (0.009)	0.578 (0.008)	0.227 (0.009)	0.396 (0.009)	0.180 (0.010)	0.472 (0.008)
Losers	-	-	0.060 (0.006)	0.422 (0.008)	0.250 (0.010)	0.604 (0.009)	0.186 (0.010)	0.528 (0.008)
Indifferent	-	-	0.843 (0.012)	0.000 (0.000)	0.522 (0.014)	0.000 (0.000)	0.634 (0.016)	0.000 (0.000)
<i>Panel C. Welfare measured by willingness to travel (in kilometers)</i>								
Average	1.593 (0.032)	1.579 (0.029)	0.107 (0.021)	0.254 (0.025)	-0.336 (0.042)	-0.492 (0.039)	-0.044 (0.032)	0.069 (0.035)
Ability Q1	0.755 (0.057)	0.793 (0.053)	0.063 (0.040)	-0.008 (0.044)	0.193 (0.058)	-0.023 (0.061)	0.136 (0.056)	-0.118 (0.060)
Ability Q2	0.343 (0.061)	-0.005 (0.053)	0.070 (0.046)	-0.055 (0.048)	0.434 (0.086)	0.688 (0.076)	0.157 (0.080)	0.274 (0.078)
Ability Q3	1.486 (0.066)	1.442 (0.058)	0.116 (0.049)	0.202 (0.050)	-0.282 (0.121)	-0.360 (0.097)	-0.213 (0.092)	-0.242 (0.086)
Ability Q4	3.791 (0.065)	4.090 (0.064)	0.178 (0.036)	0.879 (0.053)	-1.689 (0.163)	-2.277 (0.128)	-0.256 (0.086)	0.362 (0.097)
Low SES	1.203 (0.039)	1.101 (0.035)	0.013 (0.028)	0.039 (0.030)	-0.215 (0.055)	-0.320 (0.047)	-0.127 (0.041)	-0.126 (0.043)
high SES	2.007 (0.053)	2.088 (0.049)	0.207 (0.036)	0.483 (0.047)	-0.464 (0.089)	-0.675 (0.074)	0.045 (0.063)	0.275 (0.067)

Notes: This table reports the effects of different admission criteria on student sorting across high schools (by ability and by SES) and on student welfare in the Southern District of Paris. The effects are measured relative to the current system, in which priority indices are based on students' grades and low-income status. The three admission criteria considered in this table are: (i) Grades only; (ii) Random priorities; and (iii) Mixed priorities (the "top two" schools rank students by grades while the other schools use random priorities). Short-run effects measure what would happen in the first year of the alternative policy relative to the current system; long-run effects represent those in steady state. Sorting by ability (SES) is the fraction of total variance of ability (SES) explained by the between-school variance. Welfare is measured in terms of willingness to travel (in kilometers). The quartiles of ability are computed using students' composite score on the DNB exam. All effects are based on estimates from stability and undominated strategies (Table 3, column 4).