Conciliating Absolute and Relative Poverty: Income Poverty Measurement Beyond Sen's Model

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Abstract

The properties of poverty measures based on absolute poverty lines are well-known. Their properties have been extensively studied in the model proposed by Sen (1976). In contrast, the properties of relativist poverty measures – measures based on non-absolute poverty lines – have never been rigorously studied. This is not merely a theoretical issue: relativist measures do provide highly debatable poverty comparisons. This becomes increasingly problematic as relativist poverty measures are more and more used by policy makers. This paper proposes an extension of Sen's model designed for the study of relativist poverty measures. Several results show that classical properties have different implications in the extended model and in Sen's model. Finally, the paper characterizes an index specifically designed for non-absolute poverty lines (Decerf, 2015a). This result provides the first characterization of a relativist poverty measure.

JEL: D63, I32.
Keyworks: Income Poverty Measurement, Poverty Line, Relative Poverty, Absolute poverty, Additive Indices.

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1 Introduction

There are two different approaches to income poverty measurement: the absolute approach and the relativist approach. In the former approach, the poverty line below which an individual is considered as poor is absolute. The threshold of an absolute line does not depend on standards of living. This is for instance the approach underlying the extreme poverty line of the World Bank, set at 1.25 $ per person per day (Ravallion et al., 2009). The nominal amount associated to an absolute line may depend on purchasing power and inflation, but this amount corresponds to a fixed income in real terms. Typically, this fixed amount is defined from the price of a reference bundle of necessities.

Critics against the absolute approach emerged a couple of decades ago. Townsend (1979) and his followers argue that an individual whose income is not sufficient to engage in the everyday life of her society should be considered as poor. In their views, any individual whose income is too far away from her society's income standard is at risk of social exclusion and considered as relatively poor.[1] The relativist approach is based on poverty lines whose income threshold depends on the income standard. A famous example are relative lines whose income threshold corresponds to a given fraction of mean or median income.

Historically, official poverty measures used to be absolute measures. Yet, this domination is now challenged by relativist measures, which gained ground over the past decades. By the end of last century, official poverty measures were, in most developed countries, relative measures. Lately, hybrid poverty measures – which combine the absolute and relative aspects of income poverty – emerged in the specialized literature and found their way into the reflections held by policy makers (World Bank, 2015; European Commission, 2015).[2]

The increasing use of relativist poverty measures raises questions because we do not understand well how these measures compare poverty across different income distributions. Surprisingly, the properties of relativist measures have never been rigorously studied. This gap is all the more surprising that the properties of absolute measures have been extensively studied. Sen (1976) initiates the literature on income poverty measures. Sen distinguishes two components defining a poverty measure: the poverty line and the index. The index aggregates the contributions to poverty of all individuals in a distribution and, therefore, allows ranking different income distributions. The novelty of Sen is to propose a model allowing to study the properties inherent to these indices. Most of these properties define particular distributional changes that either should have no consequences for poverty or should unambiguously increase poverty. For instance, one such property is that poverty should increase when the income of a poor individual is reduced.

Sen's model is designed to study the properties that an index has when used in combination with an absolute line. Any *absolute* measure automatically inherits the desirable properties predicted for its index in Sen's model. Therefore, we can be confident that the poverty comparisons obtained from absolute measures are meaningful. The problem is that a *relativist* measure need not inherit the properties predicted for its index in Sen's model. Most poverty indices have been designed for the construction of *absolute* measures. Yet, virtually all *relativist* measures are constructed using these indices. As a result, the poverty comparisons obtained from such a relativist measure may be counterintuitive and controversial.

As shown in the literature review, this is not merely a theoretical problem. Relativist measures do lead to extremely debatable judgments. For instance, they may consider that decreasing the income of a poor individual leads to an unambiguous poverty reduction. Another issue is that they often attribute excessive importance to the relative aspect over the absolute aspect of income poverty (Decerf, 2015a). The major relativist measures systematically lead to negative evaluations of unequal growth processes. However, if the income of an individual is "sufficiently" small, one may argue that her situation improves as her income increases *even if* the other individual's income increase faster. Also, as happened in New-Zealand (Easton, 2002), relativist measures may deem that regressive policies – whose unique impact is to transfer income from the middle class and the poor to the rich – are poverty reducing.

In this paper, I investigate the properties of relativist poverty measures in three steps. First, I propose an extension of Sen's model with which the properties of relativist measures may be rigorously studied. The novelty is the introduction of an *ethical ordering*. This new object is a form of other-regarding preferences (Bolton, G., Ockenfels, 2000; Fehr and Schmidt, 1999) inspired from the Behavioral literature. Like other-regarding preferences, the ethical ordering defines the trade-offs made between the absolute

[1] See Ravallion (2008) for a review of the normative foundations of the relativist approach to poverty measurement.

[2] In 2011, the US Census Bureau released the Supplemental Poverty Measure, whose aim is to complement the US official absolute poverty measure. This new measure is based on a relativist poverty line.

and relative aspects of an individual's situation. Unlike other-regarding preferences, the ethical ordering does not correspond to the concerned individual's views but is rather the judgment of a moral observer.

Second, I adapt the main properties studied in Sen's model to the extended model and derive the constraints that they place on acceptable indices. In particular, I generalize the celebrated additive separability result of Foster and Shorrocks (1991). Their result implies that indices should aggregate the contributions to poverty of all individuals in a distribution by summing these contributions. In Sen's model, the contribution of an individual only depends on her level of income. The additive separability result derived in the extended model shows that an individual's contribution may also depend on the relative aspects of her income. Other results derive conditions under which indices satisfy key properties.

Finally, I fully characterize in the extended model a recently proposed index (Decerf, 2015a). Any relativist measure using this index automatically inherits from its properties. Hence, this result constitutes the first characterization of a relativist poverty measure.

The paper is organized as follows. I provide a critical review of the poverty measurement literature in section 2. I present the extension of Sen's model in section 3. I adapt the classical properties to the extended model and study their implications in section 4. I fully characterize a particular index in section 5. I make some concluding remarks in section 6.

2 Literature review

The literature on income poverty measurement studies indicators that rank income distributions as a function of their poverty. These indicators are called poverty measures. Any poverty measure is composed of two elements: a poverty line and an index. A poverty line specifies the income threshold below which an individual is considered as poor. An index aggregates the contributions to poverty of all individuals in a distribution. In his groundbreaking paper (Sen, 1976), Sen proposes a model allowing to study the properties inherent to these indices.

In this section, I shortly present Sen's model, the different types of non-absolute poverty lines, the main proposals of poverty measures integrating relativist considerations and expose the key reasons why Sen's model is not adequate to study the properties of relativist measures.

2.1 Sen's model

Let an income distribution $y \coloneqq (y_1, \dots, y_n)$ be a list of non-negative incomes sorted in non-decreasing order $y_1 \leq \cdots \leq y_n$. Letting $N \coloneqq \{n \in \mathbb{N} | n \geq 3\}$, the set of income distributions is

$$Y \coloneqq \cup_{n \in N} \{y \in \mathbb{R}^n_+ \mid y_i \leq y_{i+1}, \ \forall i = 1, \dots, n-1\}.$$

The number of individuals in distribution y is denoted by $n(y)$. The poverty threshold is denoted by $z^* \in \mathbb{R}_{++}$. Individual i qualifies as poor if $y_i < z^*$. The number of poor individuals is denoted by $q(y)$. As income distributions are sorted, if $i \leq q(y)$ then individual i is poor in distribution y. Let a poverty index be a real-valued function $P : Y \times \mathbb{R}_{++} \to \mathbb{R}$ representing a complete ranking on Y. For any two distributions $y, y' \in Y$ and any poverty threshold $z^* \in \mathbb{R}_{++}$, there is strictly more poverty in y than in y' if $P(y, z^*) > P(y', z^*)$, and weakly more if $P(y, z^*) \geq P(y', z^*)$.

The literature initiated by Sen (1976) aims at deriving poverty indices from a set of desirable properties. These properties are encapsulated into axioms. The major results derived from Sen's model are reviewed in Zheng (1997). A key result is the characterization of additive indices. Any index satisfying five basic properties must be ordinally equivalent to an additive index (Foster and Shorrocks, 1991):

$$P(y, z^*) \coloneqq \frac{1}{n(y)} \sum_{i=1}^{q(y)} d(y_i, z^*), \tag{1}$$

where function $d : \mathbb{R}_+ \times \mathbb{R}_{++} \to [0, 1]$ is non-increasing in y_i. Function d returns the contribution to poverty of an individual earning income y_i when the threshold is z^*. Given threshold z^*, this contribution only depends on own income. Very few restrictions are imposed on function d and, therefore, this family is very broad. Yet, most empirical applications use indices belonging to the Foster-Greer-Thorbecke (FGT) subfamily, which has an exponential expression for the contribution (Foster et al., 1984):

$$P^{FGT}(y, z^*) \coloneqq \frac{1}{n(y)} \sum_{i=1}^{q(y)} \left(1 - \frac{y_i}{z^*}\right)^{\alpha}. \tag{2}$$

The FGT family has a unique parameter $\alpha \in [0, \infty)$, which can be interpreted as poverty aversion. This family admits the Head-Count Ratio (HC) and the Poverty Gap Ratio (PGR) as particular cases when $\alpha = 0$ and $\alpha = 1$ respectively. Besides FGT indices, many other indices have been proposed and characterized, e.g. by Kakwani (1980), Chakravarty (1983) or Duclos and Gregoire (2002).

2.2 Different types of poverty lines

Income poverty measures may be classified into two main categories: absolutist and relativist measures. This classification depends on the type of poverty line used to construct the measure. An absolutist measure is based on an absolute line and a relativist measure on a non-absolute line.

The definition of a poverty line requires an additional bit of notation. Let $S : Y \to \mathbb{R}_+$ be a function returning the income standard associated to distribution y. Typical income standards are the mean or the median income. For the sake of notational simplicity, the income standard associated to y is denoted by $\overline{y} \coloneqq S(y)$. A poverty line is defined by its threshold function $z : \mathbb{R}_+ \to \mathbb{R}_+$, a continuous function specifying the income threshold $z(\overline{y})$ associated to distribution y.

A poverty line is *absolute* if its threshold function is flat. That is, its image does not depend on the income standard.[3] In contrast, the threshold of a *non-absolute* poverty line may evolve with the income standard. Typically, non-absolute lines associate a larger poverty threshold to distributions featuring a larger income standard. This implements the relativist view that an individual's relative situation matters to her well-being.[4] Relative lines are a famous example of non-absolute lines. The threshold of a relative line evolves as a constant fraction of the income standard. Relative lines only capture the relative aspect of poverty as their threshold tends to zero in low-income distributions. If that is judged non-satisfactory, an alternative is to use *hybrid* lines, which arbitrate the absolute and relative aspects of poverty. Foster (1998) proposes hybrid lines that feature a constant income elasticity.[5] The income elasticity of a poverty line is defined as the elasticity of its threshold with respect to the statistic measuring standard of living. This income elasticity can be interpreted as the extent to which poor individuals should share the benefits of economic growth. Absolute lines have an income elasticity of zero and relative lines have an income elasticity of one, representing two extreme views on this parameter.[6] Ravallion and Chen (2011) propose *weakly relative* lines, whose income threshold is constant for low-income distributions and has a constant derivative for high-income distributions. As a result, the income elasticity of weakly relative lines is zero for low-income distributions and then increases with standards of living, tending ultimately to a value of one.

2.3 Relativist measures

The most common relativist measures are relative measures, which combine a relative line with an FGT index.[7] It is well-known that relative measures completely ignore the absolute aspects of income poverty. Multiplying all incomes in a distribution by a common factor leaves relative measures unchanged. Following the premise that both the absolute and relative aspects of income matter for individual well-being, defenders of the relativist approach have proposed measures combining both aspects.

Some hybrid measures simply combine an hybrid line with an FGT index (Chen and Ravallion, 2013). Another approach proposed by Atkinson and Bourguignon (2001) is to consider two poverty lines, one absolute and one relative, and combine them with an index aggregating the gaps with respect to these two lines. Such index aggregates the absolute and relative aspects of income poverty at individual level, before aggregating the contributions to poverty of all individuals in a distribution. The two lines and the index jointly define a poverty measure. Anderson and Esposito (2013) follow this approach. Inspired by Atkinson and Bourguignon (2001), Decerf (2015a) proposes an index aggregating both aspects while always providing a minimal priority to absolutely poor individuals. Unlike all other proposals, this index is such that absolutely poor individuals always contribute more to poverty than relatively poor individuals. This property is presented below in more details.

[3] The threshold of an absolute line is often defined from the cost of a particular bundle of goods. The line is then "anchored" in that bundle and its threshold is constant *in real terms*. The *nominal* threshold of an absolute line may evolve over time with inflation or vary from one country to another as a function of purchasing power. To be sure, all incomes in this paper are expressed in real terms.

[4] See Ravallion (2008) for a normative discussion of the relativist view.

[5] For a given income standard, letting z^r denote a relative line and z^a the threshold of an absolute line, the hybrid threshold is given by $z(\overline{y}) = \left(z^a\right)^{1-\rho}\left(z^r(\overline{y})\right)^{\rho}$, where $\rho \in [0, 1]$ denotes the line's constant elasticity.

[6] Madden (2000) estimates empirically an upper-bound for the value of this parameter using Irish data.

[7] Relative measures can also be constructed using indices outside the FGT family. Nevertheless, I focus only on FGT indices for ease of exposition and given their prevalence in empirical applications.

2.4 Properties of relativist measures

The literature initiated by Sen (1976) studies the properties of poverty indices. As Sen's model is based on a fixed poverty threshold, absolutist measures automatically inherit the properties of their index. Unfortunately, this is not the case of relativist measures. The reason is that Sen's model disregards the endogenous link that a non-absolute poverty line defines between an income distribution and its associated poverty threshold. Sen's model ranks income distributions for an exogenously given poverty threshold. As a consequence, the properties of relativist measures are to a great extent still unknown. Even if they are widely used in practice, no relativist measure has ever been characterized. Only Decerf (2015a) proposes a partial characterization that accounts for this endogenous link.

I illustrate that relative measures need not inherit from the properties of their index. The two properties I investigate are satisfied by FGT indices (Foster and Shorrocks, 1991). First, *Focus* requires that the index is not sensitive to the income level of non-poor individuals. Second, *Monotonicity in Income* requires that decreasing the income of some *poor* individual never leads to an unambiguous poverty reduction.[8]

Relativist measures violate *Focus* if the income standard depends on the income of non-poor individuals. Relativist measures may violate *Monotonicity in Income* if the income standard depends on the income of poor individuals. If the income standard is mean income, then both properties may be violated. If the income standard is median income and the poverty line is relative, then only *Focus* is violated.

Table 3 illustrates that the HC violates both properties when the relative poverty line is mean-sensitive. Distribution B is constructed from distribution A by decreasing the income of the poorest individual. Yet, the value taken by the HC based on a relative line is strictly larger in A than in B, which violates *Monotonicity in Income*. Distribution C is constructed from distribution B by increasing the income of the non-poor individual 3. The value taken by the HC is strictly larger in C than in B, which violates *Focus*.

Table 1: The Head-Count Ratio based on a mean-sensitive line violates *Focus* and *Monotonicity in Income*.

	y_1	y_2	y_3	$z^r(\overline{y})$	$HC(y)$
Distribution A	1.1	2	9	2.02	$\frac{2}{3}$
Distribution B	0.6	2	9	1.93	$\frac{1}{3}$
Distribution C	0.6	2	9.5	2.02	$\frac{2}{3}$

Note: The threshold of the relative line z^r is 50% of mean income $\overline{y}$.

The violation of *Focus* raises important normative questions. For instance, relative measures based on median-sensitive lines are plagued by several paradoxes (de Mesnard, 2007). In practice, policies inducing regressive balanced transfers from the middle class to the rich may decrease median income. As a result, the value taken by median-sensitive measures may unambiguously decrease, even if such policies decrease the income of poor individuals. Easton (2002) emphasizes that this scenario happened in New-Zealand between 1981 and 1992. What is more, some institutions used the drop in the median-sensitive measure to argue that the regressive policies were a success. This illustrates that the limitations of relativist measures constructed with an index designed for absolute measures are not merely theoretical issues.

Relativist measures may also provide debatable poverty comparisons due to the unintended implications that some robustness properties have in Sen's model. I illustrate this using hybrid measures. FGT indices satisfy *Scale Invariance*. This robustness property is often defended on the grounds that it renders the currency units in which income is measured irrelevant. Formally, *Scale Invariance* requires the index not to be affected when the income of all individuals are multiplied by the same factor as the income threshold.[9] In Sen's model, *Scale Invariance* implies that an individual's contribution to poverty only depend on her normalized income $\frac{y_i}{z^*}$, her income divided by the income threshold. As a result, hybrid measures may implicitly consider that an absolutely poor individual in a low-income distribution is *better-off* than another individual who is only relatively poor when the latter lives in a high-income distribution (Decerf, 2015a). In that sense, hybrid measures deny a minimal priority to absolutely poor

[8] These two axioms are formally defined in Appendix 7.1.1.
[9] This axiom is formally defined in Appendix 7.1.1.

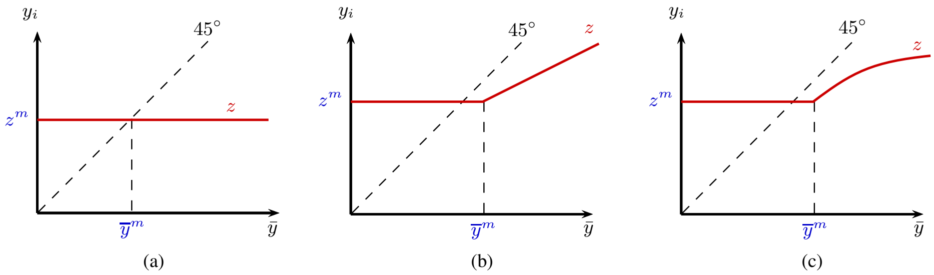

Figure 1: Illustration of the domain $\mathcal{Z}$ of poverty lines.
Note: (a) Flat line. (b) Weakly relative line. (c) Generic line in $\mathcal{Z}$.

individuals.

3 The model

I showed in Section 2 that the model of Sen (1976) is not suited for the study of income poverty measures based on non-absolute poverty lines. In this section, I extend Sen's model by the introduction of a new object, which I call an Ethical Ordering. This object allows the extended model to account for the endogenous link between an income distribution and its associated poverty threshold.

The basic notation is identical to the one presented above. I first present the restrictions imposed on poverty lines, then I introduce the concept of an ethical ordering and define poverty indices.

3.1 Poverty lines

Let mean income, denoted by $\overline{y} := \frac{\sum y_i}{n}$, be the income standard to which the poverty line is sensitive.[10] A poverty line is defined by its threshold function $z : \mathbb{R}_+ \to \mathbb{R}_+$, a continuous function specifying the income threshold associated to $\overline{y}$. The set $\mathcal{Z}$ of acceptable poverty lines, illustrated in Figure 1, is defined by three restrictions.

Non-zero Threshold requires the poverty threshold to be everywhere strictly positive. Meeting one's basic needs has a positive cost, even in low-income societies. Relative lines are hence excluded. This restriction is necessary for Theorem 4 to hold. Nevertheless, the set $\mathcal{Z}$ contains lines that are arbitrarily close to relative lines. Furthermore, when combined with a relative line the index characterized in Theorem 6 corresponds to the PGR, and any measure based on the PGR and a mean-sensitive relative line satisfies all the poverty axioms presented.

Line restriction 1 (Non-zero Threshold)**.**
For all $\overline{y} \geq 0$, we have $z(\overline{y}) > 0$.

Positive Slope less than One constrains the slope of the poverty line, defined by its first order derivative $\frac{\partial z}{\partial \overline{y}}$.[11] Strictly decreasing lines and lines exhibiting an excessive sensitivity to the mean are hence excluded.

Line restriction 2 (Positive Slope less than One)**.**
For all $\overline{y} \geq 0$, the slope $\frac{\partial z(\overline{y})}{\partial \overline{y}}$ belongs to $[0, 1]$.

Flat Line in Poor Societies requires that the line be flat, i.e. its slope is equal to zero, for distributions whose income standard is "low". Formally, a distribution has a "low" income standard if $\overline{y} < z(\overline{y})$. The total income in such a distribution is so small that all individuals would be poor if this total income was redistributed equally. Hence, any distribution with such mean income contains at least one poor individual.

Line restriction 3 (Flat Line in Poor Societies)**.**
For all $\overline{y} \geq 0$ with $\overline{y} < z(\overline{y})$ we have $\frac{\partial z(\overline{y})}{\partial \overline{y}} = 0$.

[10] Median-sensitive lines are often used in practice. See Decerf (2015b) for a discussion on the choice of the income standard and its implications for the robustness of the results.
[11] At points at which z is non-differentiable, its slope is defined to be the limit from the right of its first order derivative.

Two characteristics of the line will prove useful. First, the line's intercept, denoted by z^m, is the minimal value taken by the poverty threshold:

$$z^m := z(0).$$

Second, the maximal value of mean income at which the income threshold is equal to the intercept is denoted by $\overline{y}^m$:

$$\overline{y}^m := \max\{\overline{y} \geq 0 \mid z(\overline{y}) = z^m\}.$$

A line $z \in \mathcal{Z}$ is *flat* if $z(\overline{y}) = z^m$ for all $\overline{y} \geq 0$. The value of $\overline{y}^m$ is not well-defined for flat lines. For flat lines, I adopt the convention that $\overline{y}^m = z^m$.

Finally, I define weakly relative lines (Ravallion and Chen, 2011) that constitute a particular subset of $\mathcal{Z}$, illustrated in Figure 1.b.

Definition 1 (Weakly relative line)**.**
The poverty line $z \in \mathcal{Z}$ is weakly relative if

$$z(\overline{y}) = \max\{z^m, a + s\overline{y}\},$$

where $z^m \in \mathbb{R}_{++}$, $a \in \mathbb{R}$, $s \in [0, 1]$ and $z^m(1 - s) \geq a$.[12]

If a weakly relative line is such that $a = 0$, then the line is the upper contour of two lines: an absolute and a relative. Alternatively, if a weakly relative line is such that $s = 0$, then it is a flat line.

3.2 Ethical ordering

According to proponents of the relativist approach, the well-being of a poor individual is also affected by her relative situation. Atkinson and Bourguignon (2001) suggest to aggregate the absolute and relative aspects of income poverty at individual level, before aggregating the poverty contributions of all individuals. To perform the first aggregation, I define a new object, which I call an *ethical ordering* (EO). An EO is an ethical observer's preferences relation over individual situations. Introducing an EO in the model allows distinguishing the comparisons of individual situations from the aggregation of the contributions to poverty over the whole population.

The bundle consumed by an individual summarizes the absolute and relative aspects of her income and is therefore two-dimensional. The set of bundles for a generic individual i is

$$X := \{(y_i, \overline{y}) \in \mathbb{R}_+ \times \mathbb{R}_{++}\}.$$

An EO, denoted by $\succeq$, is an ordering on the space of bundles.

Definition 2 (Ethical Ordering)**.**
An ethical ordering $\succeq$ is a continuous ordering on X.[13]

Formally, an EO is a form of other-regarding preferences, as defined by the Behavioral literature (Fehr and Schmidt, 1999; Bolton, G., Ockenfels, 2000). There is a key difference between other-regarding preferences and an EO. The former are the preferences held by an individual over her own situation, whereas the latter summarizes the normative judgments formed by an external observer when comparing the well-being associated to different individual situations.

I draw the reader's attention on an important terminology aspect. In this paper, the word *well-being* has nothing to do with the utility experienced by an individual. Rather, I use well-being to refer to a cardinalization of the EO considered.

The set $\mathcal{R}$ of acceptable EOs is defined by three restrictions. This set is illustrated in Figure 2, which shows three different EOs. Graphically, an EO is illustrated by its associated iso-poverty map, which is composed of iso-poverty curves. An iso-poverty curve connects all the bundles that the EO deems equivalent in terms of well-being.

The first restriction, Strict Monotonicity up to line, requires that each EO in $\mathcal{R}$ has a threshold iso-poverty curve that plays the role of a poverty line. Below its associated poverty line, the EO deems that a larger income is an improvement. Above its poverty line, the EO deems that the level of income is irrelevant.

[12] The inequality $z^m(1 - s) \geq a$ ensures that weakly relative lines meet Flat Line in Poor Societies.
[13] An ordering is a reflexive, transitive and complete binary relation.

EO restriction 1 (Strict Monotonicity up to line)**.**
There exists $z \in \mathcal{Z}$ *such that for all* $(y_i, \overline{y}) \in X$ *with* $y_i = z(\overline{y})$, $a \in (0, z(\overline{y})]$ *and* $b > 0$ *we have*

$$(y_i - a, \overline{y}) \prec (y_i, \overline{y}) \sim (y_i + b, \overline{y}).$$

Let $Z : \mathcal{R} \to \mathcal{Z}$ be the function returning the poverty line $z = Z(\succeq)$ associated to $\succeq$. For a given $\succeq$, the set of bundles at which individual i qualifies as poor is

$$X_p(\succeq) := \left\{ (y_i, \overline{y}) \in X \,\middle|\, y_i < z(\overline{y}) \text{ where } z = Z(\succeq) \right\},$$

The other two restrictions limit the importance of a poor individual's relative situation in the definition of her well-being. Priority to Income below z^a requires that each EO in $\mathcal{R}$ is associated to a level of income below which mean income is irrelevant for well-being. This level of income is interpreted as the threshold for absolute poverty. All the EO's iso-poverty curves are flat up to this absolute threshold. In other words, the well-being attributed to an absolutely poor individual only depends on her own income.[14]

EO restriction 2 (Priority to Income below z^a)**.**
There exists $z^a \in [0, z^m]$ *where* $z = Z(\succeq)$ *such that for all* $(y_i, \overline{y}), (y_i', \overline{y}') \in X$ *with* $y_i = y_i' \leq z^a$ *we have*

$$(y_i, \overline{y}) \sim (y_i', \overline{y}').$$

Priority to Income below z^a implies that the EO always deems that an absolutely poor individual is worse-off than a relatively poor individual. This intuition was expressed in Atkinson and Bourguignon (2001) and is largely shared in the population, as appeared from questionnaire studies run all over the world (Corazzini et al., 2011).

Let $Z^a : \mathcal{R} \to [0, z^m]$ be the function returning the absolute threshold $z^a = Z^a(\succeq)$ associated to $\succeq$, where z^m is the intercept of $Z(\succeq)$. For a given $\succeq$, the set of bundles at which individual i qualifies as absolutely poor is

$$X_a(\succeq) := \left\{ (y_i, \overline{y}) \in X_p(\succeq) \,\middle|\, y_i < z^a \text{ where } z^a = Z^a(\succeq) \right\}.$$

I call relatively poor any poor individual that is not absolutely poor. When no confusion is possible, z is assumed to be the line associated to $\succeq$ and z^a its absolute threshold.

Finally, Homotheticity above z^a requires the iso-poverty curves between the absolute threshold and the poverty line to be homothetic. This simplifying assumption is a natural default option.[15] Given that the slope of the poverty line is smaller than one, this restriction implies that the slope of iso-poverty curves is nowhere larger than one.[16]

EO restriction 3 (Homotheticity above z^a)**.**
For all $(y_i, \overline{y})$, $(y_i', \overline{y}') \in X_p(\succeq) \backslash X_a(\succeq)$, *if* $\frac{y_i - z^a}{z(\overline{y}) - z^a} = \frac{y_i' - z^a}{z(\overline{y}') - z^a}$, *then* $(y_i, \overline{y}) \sim (y_i', \overline{y}')$.

The three EO restrictions presented strongly constrain $\mathcal{R}$.[17] Given a poverty line, selecting an EO in $\mathcal{R}$ amounts to selecting a unique parameter: its absolute threshold. As defined above, this absolute threshold is weakly smaller than the poverty line's intercept. If the poverty line is not flat, then Strict Monotonicity up to line implies that the absolute threshold is strictly smaller than the intercept.

This definition of $\mathcal{R}$ "conciliates" absolute and relative poverty in a specific sense. The well-being attributed only depends on own-income below a minimal threshold z^a. This is true both in low and high-income distributions. In contrast, the well-being above this threshold depends on both own-income and the income standard.[18]

[14] This restriction implies that EOs in $\mathcal{R}$ satisfy Minimal Absolute Concern, a weaker EO restriction presented in Appendix 7.1.2. Minimal Absolute Concern limits the importance of the relative aspect of income. It requires that there is no worse bundle than a bundle with zero income.

[15] Decerf (2015b) shows that if the iso-poverty curves above z^a deviate too much from homotheticity, then the existence of additive poverty indices satisfying *Monotonicity in Income* and *Transfer* is not guaranteed (Theorem 4 page 73). This illustrates the trade-offs that may appear between EO restrictions and poverty axioms.

[16] Homotheticity above z^a and Positive Slope less than One together imply that EOs in $\mathcal{R}$ satisfy Translation Monotonicity, a rather weak EO restriction presented in Appendix 7.1.2. Translation Monotonicity limits the importance of the relative aspect of income. It requires that an equal distribution of an extra amount of income always makes the individual weakly better-off. Such judgment corresponds to the so-called "leftist" view on income poverty (see Kolm (1976)). Hence, Translation Monotonicity implies that the "leftist" view is the maximal importance that one can give to the relative aspect.

[17] Priority to Income below z^a and Homotheticity above z^a are rather strong restrictions, but as explained in footnotes 14 and 16, they can be weakened without affecting the general results of Section 4.

[18] Strictly speaking, in all distributions with low standards of living ($\overline{y} < \overline{y}^m$), the well-being attributed to relatively poor individuals only depends on their level of income.

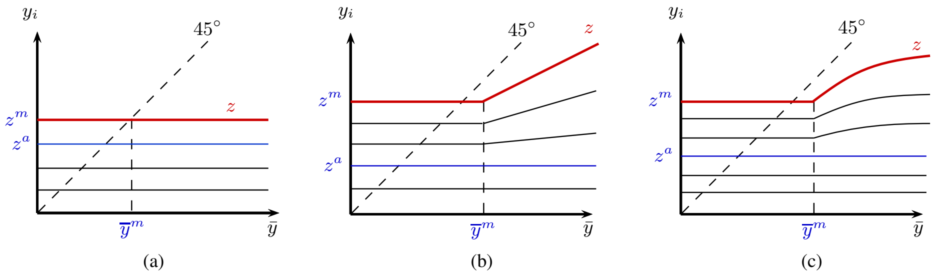

Figure 2: Iso-poverty curves of ethical orderings.
Note: (a) EO below a flat line. (b) EO below a weakly relative line. (c) EO below a generic line.

3.3 Poverty Indices

For any $\succeq \in \mathcal{R}$, the set of income distributions considered is

$$Y(\succeq) := \left\{ y \in Y \;\middle|\; y_n \geq z(\overline{y}) \right\}.$$

For technical reasons, this set excludes distributions for which all individuals are poor.[19] Let $\mathcal{P}$ be the domain on which a poverty index is defined:

$$\mathcal{P} := \{(y, \succeq) | \succeq \in \mathcal{R} \text{ and } y \in Y(\succeq)\}.$$

A poverty index is a real-valued function $P : \mathcal{P} \to \mathbb{R}$ that represents, for any $\succeq \in \mathcal{R}$, a complete ranking on $Y(\succeq)$. For any two $(y, \succeq), (y', \succeq) \in \mathcal{P}$, there is strictly more poverty in $(y, \succeq)$ than in $(y', \succeq)$ if $P(y, \succeq) > P(y', \succeq)$, and weakly more if $P(y, \succeq) \geq P(y', \succeq)$.

This model simply extends Sen's model by defining the domain of indices using the concept of EO rather than using a fixed threshold z^*. Given that the EO's definition is based on a poverty line, the EO accounts for the endogenous link between a given distribution and its specific poverty threshold.

4 Basic properties in the extended model

Zheng (1997) reviews the basic properties imposed to poverty indices and their implications in Sen's model. In this section, I adapt these basic properties to the extended model and investigate their implications.

4.1 Additive family

I extend the characterization of additive indices based on absolute lines to additive indices based on non-absolute lines. This extended characterization requires to weaken some of the basic axioms used by Foster and Shorrocks (1991).

An EO defines a specific way to perform individual well-being comparisons. *Domination* requires poverty indices to respect the individual well-being comparisons encapsulated in the EO. It does so by imposing a monotonicity requirement in the space of individual well-being distributions. If the well-being of one individual increases, while the individual well-being of all other individuals do not decrease, then poverty must decrease.

Poverty axiom 1 (*Domination*)**.**
For all $\succeq \in \mathcal{R}$ and all $y, y' \in Y(\succeq)$ with $n(y) = n(y')$, if $(y'_i, \overline{y}') \succeq (y_i, \overline{y})$ for all $i \leq n(y)$, then $P(y, \succeq) \geq P(y', \succeq)$.
If in addition there is $j \leq n(y)$ such that $(y'_j, \overline{y}') \succ (y_j, \overline{y})$, then $P(y, \succeq) > P(y', \succeq)$.

By Strict Monotonicity up to line, all bundles above the poverty line are attributed the same well-being. Therefore, *Domination* implies a weak version of *Focus*. *Weak Focus*, presented in Appendix 7.1.2, requires that changes to the incomes of non-poor individuals are irrelevant to poverty only as long as the income standard is unchanged. As a result, only the well-being of poor individuals matters and

[19] This restriction on the set of distributions is necessary for Theorem 1 to hold.

all the relative aspects of well-being are captured via the income standard. When mean income captures the income standard, balanced transfers among non-poor individuals do not affect the poverty index.

Subgroup Consistency is a standard axiom requiring that, if poverty decreases in a subgroup while it remains constant in the rest of the distribution, overall poverty must decline. Sen (1992) questioned the desirability of this axiom by arguing that the incomes in one subgroup may affect poverty in another subgroup. Foster and Sen (1997) recommend not to use this axiom when the index aims at capturing relative aspects of income poverty. I subscribe to this point of view. The issue becomes transparent once the channel through which one subgroup affects the other is modeled. In this model, the incomes in a subgroup impact the mean income which affects the other poor individuals' well-being. If the line is flat (see Figure 2.a), the relative situation does not matter and subgroup consistency is compelling. If the relative situation does matter, then it is not always meaningful to extrapolate the judgments made on subgroups to the whole population. *Weak Subgroup Consistency* restricts such extrapolations to cases for which the two subgroups of a population have the same mean income. Then, the well-being of an individual is the same when considering the mean income in her subgroup or when considering the mean income in the total population. In such cases, poverty judgments made on subgroups are relevant for the total population.

Poverty axiom 2 (*Weak Subgroup Consistency*).
For all $\succeq \in \mathcal{R}$ and all $y^1, y^2, y^3, y^4 \in Y(\succeq)$, if

$n(y^1) = n(y^3)$, $n(y^2) = n(y^4)$,

$\overline{y}^1 = \overline{y}^2$ *and* $\overline{y}^3 = \overline{y}^4$ *and*

$P(y^1, \succeq) > P(y^3, \succeq)$ *and* $P(y^2, \succeq) = P(y^4, \succeq)$,

then $P((y^1, y^2), \succeq) > P((y^3, y^4), \succeq)$.

The remaining three auxiliary axioms are straightforward adaptations of their classic counterparts. *Symmetry* requires that individuals' identities do not matter. Working with sorted distributions is therefore without loss of generality.

Poverty axiom 3 (*Symmetry*).
For all $\succeq \in \mathcal{R}$ and all $y, y' \in Y(\succeq)$ with $n(y) = n(y')$, if $y' = y \cdot \pi_{n(y)\times n(y)}$ for some permutation matrix $\pi_{n(y)\times n(y)}$, then $P(y, \succeq) = P(y', \succeq)$.

Symmetry implies that the preferences over bundles of the concerned individuals are irrelevant to the poverty index. This property generates little debate in Sen's model where only the level of income is relevant. If preferences are monotonic, then the property does not override individual preferences. In the extended model, individual preferences may differ from the EO and *Symmetry* explicitly requires to completely disregard them. This form of paternalism can be defended on the ground that it prevents poverty indices from giving priority to individuals that are more other-regarding.[20]

Continuity requires indices to be continuous in incomes. This is important for empirical applications in order to avoid that measurement errors have an excessive impact on poverty judgments.

Poverty axiom 4 (*Continuity*).
For all $\succeq \in \mathcal{R}$ and all $y \in Y(\succeq)$, P is continuous in y.

Replication Invariance specifies how to compare poverty in distributions of different population sizes. If a distribution is obtained by replicating another one several times, then the latter's poverty equals that of the original distribution.

Poverty axiom 5 (*Replication Invariance*).
For all $\succeq \in \mathcal{R}$ and all $y, y' \in Y(\succeq)$, if $n(y') = kn(y)$ for some positive integer k and $y' = (y, y, \ldots, y)$, then $P(y, \succeq) = P(y', \succeq)$.

[20] For an illustration of the issue, consider two poor individuals living in the same society. Assume that the individual with a larger income has preferences that are more affected by relative income than the preferences of the individual with a smaller income. If individual preferences matter for the poverty index, it could be that the relatively richer contributes more to poverty than the second individual. Potentially, giving an extra unit of income to the relatively richer has more impact on poverty than giving an extra unit of income to the second individual. Such conclusion is highly debatable: as they live in the same society, the bundle of the first individual dominates the bundle of the second individual.

The combination of these five axioms allows to derive an extension of the additive separability result of Foster and Shorrocks (1991). Two definitions simplify its formal statement. First, a *numerical representation* is a continuous function representing an EO.

Definition 3 (Numerical Representation $d^{\succeq}$)**.**
The continuous function $d^{\succeq} : X \to [0,1]$ *is a numerical representation of* $\succeq$ *if*

- *for all* $(y_i, \overline{y}), (y'_i, \overline{y}') \in X_p(\succeq)$ *we have*

$$(y_i, \overline{y}) \succeq (y'_i, \overline{y}') \Leftrightarrow d^{\succeq}(y_i, \overline{y}) \leq d^{\succeq}(y'_i, \overline{y}'),$$

- *for all* $(y_i, \overline{y}) \in X \backslash X_p(\succeq)$ *we have* $d^{\succeq}(y_i, \overline{y}) = 0$,
- *for all* $(0, \overline{y}) \in X_p(\succeq)$ *we have* $d^{\succeq}(0, \overline{y}) = 1$.

Observe that a numerical representation differs from a utility representation of the relevant EO. The value returned by the numerical representation *decreases* when individual well-being increases. The value returned by this function corresponds to the individual contribution to poverty, i.e. the opposite of utility.

Next, additive poverty indices aggregate individuals' well-being by summing them.

Definition 4 (Additive Poverty Index)**.**
Index P *is an additive poverty index if it is ordinally equivalent to an index* $\hat{P} : \mathcal{P} \to [0,1]$ *defined by*

$$\hat{P}(y, \succeq) := \frac{1}{n(y)} \sum_{i=1}^{n(y)} d^{\succeq}(y_i, \overline{y}), \tag{3}$$

where $d^{\succeq}$ *is a numerical representation of* $\succeq$.

Theorem 1 characterizes the family of additive poverty indices based on non-absolute lines.

Theorem 1 (Characterization of additive poverty indices)**.**
The following two statements are equivalent.

1. P *is an additive poverty index.*
2. P *satisfies Domination, Weak Subgroup Consistency, Symmetry, Continuity and Replication Invariance.*

Proof. It is easy to check that additive poverty indices satisfy these five axioms, so the proof that statement 1 implies statement 2 is omitted. The proof of the converse implication is in Appendix 7.2. In a nutshell, the proof shows that the result on additive separability of Gorman (1968) applies. The crucial assumption to verify is that the index satisfies a separability property. This separability property follows from the repeated application of *Weak Subgroup Consistency*. This repeated application is made possible by the flexibility obtained from one restriction imposed on the domain $Y(\succeq)$ of distributions, namely that individual n is non-poor. After applying Theorem 1 in Gorman (1968), the remaining part of the proof is an adaptation of Foster and Shorrocks (1991). ■

The main difference with the result of Foster and Shorrocks (1991) is that an individual's contribution to poverty depends on both her level of income and mean income. Her contribution to poverty is returned by a numerical representation of the relevant EO. If the poverty line is flat, then her well-being only depends on her income and so does her contribution to poverty. Therefore, the result of Foster and Shorrocks (1991) is nested in Theorem 1.

The validity of Theorem 1 can be extended in at least two directions.[21] First, it still holds if the EO restrictions Priority to Income below z^a and Homotheticity above z^a are replaced by the weaker restrictions Minimal Absolute Concern and Translation Monotonicity (defined in Appendix 7.1.2). Second, it is possible to extend its domain of application to indices based on relative lines, if one excludes null distributions from the domain of distributions.

[21] The proof can be found in an earlier version of this paper (Decerf, 2015b).

4.2 Two Key Fairness Axioms

Theorem 1 characterizes poverty indices that sum the contributions to poverty returned by a numerical representation of the EO. This theorem places almost no restriction on acceptable numerical representations. Such restrictions emerge from properties that define the particular cases for which it is deemed justified to improve the bundle of one individual at the detriment of another. I present here two key properties of this kind, *Monotonicity in Income* and *Transfer*, and study the constraints they place on the numerical representation.

The first property is *Monotonicity in Income*. Poverty indices based on non-absolute lines capture the relative aspects of income. The increase in the income of a poor individual has opposing impacts. On the one hand, her well-being increases as both her absolute and relative situations improve. On the other hand, mean income increases. If the income threshold increases, then the well-being of the other relatively poor individuals decreases.[22] Moreover, some individuals who were non-poor might have become poor. *Monotonicity in Income* requires that the indirect adverse impacts are dominated by the direct impact. Hence, this property imposes that decreasing the income of some *poor* individual never leads to an unambiguous poverty reduction. Observe that the larger the number of individuals, the lower is the impact of such an income increase on mean income and, hence, on the well-being of others.

Poverty axiom 6 (*Monotonicity in Income*)**.**
For all $\succeq \in \mathcal{R}$ and all $y, y' \in Y(\succeq)$ with $n(y) = n(y')$, if $y_i < y'_i < z(\overline{y}')$ and $y'_j = y_j$ for all $j \neq i$, then $P(y, z^*) \geq P(y', z^*)$.

Theorem 2 provides a necessary and sufficient condition for *Monotonicity in Income*. This condition links the two partial derivatives of the numerical representation. I denote by $\overline{y}^m(\succeq)$ the mean income $\overline{y}^m = \max_{\overline{y} \geq 0}\{\overline{y} \mid z(\overline{y}) = z^m\}$ where $z = Z(\succeq)$ is the line associated to the EO.

Theorem 2 (Condition for *Monotonicity in Income*)**.**
Let P be an additive poverty index whose numerical representation is almost everywhere differentiable. P satisfies Monotonicity in Income if and only if for all $\succeq \in \mathcal{R}$, $\overline{y} \geq \overline{y}^m(\succeq)$, $a \in [0, z(\overline{y}))$ and $b \in (z^a, z(\overline{y})]$, we have:[23]

$$\partial_2 d^{\succeq}(b, \overline{y}) \leq -\partial_1 d^{\succeq}(a, \overline{y}) \tag{4}$$

Proof. Consider any additive index P and any $\succeq \in \mathcal{R}$. Given $\succeq$, P satisfies *Monotonicity in Income* if and only if for all $y \in Y(\succeq)$ and $i \leq q(y)$ we have

$$\frac{\partial P((y_1, \ldots, y_n), \succeq)}{\partial y_i} \leq 0.$$

By the additively separable form of P, this inequality becomes by chain derivation:

$$\partial_1 d^{\succeq}(y_i, \overline{y}) + \sum_{j=1}^{n(y)} \partial_2 d^{\succeq}(y_j, \overline{y}) \partial_i \overline{y} \leq 0. \tag{5}$$

From the definition of the mean, we have $\partial_i \overline{y} = \frac{1}{n(y)}$. Inequality (5) becomes:

$$\underbrace{\partial_1 d^{\succeq}(y_i, \overline{y}) + \frac{1}{n(y)} \sum_{j=1}^{n(y)} \partial_2 d^{\succeq}(y_j, \overline{y}) \leq 0.}_{L_6} \tag{6}$$

The proof shows that (6) implies the necessary and sufficient condition linked to (4). Inequality (4) is equivalent to:

$$\underbrace{\partial_1 d^{\succeq}(a, \overline{y}) + \partial_2 d^{\succeq}(b, \overline{y})}_{L_7} \leq 0. \tag{7}$$

[22] The well-being of an absolutely poor individual only depends on her level of income and is therefore not affected by the income of other poor individuals.

[23] If function $d^{\succeq}$ is not differentiable at $(a, \overline{y})$ or at $(b, \overline{y})$, then I define these derivatives as

$$\partial_1 d^{\succeq}(a, \overline{y}) := \lim_{x \to a^+} \partial_1 d^{\succeq}(x, \overline{y}) \qquad \text{and} \qquad \partial_2 d^{\succeq}(b, \overline{y}) := \lim_{x \to \overline{y}^+} \partial_2 d^{\succeq}(b, x).$$

Necessity is proven by contradiction. If inequality (4) is violated for some values of a, b and $\overline{y}$, then one can construct a distribution y at which *Monotonicity in Income* does not hold, provided that $n(y)$ is large enough. Distribution y is such that one poor individual earns a, the $n(y)-2$ other poor individuals earn b and the non-poor individual $n(y)$ earns an amount computed to obtain the desired mean income. Sufficiency is proven by showing that if there exists a distribution y at which *Monotonicity in Income* does not hold, then inequality (4) is violated as well for at least a pair of poor individuals i and j in distribution y. The detailed proof is in Appendix 7.3. ■

The condition derived in Theorem 2 has strong discriminative power, as shown below in Theorem 6. Applying the condition may seem cumbersome since it requires checking an infinity of inequalities of the form given in (4). Nevertheless, the relationships existing between the two partial derivatives (derived in Lemmas 2 and 3 in Appendix 7.4.1) simplify its application. In particular, if the poverty line is weakly relative, then having inequalities (4) holding at $\overline{y} = \overline{y}^m$ is necessary and sufficient for *Monotonicity in Income*, which brings a further simplification.[24]

Observe that *Domination* and *Monotonicity in Income* are logically related when the poverty line is flat. if the line is flat, then the partial derivative of the numerical representation with respect to mean income equals zero, and the condition of Theorem 2 is trivially verified. More generally, *Domination* implies *Monotonicity in Income* when the line is flat. In Sen's model, the poverty line is automatically flat and any increase in a poor individual's income has not adverse impact.

The second property is *Transfer*. *Transfer* requires that a Pigou-Dalton transfer taking place between two poor individuals never unambiguously increases poverty.[25] This property conserves its normative appeal when mean income is the income standard since balanced transfers do not alter the mean. As a result, the poverty contributions of individuals not involved in the transfer are unchanged.

Poverty axiom 7 (*Transfer*)**.**
For all $\succeq \in \mathcal{R}$, all $y, y' \in Y(\succeq)$ with $n(y) = n(y')$ and all $\lambda > 0$, if $y_j - \lambda = y'_j > y'_k = y_k + \lambda$, $z(\overline{y}) > y_j$ and $y'_i = y_i$ for all $i \neq j, k$, then $P(y, \succeq) \geq P(y', \succeq)$.

As in Sen's model, poverty indices satisfying *Transfer* are based on convex numerical representations.

Theorem 3 (Condition for *Transfer*)**.**
Let P be an additive poverty index whose numerical representation is almost everywhere differentiable. P satisfies Transfer if and only if for all $\succeq \in \mathcal{R}$ and all $a, b \geq 0$ with $a < b < z^m$ we have:[26]

$$\partial_1 d^{\succeq}(a, \overline{y}^m) \leq \partial_1 d^{\succeq}(b, \overline{y}^m) \tag{8}$$

Proof. The proof is in the Appendix 7.4. ■

The relationship existing between the two partial derivatives imply that having inequalities (8) holding at $\overline{y} = \overline{y}^m$ is sufficient for *Transfer*.

Together, *Monotonicity in Income* and *Transfer* strongly constrain acceptable numerical representations. These constraints are best illustrated by applying the conditions derived in theorems 2 and 3 to parametric families of numerical representations. In particular, Decerf (2015a) shows that these two properties jointly characterize a unique index in a version of the Foster-Greer-Thorbecke family adapted for the extended model.

In the next section, I present robustness properties, which constrain the functional expression of acceptable numerical representations. In Section 5.3, I apply to this functional expression the conditions derived in Theorems 2 and 3 and obtain sharp implications.

[24] A proof for this claim is a straightforward adaptation from Step 1 of the proof of Theorem 3 in Decerf (2015b) page 104.

[25] A Pigou-Dalton transfer is a progressive balanced transfer preserving the relative ranks of the two individuals involved in the transfer.

[26] If function $d^{\succeq}$ is not differentiable at $(a, \overline{y})$ or at $(b, \overline{y})$, then I define these as

$$\partial_1 d^{\succeq}(a, \overline{y}) := \lim_{x \to a^+} \partial_1 d^{\succeq}(x, \overline{y}) \qquad \text{and} \qquad \partial_1 d^{\succeq}(b, \overline{y}) := \lim_{x \to b^-} \partial_1 d^{\succeq}(x, \overline{y}).$$

5 Characterization of a particular index

Decerf (2015a) proposes an additive index satisfying *Monotonicity in Income* and *Transfer* and whose underlying EO satisfies all restrictions presented. The remainder of this paper proposes a full characterization of that index. The novelty of this characterization is that the properties defining the index are automatically inherited by any relativist measure based on this index.

The additional properties necessary for its characterization are four robustness properties. Robustness properties require that the judgments made by the poverty index stay consistent when the EO undergoes specific modifications. For some of these robustness properties, these modifications are also applied to the distributions considered. Two properties are specific to the extended model whereas the other two are adaptations of robustness properties studied in Sen's model.

5.1 Specific Properties

The first two robustness properties are new and specific to this extended model. Each investigates how the comparison of two distributions evolves when the EO is altered in a particular way.

Line relativism independence in poor societies is restricted to the comparison of two distributions sharing the same "low" value of income standard. Mean income is considered as "low" when smaller than $\overline{y}^m$, the smallest value of mean income at which the poverty line is non-flat. *Line relativism independence in poor societies* requires that if two EOs share the same absolute threshold and their poverty thresholds are equal for low values of mean income, then their ranking of two low-income distributions is the same. In other words, how relative the line is defined for high-income distributions is irrelevant to poverty comparisons of low-income distributions, for which the line is flat.

Poverty axiom 8 (*Line relativism independence in poor societies*).
For all $\succeq, \succeq' \in \mathcal{R}$ *with*

$z^m = z^{m'}$ *where* $z = Z(\succeq)$ *and* $z' = Z(\succeq')$, *and*

$Z^a(\succeq) = Z^a(\succeq')$,

and all $y, y' \in Y(\succeq) \cap Y(\succeq')$ *with* $\overline{y} = \overline{y}' \leq \min\{\overline{y}^m(\succeq), \overline{y}^m(\succeq')\}$ *and* $n(y) = n(y')$ *we have*

$$P(y, \succeq) \geq P(y', \succeq) \Leftrightarrow P(y, \succeq') \geq P(y', \succeq')$$

Weak monotonicity in absolute threshold links the judgments of EOs that share the same poverty line. It requires that the poverty attributed to a given distribution cannot decrease when the absolute threshold is increased. A larger absolute threshold implies that absolutely poor individuals are further away from the absolute threshold and relatively poor individuals are closer to the absolute threshold (more at risk of becoming absolutely poor). Also, some relatively poor individuals may become absolutely poor. *Weak monotonicity in absolute threshold* restricts this requirement to distributions whose mean income is below a certain value in order to escape the impossibility for additive axioms satisfying *Transfer* to satisfy the strong version of this axiom.[27]

Poverty axiom 9 (*Weak monotonicity in absolute threshold*).
For all $\succeq, \succeq' \in \mathcal{R}$ *with*

$Z(\succeq) = Z(\succeq')$ *and*

$Z^a(\succeq) < Z^a(\succeq')$,

and for all $y \in Y(\succeq) \cap Y(\succeq')$ *with* $\overline{y} \leq \overline{y}^m(\succeq) = \overline{y}^m(\succeq')$ *we have*

$$P(y, \succeq) \leq P(y, \succeq').$$

Theorem 4 exposes the joint implications of these two axioms. Its presentation requires the introduction of the concept of *equivalent income function*. The definition of this function is built on the poverty line and the absolute threshold of the EO considered. For a poor individual, the equivalent income at mean income $\overline{y}^r$ is the level of income yielding – when mean income is $\overline{y}^r$ – the same well-being as

[27] See appendix 7.5 for a proof of this impossibility.

her current bundle. Formally, for a given EO, a given value of mean income $\overline{y}^r$ and a given bundle for individual i, this means

$$(y_i, \overline{y}) \;\sim\; \left(e^r_{\succeq}(y_i, \overline{y}), \overline{y}^r\right).$$

The definition of the equivalent income function is given in equation (9). This definition is in three parts. These parts specify the equivalent income for absolutely, relatively and non-poor individuals respectively.

Definition 5 (Equivalent income function at $\overline{y}^r$)**.**
Take any $\succeq \in \mathcal{R}$ and any $\overline{y}^r > 0$. For all $y \in Y(\succeq)$ and any individual $i \leq n(y)$, the equivalent income function at $\overline{y}^r$ is $e^r_{\succeq} : X \to [0, z(\overline{y}^r)]$ defined by

$$e^r_{\succeq}(y_i, \overline{y}) \coloneqq \begin{cases} y_i & \text{if } \; y_i < z^a, \\ z^a + (y_i - z^a)\frac{z(\overline{y}^r) - z^a}{z(\overline{y}) - z^a} & \text{if } y_i \in [z^a, z(\overline{y})), \\ z(\overline{y}^r) & \text{if } \; y_i \geq z(\overline{y}). \end{cases} \tag{9}$$

The first two robustness properties force the numerical representation of the EO to depend only on two parameters: the line's intercept and the absolute threshold. Remember that the intercept of the poverty line $z = Z(\succeq)$ is denoted by $z^m = z(\overline{y}^m)$.

Theorem 4 (Numerical representation based on two parameters)**.**
The following two statements are equivalent.

1. *P is an additive index that satisfies Line relativism independence in poor societies and Weak monotonicity in absolute threshold.*
2. *P is ordinally equivalent to an index $\hat{P} : \mathcal{P} \to [0, 1]$ defined by*

$$\hat{P}(y, \succeq) \coloneqq \frac{1}{n(y)} \sum_{i=1}^{n(y)} f\left(e^m_{\succeq}(y_i, \overline{y}), z^m, z^a\right), \tag{10}$$

where

- *$z^a = Z^a(\succeq)$ and $z = Z(\succeq)$,*
- *f is continuous in its first argument, non-decreasing in its third argument, strictly decreasing in its first argument on $[0, z^m)]$ and $f(0, z^m, z^a) = 1$ and $f(b, z^m, z^a) = 0$ for all $b \geq z^m$.*

Proof. Here is the intuition why statement 1 implies statement 2. Take any numerical representation of any EO. Its mathematical expression depends on the value of mean income at which it is expressed. Select (arbitrarily) the low value $\overline{y}^m$, as the value of mean income at which it is expressed. The preconditions of property *Line relativism independence in poor societies* forces any two EOs featuring the same z^a and z^m to perform the same poverty comparisons among distributions with low-income. Their associated indices are therefore based on the same function f. Function f completely defines a unique numerical representation. The evaluation of distributions whose mean income is different from $\overline{y}^m$ is done by evaluating "equivalent" distributions. For any distribution y, an equivalent distribution with mean income equal to $\overline{y}^m$ is obtained by considering for each individual the equivalent income at $\overline{y}^m$, which is returned by function $e^m_{\succeq}$. See Appendix 7.6 for the detailed proof. ■

Theorem 4 relates the numerical representation of EOs sharing the same values for the two parameters. These numerical representations have the same mathematical expression at mean income equal to $\overline{y}^m$. When the two parameters are equal, the contribution attributed to a given bundle depends on the EO considered only to the extent that it changes the equivalent income at $\overline{y}^m$. These implications are directly derived from *Line relativism independence in poor societies*. In turn, *Weak monotonicity in absolute threshold* implies that the contribution weakly increases with the absolute threshold.[28]

[28] The contribution must weakly increase with the intercept as well if one imposes *Monotonicity in line*, an additional property presented in Appendix 7.1.2.

5.2 Adapted Properties

The last two properties are *Scale consistency* and *Translation consistency*. These properties constrain the poverty judgments when the incomes of the poor individuals and the EO are submitted to the same translation or when they are scaled using the same factor. These properties are adaptations for the extended model of the homonymous properties studied in Sen's model. There is a key difference between the implications of the original properties in Sen's model and the implications of their adapted versions in the extended model: the adapted versions do not constrain the comparisons of individual well-being across distributions with different income standard. Such contraints are unavoidable in Sen's model, as shown by Zheng (2007). In other words, the original versions constrain the comparison of individual bundles, i.e. constrain the EO.

In Sen's model, a bundle's contribution to poverty may only depend on its absolute distance to the poverty threshold (Translation consistency) or on its normalized distance to the poverty threshold (Scale consistency). In the extended model, Theorem 5 shows that the adapted versions imply an FGT expression for function f in the particular cases for which the absolute threshold is zero or equal to the intercept. This result is weaker than a similar result derived in Sen's model (Ebert and Moyes, 2002). Since function f is non-decreasing in the absolute threshold, Theorem 5 provides for function f an upper and a lower-bound with exponential expression.

The adaptation of these two properties to the extended framework requires to introduce specific transformations of income distributions and EOs. The definition of these transformations is a bit technical.

Scale consistency is a robustness axiom requiring that if the poverty threshold and the income level of poor individuals are multiplied by the same amount and the EO is transformed accordingly, then the judgments are consistent. In other words, the contributions to poverty depend only on the normalized distance to the poverty threshold.

I define a transformation of any distribution y denoted by $y^{\times\lambda}$. The incomes of all poor individuals in $y^{\times\lambda}$ are equal to λ times the income of their counterpart in y and the incomes of non-poor individuals in $y^{\times\lambda}$ are adapted in order to preserve mean income. For all $y \in Y$ and all $\lambda > 0$ we have

$$y^{\times\lambda} := \left(\lambda y_1, \ldots, \lambda y_q, \frac{n\overline{y} - \sum_{j=1}^{q} \lambda y_j}{n-q}, \ldots, \frac{n\overline{y} - \sum_{j=1}^{q} \lambda y_j}{n-q}\right).$$

For all $\lambda > 0$ and all $\succeq \in \mathcal{R}$, I define the transformation $\succeq^{\times\lambda}$ by:

$$(y_i, \overline{y}) \succeq (y_i', \overline{y}') \Leftrightarrow (\lambda y_i, \overline{y}) \succeq^{\times\lambda} (\lambda y_i', \overline{y}') \qquad \text{for all } (y_i, \overline{y}), (y_i', \overline{y}') \in X.$$

The definition of $\succeq^{\times\lambda}$ implies that for all $(y_i, \overline{y}) \in X$ and all $\overline{y}^r > 0$ we have

$$e^r_{\succeq^{\times\lambda}}(\lambda y_i, \overline{y}) = \lambda e^r_{\succeq}(y_i, \overline{y}).$$

Observe that there is no guarantee that $y^{\times\lambda}$ belongs to $Y(\succeq^{\times\lambda})$ when y belongs to $Y(\succeq)$, although $\lambda \leq 1$ is a sufficient condition for this implication.

Consider the following important technical remark. If $Z(\succeq)$ is flat, then $Z^a(\succeq^{\times\lambda})$ is not well-defined, given that any $z^a \in [0, z^m(\succeq^{\times\lambda})]$ is such that $\succeq^{\times\lambda}$ meets Priority to Income below z^a when we have $Z^a(\succeq^{\times\lambda}) = z^a$. I relate the definition of $Z^a(\succeq^{\times\lambda})$ to the definition of $Z^a(\succeq)$. This definition requires that an individual that is relatively poor in y has its counterpart relatively poor in $y^{\times\lambda}$. The same holds true for absolutely poor individuals. Hence, $Z^a(\succeq^{\times\lambda})$ is the unique value in $[0, z^m(\succeq^{\times\lambda})]$ such that for all $y \in Y(\succeq)$ and $y^{\times\lambda} \in Y(\succeq^{\times\lambda})$ we have that $q^a(y) = q^a(y^{\times\lambda})$ given $\succeq$ and $\succeq^{\times\lambda}$ respectively.

Poverty axiom 10 (*Scale consistency*)**.**
For all $\succeq \in \mathcal{R}$, all $\lambda > 0$ and all $x, y \in Y(\succeq)$ with $\overline{x} = \overline{y}$, if

$\succeq^{\times\lambda} \in \mathcal{R}$ *and*

$x^{\times\lambda}, y^{\times\lambda} \in Y(\succeq^{\times\lambda})$,

then we have that

$$P(x, \succeq) \geq P(y, \succeq) \Leftrightarrow P(x^{\times\lambda}, \succeq^{\times\lambda}) \geq P(y^{\times\lambda}, \succeq^{\times\lambda}).$$

Analogously, *Translation consistency* is a robustness axiom requiring that if the poverty threshold and the income level of poor individuals are translated by the same amount and the EO is transformed accordingly, then the judgments are consistent. In other words, the contributions to poverty depend only on the absolute distance to the poverty threshold.

I define a transformation of any distribution y denoted by $y^{+\delta}$. The income of all poor individuals in $y^{+\delta}$ are equal to δ plus the income of their counterpart in y and the incomes of non-poor individuals in $y^{+\delta}$ are adapted in order to preserve mean income. For all $y \in Y$ and all $\delta \in \mathbb{R}$ we have

$$y^{+\delta} := \left(y_1 + \delta, \dots, y_q + \delta, \frac{n\overline{y} - \sum_{j=1}^{q}(y_j + \delta)}{n-q}, \dots, \frac{n\overline{y} - \sum_{j=1}^{q}(y_j + \delta)}{n-q} \right).$$

For all $\delta \in \mathbb{R}$ and all $\succeq \in \mathcal{R}$, I define the transformation $\succeq^{+\delta}$ by:

$$(y_i, \overline{y}) \succeq (y_i', \overline{y}') \Leftrightarrow (y_i + \delta, \overline{y}) \succeq^{+\delta} (y_i' + \delta, \overline{y}') \qquad \text{for all } (y_i, \overline{y}), (y_i + \delta, \overline{y}), (y_i', \overline{y}') \text{ and } (y_i' + \delta, \overline{y}') \in X.$$

The definition of $\succeq^{+\delta}$ implies that for all $(y_i, \overline{y}) \in X$ and all $\overline{y}^r > 0$ we have

$$e^r_{\succeq^{+\delta}}(y_i + \delta, \overline{y}) = e^r_{\succeq}(y_i, \overline{y}) + \delta.$$

Observe that $\succeq^{+\delta} \in \mathcal{R}$ for all $\delta > -Z^a(\succeq)$. If the line of $\succeq$ is non-flat, then $\succeq^{+\delta} \notin \mathcal{R}$ for all values $\delta \in (-z^m(\succeq), -Z^a(\succeq))$, because its iso-poverty curves violate Homotheticity above z^a.[29] The same technical remark about the definition of $Z^a(\succeq^{+\delta})$ as the one made above about $Z^a(\succeq^{\times\lambda})$ holds for EOs having flat lines.

Observe that there is no guarantee that $y^{+\delta}$ belongs to $Y(\succeq^{+\delta})$ when y belongs to $Y(\succeq)$, although $\delta \in [-\min\{y_1, Z^a(\succeq)\}, 0]$ is a sufficient condition for this implication.

Poverty axiom 11 (*Translation consistency*).
For all $\succeq \in \mathcal{R}$, all $\delta \in \mathbb{R}$ and all $x, y \in Y(\succeq)$ with $\overline{x} = \overline{y}$, if

$\succeq^{+\delta} \in \mathcal{R}$ *and*

$x^{+\delta}, y^{+\delta} \in Y(\succeq^{+\delta})$,

then we have that

$$P(x, \succeq) \geq P(y, \succeq) \Leftrightarrow P(x^{+\delta}, \succeq^{+\delta}) \geq P(y^{+\delta}, \succeq^{+\delta}).$$

Theorem 5 shows that *Scale consistency* and *Translation consistency* imply exponential bounds for the function f defining the individual contributions to poverty.

Theorem 5 (FGT bounds on poverty indices).
Any index P that is ordinally equivalent to (10) satisfies Scale consistency and Translation consistency only if for all $\succeq \in \mathcal{R}$ with $Z^a(\succeq) = 0$ and all $(y_i, \overline{y}) \in X_p$ we have

$$f\left(e^m_{\succeq}(y_i, \overline{y}), z^m, 0\right) = \left(1 - \frac{e^m_{\succeq}(y_i, \overline{y})}{z^m}\right)^{\alpha_0}, \tag{11}$$

and if for all $\succeq \in \mathcal{R}$ with $Z^a(\succeq) = z^m$ and all $(y_i, \overline{y}) \in X_p$ we have

$$f\left(e^m_{\succeq}(y_i, \overline{y}), z^m, z^m\right) = \left(1 - \frac{e^m_{\succeq}(y_i, \overline{y})}{z^m}\right)^{\alpha_1}, \tag{12}$$

where $\alpha_0 \geq \alpha_1 > 0$.

Proof. The proof is based on the subset of EOs whose poverty line is flat. On this subset, the extended model is equivalent to the framework studied by Ebert and Moyes (2002). These authors show in the model of Sen that these two properties lead to poverty orderings representable by FGT indices. The relative size of α_0 and α_1 follows from the fact that function f is non-decreasing in its third argument. The detailed proof is in Appendix 7.7. ■

[29] More fundamentally, $\succeq^{+\delta} \notin \mathcal{R}$ for all $\delta \in (-z^m(\succeq), -Z^a(\succeq))$ because its iso-poverty curves violate Minimal Absolute Concern.

Theorem 5 proposes two necessary conditions for *Scale consistency* and *Translation consistency*. These conditions respectively specify, up to a parameter α, the numerical representation for EOs whose absolute threshold is equal to zero and for EO's whose absolute threshold is equal to the intercept of their poverty line. What is more, the relative size of parameters α_0 and α_1 is constrained. Given that function f is non-decreasing in its third argument, these conditions provide an upper- and a lower-bound for the numerical representation of EOs whose absolute threshold has intermediary value.

The conditions in Theorem 5 are necessary but not sufficient for these two properties. One may wonder whether there exists an index ordinally equivalent to (10) satisfying *Scale consistency* and *Translation consistency*. In Appendix 8, I show that any index that has the FGT form with exponent $\alpha_0 = \alpha_1 = \alpha > 0$ is an example of such index.

5.3 Characterization

The robustness properties imply that the individual contribution to poverty is given by a unique function f that depends on two parameters. Moreover, function f has exponential form for the two extreme values of the absolute threshold parameter. Theorem 6 shows that there is a unique index meeting these exponential bounds that satisfies *Monotonicity in Income* and *Transfer*.[30] In other words, Theorem 6 shows that all the properties jointly characterize a unique index. This index corresponds to the index proposed in Decerf (2015a). This result is the first characterization of an index designed for non-absolute poverty lines. The index is a version of the Poverty Gap Ratio that compares individual well-beings according to the EO considered.

Theorem 6 (Characterization of an index)**.**
The following two statements are equivalent.

1. *Index P is ordinally equivalent to* (10) *and satisfies Scale consistency, Translation consistency, Monotonicity in Income and Transfer.*
2. *P is ordinally equivalent to index $\hat{P} : \mathcal{P}_{\rightarrow}[0,1]$ defined by*

$$\hat{P}(y, \succeq) := \frac{1}{n(y)} \sum_{i=1}^{n(y)} \left(1 - \frac{e^m_{\succeq}(y_i, \overline{y})}{z^m}\right),$$

where

$$e^m_{\succeq}(y_i, \overline{y}) := \begin{cases} y_i & \text{if } \ y_i < z^a, \\ z^a + (y_i - z^a)\frac{z^m - z^a}{z(\overline{y}) - z^a} & \text{if } y_i \in [z^a, z(\overline{y})), \\ z^m & \text{if } \ y_i \geq z(\overline{y}). \end{cases}$$

Proof. The proof is in three steps, whose respective proofs are inspired from the proof of Theorem 1 in Decerf (2015a). First, I show that the index satisfies *Monotonicity in Income* only if the lower-bound on f given by (11) is such that $\alpha_0 = 1$. That is, function f is linear in its first argument when $z^a = 0$. Second I show that, when the lower-bound on f is given by (11) with $\alpha_0 = 1$, the index satisfies *Transfer* only if

$$f\left(e^m_{\succeq}(y_i, \overline{y}), z(\overline{y}^m), z^a\right) = f\left(e^m_{\succeq}(y_i, \overline{y}), z(\overline{y}^m), 0\right) \qquad \text{for all } z_a \in [0, z^m],$$

i. e. function f takes the expression of its lower-bound for all values of the absolute threshold. Here is the intuition for this result. By Theorem 4, we have $f(0, z^m, z^a) = 1$ and $f(z^m, z^m, z^a) = 0$ for all $z_a \in [0, z^m]$. *Transfer* forces f to be convex in its first argument. If function f has a linear expression when $z^a = 0$ and function f is increasing in its parameter z^a, the only way for f to stay convex in its first argument when z^a increases is to maintain its linear expression. Therefore, function f is independent on the absolute threshold. Third, I show that this index satisfies both *Monotonicity in Income* and *Transfer*. The detailed proof is in Appendix 9. ■

[30] As shown in Appendix 8, the index characterized satisfies *Scale consistency* and *Translation consistency*.

Theorem 6 illustrates the strong discriminative power of *Monotonicity in Income* and *Transfer*. In order to get a sense of their combined power, remember that Theorem 5 only requires function f to evolve between two very wide bounds defined by (11) and (12). A detailed intuition for their combined power is provided elsewhere (Decerf, 2015a).

To some extent, the characterization also holds even if the index is used in combination with a flat line. The reason is that the ranking of distributions must be independent on how the line behaves beyond $\overline{y}^m$. However, this characterization does not hold if the set of poverty lines $\mathcal{Z}$ is restricted to only include flat poverty lines.

One could argue that the characterization is not complete in the sense that there remains one parameter: the absolute threshold. That is true. However, another perspective is to consider that any relativist measure based on this index needs two poverty lines: one hybrid line and one absolute line whose threshold is everywhere below the hybrid line.

6 Concluding remarks

Income inequalities have recently attracted more attention. Abstracting from the impacts that inequalities may have on behavior, there exists two main normative reasons why one may care about inequalities. The first is fairness. An external observer may prefer more equal distributions of resources. The second is that inequalities may have intrinsic value for the concerned individual. For instance, people's preferences may depend on both their absolute income and their relative income. Alternatively, the social functionings provided by a given amount of resources may depend on the society's standards of living. The second reason is the mainstream foundation used to defend relativist poverty measures. Any poverty measure endorsing such foundation must *first* aggregate the absolute and relative aspects of income at the individual level and *second* aggregate individual contributions over the whole population, as proposed by Atkinson and Bourguignon (2001). As these authors suggest, taking onboard the relative aspect of poverty is not only a matter of picking the right poverty line(s) but also a matter of selecting an appropriate index. Following their approach, Decerf (2015a) stresses the importance of the iso-poverty maps associated to relativist poverty measures. This paper integrates iso-poverty maps, formalized by the concept of ethical ordering, in the model used to study the properties of poverty measures.

More generally, this research participates to recent attempts at introducing relative aspects in normative judgments (Decerf and Van der Linden, 2014; Treibich, 2014). There are many potential issues linked to the such attempts. I mention two of them. First, normative judgments should not give excessive importance to the relative over the absolute aspect of one's situation. The imposition of appropriate restrictions to the relevant EO constitutes an obvious solution for relativist poverty measures. Second, one should not give more priority to jealous individuals than to self-centered or altruistic individuals. The introduction of an ethical ordering, which makes the same trade-offs for all individuals, is one way out of this second issue.

7 Appendix

7.1 Additional Axioms

7.1.1 Sen's model Axioms

Classical poverty axiom 1 (*Scale Invariance*).
For all $y \in Y$, $z^* \in \mathbb{R}_{++}$ and $\lambda > 0$, $P(y, z^*) = P(\lambda y, \lambda z^*)$.

Focus requires that the index is not sensitive to the income level of non-poor individuals.

Classical poverty axiom 2 (*Focus*).
For all $y, y' \in Y$ and $z^* \in \mathbb{R}_{++}$, if $n(y) = n(y')$, $q(y) = q(y')$ and $y_i = y'_i$ for all $i \leq q(y)$, then $P(y, z^*) = P(y', z^*)$.

Classical poverty axiom 3 (*Subgroup Consistency*).
For all $z^* \in \mathbb{R}_{++}$ and all $y^1, y^2, y^3, y^4 \in Y$ with $n(y^1) = n(y^3)$ and $n(y^2) = n(y^4)$, if $P(y^1, z^*) > P(y^3, z^*)$ and $P(y^2, z^*) = P(y^4, z^*)$, then $P((y^1, y^2), z^*) > P((y^3, y^4), z^*)$.

Monotonicity in Income requires that decreasing the income of some *poor* individual never leads to an unambiguous poverty reduction.

Classical poverty axiom 4 (*Monotonicity in Income*).
For all $y, y' \in Y$ and $z^* \in \mathbb{R}_{++}$, if $n(y) = n(y')$, $y_i < y'_i < z(\overline{y}')$ and $y'_j = y_j$ for all $j \neq i$, then $P(y, z^*) \geq P(y', z^*)$.

7.1.2 New model Axioms & Restrictions

Translation Monotonicity requires that any poor individual is considered weakly better-off after the equal distribution of an extra amount of income. Intuitively, an equal distribution of income cannot deteriorate the relative situation of a poor individual. The corollary of Translation Monotonicity is that the slope of the EO's iso-poverty curves is nowhere larger than one.

EO restriction 4 (Translation Monotonicity).
For all $(y_i, \overline{y}) \in X_p(\succeq)$ and $a > 0$, we have $(y_i + a, \overline{y} + a) \succeq (y_i, \overline{y})$.

Minimal Absolute Concern requires that an individual with zero income is considered strictly worse-off than another individual with non-zero income, regardless of the mean incomes in their respective societies.

EO restriction 5 (Minimal Absolute Concern).
For all $(y_i, \overline{y}), (0, \overline{y}') \in X_p(\succeq)$ with $y_i > 0$, we have $(y_i, \overline{y}) \succ (0, \overline{y}')$.

Weak Focus requires that the incomes of non-poor individuals is irrelevant to poverty, only as long as the income standard is unchanged.

Poverty axiom 12 (*Weak Focus*).
For all $\succeq \in \mathcal{R}$ and all $y, y' \in Y(\succeq)$, if $n(y) = n(y')$, $q(y) = q(y')$, $\overline{y} = \overline{y}'$ and $y_i = y'_i$ for all $i \leq q(y)$, then $P(y, \succeq) = P(y', \succeq)$.

Monotonicity in line links the judgments of EOs that share the same iso-poverty curves, but not necessarily the same line. It requires that an increase in the line that does not affect the individual comparisons below the old line cannot decrease poverty.

Poverty axiom 13 (*Monotonicity in line*).
For all $\succeq \in \mathcal{R}$, and all $\lambda > 1$, if $\succeq' \in \mathcal{R}$ is such that

$$Z^a(\succeq') = Z^a(\succeq)$$

$$z'(\overline{y}) = \lambda(z(\overline{y}) - z^a) + z^a \text{ for all } \overline{y} \geq 0,$$

where $z = Z(\succeq)$, $z' = Z(\succeq')$ and $z^a = Z^a(\succeq)$, then for all $y \in Y(\succeq)$ we have

$$P(y, \succeq) \leq P(y, \succeq').$$

7.2 Proof of Theorem 1

I only show that statement 2 implies statement 1 as the reverse implication is easily verified. Consider any poverty index P satisfying the axioms listed in statement 2. Take any EO $\succeq \in \mathcal{R}$.

STEP 1: Definition of a poverty ordering on a product space that is equivalent to the poverty ordering on $Y(\succeq)$.

In step 1, I show that the continous and complete ordering on $Y(\succeq)$ represented by P is equivalent to a continuous and complete poverty ordering on a product space

$$V_d := \cup_{n \in N'} [0, 1]^n,$$

where $N' := \{n \in \mathbb{N} | n \geq 2\}$, represented by P^V.

To do so, I construct a continuous mapping $m : Y(\succeq) \to V_d$ such that for any two $\nu, \nu' \in V_d$ and $y, y' \in Y(\succeq)$ with $m(y) = \nu$ and $m(y') = \nu'$ we have that

$$P^V(\nu) \geq P^V(\nu') \Leftrightarrow P(y, \succeq) \geq P(y', \succeq). \tag{13}$$

Definition of the mapping

Consider any numerical representation $d_1^{\succeq}$ of $\succeq$. I define a continuous mapping $m : Y(\succeq) \to V_d$. For all $y \in Y(\succeq)$, mapping m is defined by

$$m(y) = (d_1^{\succeq}(y_1, \overline{y}), \ldots, d_1^{\succeq}(y_{n-1}, \overline{y})) \coloneqq (\nu_1, \ldots, \nu_{n-1}) = \nu.$$

I emphasize two particularities of mapping m. First, the size of distribution y is $n(y)$, whereas the size of its image $\nu = m(y)$ is $n(y) - 1$. Second, the mapping m is not a one-to-one mapping as several income distributions may be mapped to the same image in V_d.[31] As I show below, these particularities do not prevent the ordering on V_d to have the desired properties.

Mapping m is continuous as $d_1^{\succeq}$ is continuous in both its arguments and the mean is a continuous function of its arguments.

I show that the mapping defined is such that

$$m(Y(\succeq)) = V_d.$$

That is, the domain of images of $Y(\succeq)$ through mapping m is the product space V_d. This means that (i) $m(Y(\succeq)) \subseteq V_d$ and (ii) $V_d \subseteq m(Y(\succeq))$. If (i) follows directly from the definition of mapping m, (ii) remains to be proven. Proving (ii) amounts to prove Lemma 1.

Lemma 1. *For all $\nu \in V_d$, there exists $y \in Y(\succeq)$ such that $\nu = m(y)$.*

Proof. Take any $\nu \in V_d$, sorted in non-increasing order $\nu_1 \geq \cdots \geq \nu_i \geq \cdots \geq \nu_{n-1}$, where $n - 1 \in N'$ denote the size of ν. Let $q \leq n-1$ denote the largest integer for which $\nu_q > 0$. I construct a distribution y such that $n(y) = n$, $q(y) = q$, $m(y) = \nu$ and show that $y \in Y(\succeq)$. The poverty line $z = Z(\succeq)$ belongs to $\mathcal{Z}$ by assumption as $\succeq \in \mathcal{R}$. Therefore, z meets Flat Line in Poor Societies and, hence, there exists $\overline{y}' > 0$ such that $\overline{y}' \geq z(\overline{y}')$. I construct distribution y such that $\overline{y} = \overline{y}'$:

- $y_i \coloneqq a_i$ defined implicitly by $\nu_i = d_1^{\succeq}(a_i, \overline{y}')$ for all $i \leq q$,
- $y_j \coloneqq \frac{n\overline{y}' - \sum_{k=1}^{q} y_k}{n-q}$ for all j with $q+1 \leq j \leq n$.

By construction, we have that $n(y) = n$, $q(y) \geq q$ and $\overline{y} = \overline{y}'$. I show that $y_j \geq z(\overline{y}')$, which implies that $q(y) = q < n(y)$ and, hence, $y \in Y(\succeq)$. By restrictions Priority to Income below z^a and Homotheticity above z^a and the continuity of $d_1^{\succeq}$, we have that $a_i \in [0, z(\overline{y}'))$ for all $i \leq q$. Therefore, we have that

$$\sum_{k=1}^{q} y_k \leq q\overline{y}'.$$

This inequality implies that $y_j \geq z(\overline{y}')$ since we have $\overline{y}' \geq z(\overline{y}')$. In words, all individuals j with $q+1 \leq j \leq n$ are non-poor and are such that $m(y_j) = 0$ by constuction of the mapping. By construction of y, we have that $m(y) = \nu$. ∎

The equivalent ordering on V_d

I define an ordering on V_d from the ordering on $Y(\succeq)$ represented by P. Let this ordering on V_d be represented by $P^V : V_d \to [0,1]$ defined by

$$P^V(\nu) = P^V\big(m(y)\big) = P(y, \succeq) \qquad \text{for all } \nu \in V_d,$$

where $y \in Y(\succeq)$ is such that $\nu = m(y)$. If such distribution y exists and is unique, then the ordering represented by P^V satisfies (13) by construction.

By Lemma 1, such $y \in Y(\succeq)$ exists, but this distribution needs not be unique as m is not a one-to-one mapping. Nevertheless, for all $y, y' \in Y(\succeq)$ with $\nu = m(y) = m(y')$, we have $P(y, \succeq) = P(y', \succeq)$ by *Domination*.

Therefore, this ordering on V_d is equivalent to the ordering on $Y(\succeq)$ represented by $P(y, \succeq)$. Taking the size of ν to be $n(y) - 1$ instead of $n(y)$ is without loss of generality as for all $y \in Y(\succeq)$ we have by

[31] This is for example the case when two distributions of the same size have the same mean, the same number of poor individuals who have the same income as their counterparts, but non-poor individuals have different incomes than their counterparts.

assumption that individual $n(y)$ is non-poor.[32] By *Domination*, the fate of individual $n(y)$ is irrelevant to the ordering on $Y(\succeq)$.

This ordering on V_d is well-defined and complete, as P represents a complete ordering on $Y(\succeq)$. Function P^V is continuous since the ordering on V_d is continuous because the ordering on $Y(\succeq)$ is continuous by *Continuity* and mapping m is continuous.

STEP 2: Index P^V representing the equivalent ordering on the product space is additively separable.

If the assumptions of Theorem 1 in Gorman (1968) are all met, then for any $n \in N$ and any $\nu \in V_d$ that has size $n-1$, index P^V has the following functional form:

$$P^V(\nu) = \tilde{F}\left(\sum_{i=1}^{n-1} \tilde{\phi}(\nu_i)\right) \tag{14}$$

where $\tilde{F}$ and $\tilde{\phi}$ are strictly increasing functions.

Take any $n \in N$. For the remaining part of Step 2, I abuse slightly notation by denoting $[0,1]^{n-1} \subset V_d$ directly by V_d. The three assumptions required for this theorem are the following:

Assumption 1: The index P^V represents a *complete* and *continuous* ordering on a *product space* V_d.

V_d is a product space since $V_d = \times_{i=1}^{n-1}[0,1]_i$. I proved in Step 1 that the ordering on V_d is complete and continuous.

Assumption 2: Each sector $[0,1]_i$ of V_d has a *countably dense subset*, is *arc-connected* and is *strictly essential*. *Strict essentiality* means that given any subdistribution $(\nu_1, \ldots, \nu_{i-1}, \nu_{i+1}, \ldots, \nu_{n-1}) \in \times_{j=1}^{n-2}[0,1]_j$, not all elements of $[0,1]_i$ are indifferent for the ordering on V_d.

As all sectors are real intervals, any sector has a *countably dense subset* and is *arc-connected*. Strict essentiality follows directly from *Domination* and Lemma 1, which implies that for any subdistribution

$$(\nu_1, \ldots, \nu_{i-1}, \nu_{i+1}, \ldots, \nu_{n-1}) \in [0,1]^{n-2},$$

the value of $\nu_i \in [0,1]$ is not constrained by the subdistribution.[33]

Assumption 3: Let $S := \{[0,1]_1, \ldots, [0,1]_{n-1}\}$ be the set of sectors in V_d. Any subset of sectors $A \subseteq S$ is *separable*. *Separability* means that for all $(u,w), (v,w), (u,t), (v,t) \in V_d$, we have

$$P^V(u,w) \geq P^V(v,w) \;\Leftrightarrow\; P^V(u,t) \geq P^V(v,t).$$

Separability is proven in two substeps.

Substep 1:
Construct for each of these four elements of V_d a particular income distribution in $Y(\succeq)$ that has the same poverty and whose subgroups have the same value of mean income.

Construct $y^1, y^2, y^3, y^4 \in Y(\succeq)$ such that $m(y^1) = (u,w)$, $m(y^2) = (v,w)$, $m(y^3) = (u,t)$, $m(y^4) = (v,t)$ and $\overline{y}^1 = \overline{y}^2 = \overline{y}^3 = \overline{y}^4 = \overline{y}'$ with $\overline{y}' \geq z(\overline{y}')$. Such distributions exist and are constructed with the procedure given in Lemma 1.

The next operations aim at constructing from y^1 another income distribution y^{1*} whose poverty is the same as y^1 but meeting the restrictions necessary to apply *Weak Subgroup Consistency*.

Decompose y^1 in subgroups $y^1 = (y^1_A, y^1_B, y^1_n)$, such that subdistributions y^1_A and y^1_B are associated – via the numerical representation $d_1^{\succeq}$ – to the subdistributions u and w respectively. That is, for each element $u_i \in u$ there exists $y^1_i \in y^1_A$ such that $u_i = d_1^{\succeq}(y^1_i, \overline{y}^1)$. The same holds for w and y^1_B. Typically, $\overline{y}^1_A \neq \overline{y}^1_B \neq \overline{y}'$ but the next operations aims at obtaining such equality.

Triplicate y^1 and re-organize the subgroups to obtain at least one non-poor individual per subgroup. Let $y^{1'}$ denote the distribution obtained from this triplication

$$y^{1'} := (y^1, y^1, y^1) = (y^1_A, y^1_A, y^1_A, y^1_B, y^1_B, y^1_B, y^1_n, y^1_n, y^1_n)$$

[32] As individual $n(y)$ is non-poor we have by definition that $d_1^{\succeq}(y_n, \overline{y}) = 0$.

[33] In the definition and the proof of strict essentiality, the indices are not sorted by income level but refer to the identities.

This triplication does not affect the mean: $\overline{y}^{1'} = \overline{y}^1$. Reorganize subgroups: $y^{1'} = (y^1_{A'}, y^1_{B'}, y^1_n)$ with $y^1_{A'} := (y^1_A, y^1_A, y^1_A, y^1_n)$ and $y^1_{B'} := (y^1_B, y^1_B, y^1_B, y^1_n)$. Letting $u' := (u,u,u)$ and $w' := (w,w,w)$, we have that

$$m(y^{1'}) = (u,u,u,0,w,w,w,0) = (u',0,w',0),$$

as $d^{\succeq}_1(y_i, \overline{y}') = 0$ for any $y_i \geq z(\overline{y}')$.

Construct $y^{1*}_{A'}$ such that $m(y^{1*}_{A'}) = u'$ with $\overline{y}^{1*}_{A'} = \overline{y}'$ and $y^{1*}_{B'}$ such that $m(y^{1*}_{B'}) = w'$ with $\overline{y}^{1*}_{B'} = \overline{y}'$. Those income distributions exist as proven in Lemma 1, as each subgroup A' and B' contains at least one non-poor individual. The income distribution $y^{1*} := (y^{1*}_{A'}, y^{1*}_{B'}, \overline{y}')$ is such that $m(y^{1*}) = (u', 0, w', 0)$. This distribution belongs to $Y(\succeq)$ given that $\overline{y}' \geq z(\overline{y}')$. This distribution is also such that $\overline{y}^{1*} = \overline{y}'$ as its three subgroups have mean $\overline{y}'$.

Using the same procedure (decomposition, triplication, reorganization), construct successively $y^{2'}, y^{3'}, y^{4'}$ and y^{2*}, y^{3*}, y^{4*} such that:

$$\begin{aligned}
y^{1*} &= (y^{1*}_{A'}, y^{1*}_{B'}, \overline{y}') &\quad& \text{with } m(y^{1*}) = (u',0,w',0) = (u,u,u,0,w,w,w,0),\\
y^{2*} &= (y^{2*}_{A'}, y^{2*}_{B'}, \overline{y}') && \text{with } m(y^{2*}) = (v',0,w',0) = (v,v,v,0,w,w,w,0),\\
y^{3*} &= (y^{3*}_{A'}, y^{3*}_{B'}, \overline{y}') && \text{with } m(y^{3*}) = (u',0,t',0) = (u,u,u,0,t,t,t,0),\\
y^{4*} &= (y^{4*}_{A'}, y^{4*}_{B'}, \overline{y}') && \text{with } m(y^{4*}) = (v',0,t',0) = (v,v,v,0,t,t,t,0).
\end{aligned}$$

For all $m \in \{1,2,3,4\}$, we have $P(y^{m'}, \succeq) = P(y^m, \succeq)$ by *Replication Invariance*. As $(y^{m'}_i, \overline{y}') \sim (y^{m*}_i, \overline{y}')$ for all $i \leq q(y^{m*})$, we have $P(y^{m*}, \succeq) = P(y^{m'}, \succeq)$ by *Domination*. By Step 1, proving

$$P(y^{1*}, \succeq) \geq P(y^{2*}, \succeq) \;\Leftrightarrow\; P(y^{3*}, \succeq) \geq P(y^{4*}, \succeq)$$

is equivalent to proving

$$P^V(u,w) \geq P^V(v,w) \;\Leftrightarrow\; P^V(u,t) \geq P^V(v,t).$$

Importantly, the subgroups A' and B' in the distribution y^{m*} have their mean income equal to $\overline{y}'$ by construction. For notational simplicity, drop the symbols * and $'$ to name the new distributions and subgroups as the old ones.

Substep 2:
Prove separability from judgments on the associated income distributions:

$$P\left((y^1_A, y^1_B, \overline{y}'), \succeq\right) \geq P\left((y^2_A, y^2_B, \overline{y}'), \succeq\right) \Leftrightarrow P\left((y^3_A, y^3_B, \overline{y}'), \succeq\right) \geq P\left((y^4_A, y^4_B, \overline{y}'), \succeq\right).$$

By construction, these income distributions are such that

- $P(y^1_A, \succeq) = P(y^3_A, \succeq)$,
- $P(y^2_A, \succeq) = P(y^4_A, \succeq)$,
- $P(y^1_B, \succeq) = P(y^2_B, \succeq)$ and
- $P(y^3_B, \succeq) = P(y^4_B, \succeq)$,

by *Domination*. By assumption, we have $P(y^1, \succeq) \geq P(y^2, \succeq)$. As $P(y^1_B, \succeq) = P(y^2_B, \succeq)$, we have that $P((y^1_A, \overline{y}'), \succeq) \geq P((y^2_A, \overline{y}'), \succeq)$ by *Weak Subgroup Consistency* (remember all our subgroups have their mean equal to $\overline{y}'$). By *Weak Subgroup Consistency* again, this implies $P(y^1_A, \succeq) \geq P(y^2_A, \succeq)$.[34]

Then, combining $P(y^1_A, \succeq) \geq P(y^2_A, \succeq)$ together with $P(y^1_A, \succeq) = P(y^3_A, \succeq)$ and $P(y^2_A, \succeq) = P(y^4_A, \succeq)$ imply $P(y^3_A, \succeq) \geq P(y^4_A, \succeq)$. Two cases can arise.

- Case 1: $P(y^3_A, \succeq) > P(y^4_A, \succeq)$.
  Since $P(y^3_B, \succeq) = P(y^4_B, \succeq)$ by *Domination*, we have that $P((y^3_B, \overline{y}'), \succeq) = P((y^4_B, \overline{y}'), \succeq)$ by *Domination*. Together we obtain

  $$P((y^3_A, y^3_B, \overline{y}'), \succeq) > P((y^4_A, y^4_B, \overline{y}'), \succeq)$$

  by *Weak Subgroup Consistency*. This case is hence such that $P(y^3, \succeq) \geq P(y^4, \succeq)$, as desired.

[34] Strictly speaking *Weak Subgroup Consistency* cannot be applied again as subgroup $\overline{y}'$ contains a unique individual and hence does not belong to $Y(\succeq)$. Nevertheless, further replications of the income distributions solve the issue.

- Case 2: $P(y^3_A, \succeq) = P(y^4_A, \succeq)$.
  I show by contradiction that this case is such that $P(y^3, \succeq) \geq P(y^4, \succeq)$. Assume to the contrary that we have
  $$P((y^3_A, y^3_B, \overline{y}'), \succeq) < P((y^4_A, y^4_B, \overline{y}'), \succeq).$$
  As $P(y^3_A, \succeq) = P(y^4_A, \succeq)$, *Weak Subgroup Consistency* implies that
  $$P((y^3_A, y^3_B, y^4_A, \overline{y}'), \succeq) < P((y^4_A, y^4_B, y^3_A, \overline{y}'), \succeq).$$
  Again, as $P(y^3_B, \succeq) = P(y^4_B, \succeq)$, we obtain
  $$P((y^3_A, y^3_B, y^4_A, y^4_B, \overline{y}'), \succeq) < P((y^4_A, y^4_B, y^3_A, y^3_B, \overline{y}'), \succeq).$$
  This is a contradiction as the two distributions have equal poverty by *Symmetry*.

The two cases lead to $P(y^3, \succeq) \geq P(y^4, \succeq)$, which proves separability.

As all three assumptions hold, we can use Theorem 1 in Gorman (1968) and obtain:

$$P^V(\nu) = \tilde{F}' \left(\sum_{i=1}^{n-1} \tilde{\phi}_i(\nu_i) \right) \qquad \text{for all } \nu \in V_d,$$

where $\tilde{F}'$ and $\tilde{\phi}_i$ are strictly increasing functions. Functions $\tilde{\phi}_i$ might still depend on the rank i of the considered individual. Nevertheless, since the ordering on V_d is separable, we must have $\tilde{\phi}_i = \tilde{\phi} + f(i)$. Defining $\tilde{F}(x) := \tilde{F}'(x + \sum f(i))$, a translation of $\tilde{F}'$, we obtain (14) with function $\tilde{\phi}$ independent of rank i.

STEP 3: Link the functional form of P^V for different sizes of ν.

I show that functions $\tilde{F}$ and $\tilde{\phi}$ trivially depend on the number n of individuals. Theorem 1 in Gorman (1968) is valid for a fixed number n of individuals. Therefore, when n is allowed to vary, equation (14) must be written:

$$P^V(\nu) = \tilde{F}_n \left(\sum_{i=1}^{n-1} \tilde{\phi}_n(\nu_i) \right).$$

I modify the proof of Foster and Shorrocks (1991) in order to show that function $\tilde{\phi}_n$ is independent of n and function $\tilde{F}_n$ is inversely related to n.

Step 3.1: Define transformations of $\tilde{F}_n$ and $\tilde{\phi}_n$ for normalization purposes.
Let F_n and ϕ_n be the following transformations of $\tilde{F}_n$ and $\tilde{\phi}_n$:

$$\phi_n(\nu_i) = n\left[\tilde{\phi}_n(\nu_i) - \tilde{\phi}_n(0)\right],$$
$$F_n(x) = \tilde{F}_n\left[x + (n-1)\tilde{\phi}_n(0)\right].$$

These transformations allows rewritting last equation in the following way

$$P^V(\nu) = F_n \left(\frac{1}{n} \sum_{i=1}^{n-1} \phi_n(\nu_i) \right),$$

where $\phi_n(0) = 0$.

Since individual n is non-poor by definition, we have $d_1^{\succeq}(y_n, \overline{y}) = 0$. Therefore, we obtain – slightly abusing notation by introducing the zero contribution to poverty of individual n – that for all $n \geq 3$:

$$P^V(\nu) = F_n \left(\frac{1}{n} \left(\phi_n(0) + \sum_{i=1}^{n-1} \phi_n(\nu_i) \right) \right) = F_n \left(\frac{1}{n} \sum_{i=1}^{n} \phi_n(\nu_i) \right), \tag{15}$$

where F_n and ϕ_n are continuous, strictly increasing and $\phi_n(0) = 0$.

Step 3.2: Prove that functions F_n and ϕ_n do not depend on n using *Replication Invariance*.

From the previous step, we have $\phi_n : [0,1] \to [0,b_n]$ with $\phi_n(0) = 0$ for all $n \in N$. Take any $y \in Y(\succeq)$ with $n(y) = 5$ such that a single individual is poor in y, which is $q(y) = 1$. Let $\nu := m(y) = (t,0,0,0)$ be the image of y through mapping m, where t can be any point in $[0,1]$. Consider distribution $x := (y,\dots,y)$ a k-replication of y. Let $\nu' := m(x) = (t,\dots,t,0,\dots,0)$ be the image of x, which contains $4k-1$ zeros and k t's. The dimension of ν is $r = 4$ and the dimension of ν' is $s = 5k-1$. Therefore we have $s = k(r+1)-1 = kr+k-1$.

Denoting $F := F_4$ and $\phi := \phi_4$, the relationship between ϕ, ϕ_s, F and F_s for all $t \in [0,1]$ is computed using (15) and *Replication Invariance*:

$$P^V(\nu) = F\left[\frac{1}{5}\phi(t)\right] = F_s\left[\frac{k}{5k}\phi_s(t)\right] = P^V(\nu'),$$

$$\phi_s(t) = 5F_s^{-1}\left[F\left(\frac{1}{5}\phi(t)\right)\right]. \tag{16}$$

Replacing $\phi_s(t)$ in (15) by its value obtained in (16), we get:

$$F^{-1}[P^V(\nu')] = F^{-1}\left[F_s\left(\frac{1}{5k}\sum_{i=1}^{5k}5F_s^{-1}\left[F\left(\frac{1}{5}\phi(\nu_i')\right)\right]\right)\right] \tag{17}$$

$$= G_s^{-1}\left(\frac{1}{5k}\sum_{i=1}^{5k}5G_s\left(\frac{1}{5}\phi(\nu_i')\right)\right), \tag{18}$$

where $G_s(w) := F_s^{-1}(F(w))$ and $G_4(w) = F^{-1}(F(w)) = w$.

By *Replication Invariance*, we have that $F^{-1}[P^V(\nu)] = F^{-1}[P^V(\nu')]$, which by (18) yields:

$$\begin{aligned} G_s\left(\frac{1}{5}\phi(t)\right) &= \left(\frac{1}{5k}\sum_{i=1}^{5k}5G_s\left(\frac{1}{5}\phi(\nu_i')\right)\right) \\ &= \frac{1}{k}\sum_{i=1}^{5k}G_s\left(\frac{1}{5}\phi(\nu_i')\right) \\ &= G_s\left(\frac{1}{5}\phi(t)\right) + 4G_s(0), \end{aligned}$$

which shows that $G_s(0) = 0$.

Consider now any $y' \in Y(\succeq)$ with $n(y') = 5$ and such that $q(y') = 2$. Let $\nu := m(y') = (t,u,0,0)$ be the image of y', where t and u can be any points in $[0,1]$. Consider $x' := (y',\dots,y')$ a k-replication of y'. Let $\nu' := m(x') = (t,\dots,t,u,\dots,u,0,\dots,0)$ be the image of x, which contains $3k-1$ zeros, k t's and k u's.

By *Replication Invariance*, we have that $F^{-1}[P^V(\nu)] = F^{-1}[P^V(\nu')]$, which by (18) yields:

$$\frac{1}{5}\phi(t) + \frac{1}{5}\phi(u) = G_s^{-1}\left(G_s\left(\frac{1}{5}\phi(t)\right) + G_s\left(\frac{1}{5}\phi(u)\right)\right),$$

which can be rewritten as the Jensen equation

$$G_s(x+x') = G_s(x) + G_s(x'),$$

that admits as general solution $G_s(x) = a_s x + b_s$. As $G_s(0) = 0$ we have $b_s = 0$.

Replacing G_s by its expression in (18), we obtain

$$F^{-1}[P^V(\nu')] = \frac{1}{5k}\sum_{i=1}^{5k}\phi(\nu_i').$$

Therefore, for any $y \in Y(\succeq)$ with $n(y) = 5k$ and its image $\nu = m(y)$:

$$P^V(\nu) = F\left(\frac{1}{5k}\sum_{i=1}^{5k}\phi(\nu_i)\right) \tag{19}$$

The same expression is valid for all $y \in Y(\succeq)$ with dimension $n(y)$ as the same reasoning can be applied between $n(y)$ and the least common multiple between $n(y)$ and 5.

Finally, the transformations d' and G of the functions ϕ and F respectively guarantee that the domain of image of d' is $[0,1]$. Letting $d'(y_i, \overline{y}) := \frac{\phi(d_1^{\succeq}(y_i,\overline{y}))}{\phi(1)}$ and $G(x) := F\big(x\phi(1)\big)$, we have for all $y \in Y(\succeq)$:

$$P^V(\nu) = G\left(\frac{1}{n(y)}\sum_{i=1}^{n(y)} d'(y_i, \overline{y})\right) = P(y, \succeq) \tag{20}$$

where G is a continuous and strictly increasing function and d' is a numerical representation of $\succeq$, which needs not be the same as $d_1^{\succeq}$. The previous reasoning is valid for any $\succeq \in \mathcal{R}$. As function G is strictly increasing, P is ordinally equivalent to $P' : \mathcal{P} \to [0,1]$ with $P'(y, \succeq) = \frac{1}{n(y)}\sum_{i=1}^{n(y)} d'(y_i, \overline{y})$. This proves that P is an additive poverty index.

7.3 Proof of Theorem 2

Necessity.
Necessity is proven by contradiction. Assume (7) does not hold for some $(a^1, \overline{y}^1)$, $(b^1, \overline{y}^1) \in X_p(\succeq)$ with $a^1 \leq b^1$ and $z^a < b^1 \leq z(\overline{y}^1)$ where $\overline{y}^1$ is such that $\overline{y}^1 \geq \overline{y}^m$. Therefore, at these two bundles, we have for some $l > 0$ that $L_7 = l$.

I prove that for all $\epsilon > 0$, there exists a distribution $y^2 \in Y(\succeq)$ with $\overline{y}^2 = \overline{y}^1$ such that

$$|l - L_6(y^2)| < \epsilon$$

and hence, for $\epsilon < l$, distribution y^2 is such that $L_6(y^2) > 0$, violating *Monotonicity in Income*. Construct y^2 such that

- $y_1^2 := a^1$,
- $y_k^2 := b^1$ for all k with $2 \leq k \leq n(y^2) - 1$ and
- $y_n^2 := n(y^2)\overline{y}^1 - \sum_{h=1}^{n(y^2)-1} y_h^2$.

First, I show that distribution y^2 is well-specified and belongs to $Y(\succeq)$. As $\overline{y}^1 \geq \overline{y}^m$, we have that $z(\overline{y}^1) \leq \overline{y}^1$ by Flat Line in Poor Societies and Positive Slope less than One. Therefore, individual n is non-poor: $y_n^2 \geq z(\overline{y}^1)$ since $\overline{y}^1 \geq z(\overline{y}^1)$.

We have for distribution y^2 that:[35]

$$\begin{aligned} l - L_6(y^2) &= L_7 - L_6(y^2) \\ &= \frac{2\partial_2 d^{\succeq}(b^1, \overline{y}^1) - \partial_2 d^{\succeq}(a^1, \overline{y}^1)}{n(y^2)}. \end{aligned}$$

In order to show that $|l - L_6(y^2)| < \epsilon$, two cases must be considered:

- Case 1: $\partial_2 d^{\succeq}(b^1, \overline{y}^1)$ and $\partial_2 d^{\succeq}(a^1, \overline{y}^1)$ are finite.
  The distance $|l - L_6(y^2)|$ can be made arbitrarly small by taking $n(y^2)$ sufficiently large, implying $L_6(y^2) > 0$, which violates (6) and hence *Monotonicity in Income*.
- Case 2: $\partial_2 d^{\succeq}(b^1, \overline{y}^1) = \infty$.
  By Priority to Income below z^a, this case must be such that $b^1 > z^a$. Take any $a \in [0, z(\overline{y}^1))$ at which $\partial_1 d^{\succeq}(a, \overline{y}^1)$ is finite. Any distribution $y^3 \in Y(\succeq)$ with $\overline{y}^3 = \overline{y}^1$ constructed such that the poor individual receiving the increment earns a and any other poor individual earns b^1 violates inequality (6) and hence *Monotonicity in Income*.
- Case 3: $\partial_2 d^{\succeq}(a^1, \overline{y}^1) = \infty$.
  By Priority to Income below z^a, this case must be such that $a^1 > z^a$. A distribution constructed in the same way as y^3 is constructed in case 2 leads to the desired violation.

[35] We have $\partial_2 d^{\succeq}(y_n^2, \overline{y}^1) = 0$. If $\partial_2 d^{\succeq}(y_n^2, \overline{y}^1) > 0$, then *Monotonicity in Income* is silent because the increment produces a distribution without any non-poor individual, hence outside $Y(\succeq)$.

The case for which the two bundles are such that $b^1 < a^1 < z(\overline{y}^1)$ leads to the same contradiction. The only difference lies in the construction of y^2: $y^2_{n(y^2)-1} := a^1$, $y^2_k := b^1$ for all k with $1 \leq k \leq n(y^2)-2$. The condition is therefore necessary.

Sufficiency.
First, I show that *Monotonicity in Income* is systematically satisfied for all $y \in Y(\succeq)$ with $\overline{y} < \overline{y}^m$. By Flat Line in Poor Societies, the poverty line's slope $\frac{\partial z(\overline{y})}{\partial \overline{y}} = 0$ for all $\overline{y} < \overline{y}^m$. By Priority to Income below z^a and Homotheticity above z^a, the iso-poverty curves below the line are then all flat for all $\overline{y} < \overline{y}^m$, which implies that

$$\partial_2 d^{\succeq}(y_j, \overline{y}) = 0 \qquad \text{for all } j \leq n(y)$$

and inequality (6) is respected as its first term is strictly negative by Strict Monotonicity up to line.

Consider any $y \in Y(\succeq)$ with $\overline{y} \geq \overline{y}^m$. By Priority to Income below z^a, the iso-poverty curves below the absolute threshold z^a are all flat, which implies again that

$$\partial_2 d^{\succeq}(y_j, \overline{y}) = 0 \qquad \text{for all } j \leq q(y) \text{ with } y_j \leq z^a$$

and therefore inequality (4) is systematically respected as its right hand side is strictly positive by Strict Monotonicity up to line. As a result, if the condition holds, then we have that for all $a \in [0, z(\overline{y}))$ and $b \in [0, z(\overline{y})]$ that (7) holds.

Sufficiency follows from the fact that, if there exists an $y \in Y(\succeq)$ and individual i such that (6) is violated, then inequality (7) is violated as well when (7) is computed from the income of individual i and the income of another individual $j* \leq n(y)$. Individual $j*$ is defined to be the individual that has the largest partial derivative of $d^{\succeq}$ with respect to $\overline{y}$, in the distribution at hand. For all $y \in Y(\succeq)$ there exists $j* \leq n(y)$ such that

$$\frac{1}{n(y)} \sum_{k=1}^{n(y)} \partial_2 d^{\succeq}(y_k, \overline{y}) \leq \partial_2 d^{\succeq}(y_{j*}, \overline{y}),$$

which implies that $L_6(y) \leq L_7$ when L_7 is computed from the incomes of individuals i and $j*$. The key property for last inequality to hold is that $\partial_2 d^{\succeq}(y_k, \overline{y})$ depends on the income of other individuals only through the mean income $\overline{y}$.

7.4 Proof of Theorem 3

Consider any additive index P and any $\succeq \in \mathcal{R}$. *Transfer* is silent for all $y \in Y(\succeq)$ with $q(y) < 2$. Consider any $y \in Y(\succeq)$ with $q(y) \geq 2$, any $\lambda > 0$ and $y' \in Y(\succeq)$ with $n(y) = n(y')$, $y_j - \lambda = y'_j > y'_k = y_k + \lambda$, $z(\overline{y}) > y_j$ and $y'_i = y_i$ for all $i \neq j, k$. *Transfer* requires that $P(y, \succeq) \geq P(y', \succeq)$. By construction we have that $\overline{y}' = \overline{y}$, which implies that

$$d^{\succeq}(y_i, \overline{y}) = d^{\succeq}(y'_i, \overline{y}') \qquad \text{for all } i \neq j, k$$

since $y'_i = y_i$. Therefore, $P(y, \succeq) \geq P(y', \succeq)$ is equivalent to

$$d^{\succeq}(y_j, \overline{y}) + d^{\succeq}(y_k, \overline{y}) \geq d^{\succeq}(y'_j, \overline{y}') + d^{\succeq}(y'_k, \overline{y}'),$$

where $y_k < y'_k < y'_j < y_j$. The last inequality rewrites

$$d^{\succeq}(y'_k, \overline{y}') - d^{\succeq}(y_k, \overline{y}) \leq d^{\succeq}(y'_j, \overline{y}') - d^{\succeq}(y_j, \overline{y}).$$

Hence, index P satisfies *Transfer* if and only if for all $\overline{y} > 0$ and all $a, b \in [0, z(\overline{y}))$ with $a < b$ we have:[36]

$$\partial_1 d^{\succeq}(a, \overline{y}) \leq \partial_1 d^{\succeq}(b, \overline{y}). \tag{21}$$

I prove that the condition stated in Theorem 3 is necessary and sufficient to meet the condition expressed in (21). This proof is based on Lemma 2 and 3 that link the partial derivatives of the numerical representation.

[36] The same remark as in footnote 26 applies.

Necessity.
Necessity directly follows from the condition expressed in (21).

Sufficiency.
I show that if the condition stated in Theorem 3 holds, then the condition expressed in (21), which is sufficient for *Transfer*, also holds.

Take any $\overline{y} > 0$ and any two $a, b \geq 0$ such that $a < b < z(\overline{y})$, I show that we have

$$\partial_1 d(a, \overline{y}) \leq \partial_1 d(b, \overline{y}), \tag{22}$$

if the condition stated in Theorem 3 holds.

Consider a', b' such that $(a', \overline{y}^m) \sim (a, \overline{y})$ and $(b', \overline{y}^m) \sim (b, \overline{y})$. These a', b' exist and belong to $[0, z^m)$ by Priority to Income below z^a and Homotheticity above z^a. By Lemma 3 we have that

$$\frac{\partial_1 d(a, \overline{y})}{\partial_1 d(b, \overline{y})} = \frac{\partial_1 d(a', \overline{y}^m)}{\partial_1 d(b', \overline{y}^m)} \qquad \text{if } b \leq z^a \text{ or } a \geq z^a,$$

which implies (22) as we have $\frac{\partial_1 d(a', \overline{y}^m)}{\partial_1 d(b', \overline{y}^m)} \geq 1$ since condition (8) holds by assumption and $\partial_1 d(.,.) < 0$ by *Domination* and Strict Monotonicity up to line. By Lemma 3 again we have that

$$\frac{\partial_1 d(a, \overline{y})}{\partial_1 d(b, \overline{y})} = \left(\frac{z^m - z^a}{z(\overline{y}) - z^a}\right)^{-1} \frac{\partial_1 d(a', \overline{y}^m)}{\partial_1 d(b', \overline{y}^m)} \qquad \text{if } a < z^a < b,$$

which by Positive Slope less than One implies that

$$\frac{\partial_1 d(a, \overline{y})}{\partial_1 d(b, \overline{y})} \geq \frac{\partial_1 d(a', \overline{y}^m)}{\partial_1 d(b', \overline{y}^m)} \qquad \text{if } a < z^a < b,$$

which also implies (22) as we have $\frac{\partial_1 d(a', \overline{y}^m)}{\partial_1 d(b', \overline{y}^m)} \geq 1$ since condition (8) holds by assumption and $\partial_1 d(.,.) < 0$ by *Domination* and Strict Monotonicity up to line.

7.4.1 Relation between partial derivatives

I present two lemmas useful for the proof of Theorem 3.

Lemma 2 shows that the partial derivatives of the numerical representation of an EO meeting Priority to Income below z^a and Homotheticity above z^a are connected to one another in a simple way. The partial derivative to the mean is zero for absolutely poor individuals. For relatively poor individuals, this derivative is the opposite of the partial derivative to the income multiplied by the slope of the well-being curve at the point considered.

Lemma 2 (Partial derivatives at $(a, \overline{y})$)**.**
Let $d^{\succeq}$ be a numerical representations of $\succeq \in \mathcal{R}$. For all $(a, \overline{y}) \in X_p(\succeq)$ we have that

$$\partial_2 d^{\succeq}(a, \overline{y}) = 0 \qquad\qquad \textit{if } a \leq z^a, \tag{23}$$

$$\partial_2 d^{\succeq}(a, \overline{y}) = -\frac{\partial z(\overline{y})}{\partial \overline{y}} \frac{a - z^a}{z(\overline{y}) - z^a} \partial_1 d^{\succeq}(a, \overline{y}) \qquad \textit{if } a > z^a \textit{ and} \tag{24}$$

$d^{\succeq}$ is differentiable at $(a, \overline{y})$ with $a > z^a$.

Proof. From the definition of the equivalent income function at $\overline{y}^r$ we have that

$$d^{\succeq}(a, \overline{y}) = d^{\succeq}\left(e^r(a, \overline{y}), \overline{y}^r\right).$$

First, I prove equation (23). By chain derivation, we have that

$$\partial_2 d^{\succeq}(a, \overline{y}) = \partial_1 d^{\succeq}\left(e^r(a, \overline{y}), \overline{y}^r\right) \partial_2 e^r(a, \overline{y}). \tag{25}$$

Equation (23) follows from the fact that $\partial_2 e^r(a, \overline{y}) = 0$ when $a \leq z^a$.

Second, I prove equation (24). By chain derivation, we have

$$\partial_1 d^{\succeq}(a, \overline{y}) = \partial_1 d^{\succeq}\left(e^r(a, \overline{y}), \overline{y}^r\right) \partial_1 e^r(a, \overline{y}). \tag{26}$$

From equation (9), we obtain for $a > z^a$ that

$$\partial_1 e^r(a,\overline{y}) = \frac{z(\overline{y}^r) - z^a}{z(\overline{y}) - z^a},$$
$$\partial_2 e^r(a,\overline{y}) = -\frac{\partial z(\overline{y})}{\partial \overline{y}} \frac{a - z^a}{z(\overline{y}) - z^a} \partial_1 e^r(a,\overline{y}).$$

Replacing the last two equations inside (25) and (26), we obtain respectively that

$$\partial_2 d^{\succeq}(a,\overline{y}) = \partial_1 d^{\succeq}\left(e^r(a,\overline{y}),\overline{y}^r\right)\left(-\frac{\partial z(\overline{y})}{\partial \overline{y}} \frac{a - z^a}{z(\overline{y}) - z^a} \frac{z(\overline{y}^r) - z^a}{z(\overline{y}) - z^a}\right),$$
$$\partial_1 d^{\succeq}(a,\overline{y}) = \partial_1 d^{\succeq}\left(e^r(a,\overline{y}),\overline{y}^r\right)\left(\frac{z(\overline{y}^r) - z^a}{z(\overline{y}) - z^a}\right),$$

which yields the desired result when the last two equations are equated.

■

Lemma 3 shows that the partial derivatives at two equivalent bundles are also connected. The partial derivative to own income are equal for absolutely poor individuals. For relatively poor individuals, they are proportional to each other.

Lemma 3 (Partial derivatives at equivalent bundles)**.**
Let $d^{\succeq}$ be a numerical representations of $\succeq \in \mathcal{R}$. For all $(a,\overline{y}) \in X_p(\succeq)$ at which $d^{\succeq}$ is differentiable and $(a',\overline{y}')$ such that $(a,\overline{y}) \sim (a',\overline{y}')$ we have that

$$\partial_1 d^{\succeq}(a,\overline{y}) = \partial_1 d^{\succeq}(a',\overline{y}') \qquad \text{if } a < z^a, \tag{27}$$
$$\partial_1 d^{\succeq}(a,\overline{y}) = \frac{z(\overline{y}') - z^a}{z(\overline{y}) - z^a} \partial_1 d^{\succeq}(a',\overline{y}') \qquad \text{if } a \geq z^a. \tag{28}$$

Proof. Let the equivalent income function at $\overline{y}'$ defined in (9) be denoted e'. This definition implies that

$$d^{\succeq}(a,\overline{y}) = d^{\succeq}\left(e'(a,\overline{y}),\overline{y}'\right).$$

First, I prove equation (27). By chain derivation we have that

$$\partial_1 d^{\succeq}(a,\overline{y}) = \partial_1 d^{\succeq}\left(e'(a,\overline{y}),\overline{y}'\right)\partial_1 e'(a,\overline{y}), \tag{29}$$
$$= \partial_1 d^{\succeq}(a',\overline{y}')\partial_1 e'(a,\overline{y}). \tag{30}$$

Equation (27) follows from the fact that $\partial_1 e'(a,\overline{y}) = 1$ when $a < z^a$ because $e'(a,\overline{y}) = a$.
Second, equation (28) follows from (30) because when $z^a \leq a$ we have that

$$\partial_1 e'(a,\overline{y}) = \frac{z(\overline{y}') - z^a}{z(\overline{y}) - z^a}.$$

■

7.5 No additive index satisfies *Transfer* and *Strong monotonicity in absolute threshold*

I show that no additive index satisfying *Transfer* also satisfies *Strong monotonicity in absolute threshold*.

Poverty axiom 14 (*Strong monotonicity in absolute threshold*)**.**
For all $\succeq, \succeq' \in \mathcal{R}$ with

$Z(\succeq) = Z(\succeq')$ *and* $Z^a(\succeq) < Z^a(\succeq')$,

and all $y \in Y(\succeq) \cap Y(\succeq')$ we have

$$P(y,\succeq) \leq P(y,\succeq').$$

The proof is by contradiction. I show that any additive index P that satisfies *Transfer* and *Strong monotonicity in absolute threshold* violates *Domination* for any $\succeq \in \mathcal{R}$ with $Z^a(\succeq) = 0$ and $Z(\succeq) = z$, where z a weakly relative line with $s > 0$.

Take any $\succeq \in \mathcal{R}$ such that $Z(\succeq) = z$. Consider the subset $\mathcal{R}^z \subset \mathcal{R}$ of EOs having their associated poverty line equal to z:

$$\mathcal{R}^z := \{\succeq' \in \mathcal{R} \mid Z(\succeq') = z\}.$$

I denote by $\succeq^{z^a}$ the EO in $\mathcal{R}^z$ whose absolute threshold is equal to z^a. Furthermore, I denote by d^{z^a} the numerical representation attached to index P for the EO $\succeq^{z^a}$. In particular, this numerical representation is denoted by d^0 for the EO $\succeq^0$ whose $z^a = 0$.

I show by contradiction that index P is such that

$$d^0(a, \overline{y}^m) = 0 \qquad \text{for all } a \in (0, z^m),$$

which clearly violates *Domination* for this EO that satisfies Strict Monotonicity up to line.

Assume to the contrary that $d^0(a, \overline{y}^m) = l > 0$ for some $a < z^m$. First, I show that for any b with $a < b < z^m$, there exists $\overline{y} > 0$ such that

$$(a, \overline{y}^m) \sim^0 (b, \overline{y}). \tag{31}$$

By Homotheticity above z^a, when $z^a = 0$, the previous equivalence is obtained when

$$\frac{a}{z^m} = \frac{b}{z(\overline{y})} = \frac{b}{\max\{z^m, a' + s\overline{y}\}}.$$

Given that $s > 0$ and $a > 0$, such $\overline{y}$ necessarily exists.

Consider the income distribution $y^b = (b, b, 3\overline{y} - 2b)$, which belongs to $Y(\succeq)$ for all $\succeq \in \mathcal{R}^z$. In effect, we have $\overline{y} \geq \overline{y}^m$ and hence $\overline{y} \geq z(\overline{y})$ by Flat Line in Poor Societies and Positive Slope less than One, which implies that individual 3 is non-poor. Given that P is an additive index, we have for all $\succeq \in \mathcal{R}^z$ that

$$P(y^b, \succeq) = \frac{2}{3} d^{z^a}(b, \overline{y}),$$

where $z^a = Z^a(\succeq)$.

By *Strong monotonicity in absolute threshold*, we have that

$$P(y^b, \succeq^0) \leq P(y^b, \succeq^{z^a}),$$

for all $z^a \in [0, z^m)$, which is equivalent to

$$d^0(b, \overline{y}) \leq d^{z^a}(b, \overline{y}). \tag{32}$$

Observe that if $z^a = b$, then $\succeq^b \in \mathcal{R}^z$ as $b < z^m$ and is such that

$$(b, \overline{y}^m) \sim^b (b, \overline{y}), \tag{33}$$

by Homotheticity above z^a. Using (31) and (33), inequality (32) becomes

$$d^0(a, \overline{y}^m) \leq d^b(b, \overline{y}^m). \tag{34}$$

The previous inequality can be written for all $a \in (0, z^m)$ and all $b \in (a, z^m)$.

To complete the proof, consider the following corollary of Theorem 3: for all $\succeq \in \mathcal{R}$ and all $b' \in [0, z^m]$ we have

$$d^{\succeq}(b', \overline{y}^m) \leq \frac{z^m - b'}{z^m},$$

where $z^m = z(\overline{y}^m)$. This corollary is a direct consequence of the necessity for $d^{\succeq}$ to be convex at $\overline{y}^m$.

Given the previous corollary and as I assumed that $d^0(a, \overline{y}^m) = l > 0$, inequality (34) becomes

$$0 < l \leq d^b(b, \overline{y}^m) \leq \frac{z^m - b}{z^m}.$$

Taking any value of b such that $z^m(1 - l) < b < z^m$ leads to a violation of the previous inequality, which yields the desired contradiction.

7.6 Proof of Theorem 4

Statement 1 implies statement 2.

By definition, any additive index P is ordinally equivalent to an index $\hat{P} : \mathcal{P} \to [0,1]$ defined by

$$\hat{P}(y, \succeq) := \frac{1}{n(y)} \sum_{i=1}^{n(y)} d^{\succeq}(y_i, \overline{y}),$$

where $d^{\succeq}$ is a numerical representation of $\succeq$.

For a given line z, let $\overline{y}^B$ be the value of mean income defined by $\overline{y}^B := z^m$. By Non-zero Threshold and Flat Line in Poor Societies, this value is such that

$$z(\overline{y}^B) = z^m.$$

Any two $\succeq, \succeq' \in \mathcal{R}$ with $Z(\succeq) = Z(\succeq')$ share the same value of z^m and, therefore, the same $\overline{y}^B$.

For all $\succeq \in \mathcal{R}$, let function $e^B_{\succeq}$ be the equivalent income at $\overline{y}^B$. The definition of the equivalent income function at $\overline{y}^B$ implies that

$$(y_i, \overline{y}) \sim \left(e^B_{\succeq}(y_i, \overline{y}), \overline{y}^B\right) \qquad \text{for all } (y_i, \overline{y}) \in X.$$

Given that $d^{\succeq}$ is a numerical representation of $\succeq$, we have that

$$d^{\succeq}(y_i, \overline{y}) = d^{\succeq}\left(e^B_{\succeq}(y_i, \overline{y}), \overline{y}^B\right) \qquad \text{for all } (y_i, \overline{y}) \in X.$$

As $\overline{y}^B$ is a constant for a given $\succeq$, I can omit $\overline{y}^B$ in the list of arguments of function $d^{\succeq}$. Then, defining $f^{\succeq} : [0, z^m] \to [0,1]$ by

$$f^{\succeq}\left(e^B_{\succeq}(y_i, \overline{y})\right) := d^{\succeq}\left(e^B_{\succeq}(y_i, \overline{y}), \overline{y}^B\right) \qquad \text{for all } (y_i, \overline{y}) \in X,$$

we have

$$\hat{P}(y, \succeq) := \frac{1}{n(y)} \sum_{i=1}^{n(y)} f^{\succeq}\left(e^B_{\succeq}(y_i, \overline{y})\right). \tag{35}$$

Let $\mathcal{R}^I(\succeq)$ be the subset of EOs that share the same absolute threshold and intercept as $\succeq$:

$$\mathcal{R}^I(\succeq) := \{\succeq' \in \mathcal{R} | Z^a(\succeq') = Z^a(\succeq) \text{ and } z^{m'} = z^m\}.$$

The respetive definitions of $\overline{y}^B$ and $\overline{y}^m$ imply that $\overline{y}^B \leq \overline{y}^m$ by Flat Line in Poor Societies. Therefore, *Line relativism independence in poor societies* implies that for all $\succeq \in \mathcal{R}$, $\succeq' \in \mathcal{R}^I(\succeq)$ and all $y, y' \in Y(\succeq) \cap Y(\succeq')$ with $\overline{y} = \overline{y}' = \overline{y}^B = \overline{y}^{B'}$ we have that[37]

$$\hat{P}(y, \succeq) \geq \hat{P}(y', \succeq) \Leftrightarrow \hat{P}(y, \succeq') \geq \hat{P}(y', \succeq').$$

Given the additive expression of $\hat{P}$ shown in (35), the last property holds if and only if $f^{\succeq'}$ is an affine transformation of $f^{\succeq}$, which is

$$f^{\succeq'}(u) = a + b f^{\succeq}(u) \qquad \text{for all } u \in [0, z^m],$$

where $a \in \mathbb{R}$ and $b > 0$. As $d^{\succeq}$ is a numerical representation of $\succeq$, we have that $d^{\succeq}\left(0, \overline{y}^B\right) = 1$ and $d^{\succeq}\left(z^m, \overline{y}^B\right) = 0$. The same holds true for $d^{\succeq'}$. This implies that $a = 0$ and $b = 1$, hence

$$f^{\succeq'}(u) = f^{\succeq}(u) \qquad \text{for all } u \in [0, z^m].$$

Therefore, all EOs in $\mathcal{R}^I(\succeq)$ share the same function f. Given that the definition of $\mathcal{R}^I(\succeq)$ only depends on the absolute threshold z^a and the intercept z^m, equation (35) becomes

$$\hat{P}(y, \succeq) := \frac{1}{n(y)} \sum_{i=1}^{n(y)} f\left(e^B_{\succeq}(y_i, \overline{y}), z^m, z^a\right).$$

[37] By definition of $\overline{y}^B$, for all $\succeq \in \mathcal{R}$ and all $\succeq' \in \mathcal{R}^I(\succeq)$ we have that $\overline{y}^B = \overline{y}^{B'}$.

For a given $\succeq$, let function $e^m_\succeq$ be the equivalent income at $\overline{y}^m$. By Priority to Income below z^a and Homotheticity above z^a and the definitions of $\overline{y}^m$ and of the equivalent income function at $\overline{y}^m$ we have that

$$e^m_\succeq(y_i, \overline{y}) = e^B_\succeq(y_i, \overline{y}) \qquad \text{for all } (y_i, \overline{y}) \in X,$$

and hence

$$\hat{P}(y, \succeq) := \frac{1}{n(y)} \sum_{i=1}^{n(y)} f\left(e^m_\succeq(y_i, \overline{y}), z^m, z^a\right). \tag{36}$$

I prove the properties of function f stated in Theorem 4. Function f is strictly decreasing in its first argument on $[0, z(\overline{y}^m)]$, $f(0, z(\overline{y}^m), z^a) = 1$, $f(b, z(\overline{y}^m), z^a) = 0$ for all $b \geq z(\overline{y}^m)$ because f inherits these properties from the numerical representation $d^\succeq$.

Using *Weak monotonicity in absolute threshold*, I show that function f is non-decreasing in its third argument. Assume to the contrary that there exist $m > 0$, $a \in [0, m]$ and $e \in [0, m]$ such that f is strictly decreasing in its third argument at (e, m, a). This assumption implies that for some $a' \in (a, m)$ we have that

$$f(e, m, a) > f(e, m, a').$$

Consider any $\succeq \in \mathcal{R}$ such that $Z^a(\succeq) = a$ and $z = Z(\succeq)$ such that $z^m = m$. By Non-zero Threshold, this $\succeq$ exists. Consider now $\succeq' \in \mathcal{R}$ such that $Z(\succeq') = Z(\succeq)$ and $Z^a(\succeq') = a'$. Consider the income distribution $y^B = (e, e, 3\overline{y}^B - 2e)$, which belongs to $Y(\succeq) \cap Y(\succeq')$. By *Weak monotonicity in absolute threshold*, we must have that

$$P(y^B, \succeq) \leq P(y^B, \succeq'),$$

which by (36) yields

$$\frac{2}{3} f(e, m, a) \leq \frac{2}{3} f(e, m, a'),$$

which contradicts the assumption that $f(e, m, a) > f(e, m, a')$.

Statement 2 implies statement 1.

I only show that an index meeting the description in statement 2 satisfies *Line relativism independence in poor societies*. The proof for the other two properties follows similar lines.

Consider any two $\succeq, \succeq' \in \mathcal{R}$ where $z = Z(\succeq)$ and $z' = Z(\succeq')$ and $z^a = Z^a(\succeq)$ and $z^{a'} = Z^a(\succeq')$. If either $z^m \neq z^{m'}$ or $z^a \neq z^{a'}$, then *Line relativism independence in poor societies* is trivially satisfied for these two EOs. Therefore, I focus on the case for which $z^m = z^{m'}$ and $z^a = z^{a'}$.

Take any two $y, y' \in Y(\succeq) \cap Y(\succeq')$ with $n(y) = n(y')$ and $\overline{y} = \overline{y}' \leq \min\{\overline{y}^m, \overline{y}^{m'}\}$. *Line relativism independence in poor societies* implies that if P meets the description in statement 2, then we have

$$P(y, \succeq) \geq P(y', \succeq) \Leftrightarrow P(y, \succeq') \geq P(y', \succeq'), \tag{37}$$

which I shown in the remainder of this proof.

For all $\succeq \in \mathcal{R}$, $\overline{y} \leq \overline{y}^m$ and $a \in [0, z^m]$, the definition of the equivalent income function $e^m_\succeq$ and Flat Line in Poor Societies imply that

$$e^m_\succeq(a, \overline{y}) = a.$$

Therefore, given that $\overline{y} = \overline{y}' \leq \min\{\overline{y}^m, \overline{y}^{m'}\}$, we have that

$$e^m_\succeq(y_i, \overline{y}) = e^m_{\succeq'}(y_i, \overline{y}),$$

for all $i \leq q(y)$ and

$$e^m_\succeq(y'_i, \overline{y}') = e^m_{\succeq'}(y'_i, \overline{y}'),$$

for all $i \leq q(y')$. Remembering that $f(b, z^m, z^a) = 0$ if $b \geq z^m$, this implies that

$$\hat{P}(y, \succeq) = \frac{1}{n(y)} \sum_{i=1}^{n(y)} f\left(e^m_{\succeq}(y_i, \overline{y}), z^m, z^a\right) = \frac{1}{n(y)} \sum_{i=1}^{n(y)} f\left(e^m_{\succeq'}(y_i, \overline{y}), z^{m'}, z^{a'}\right) = \hat{P}(y, \succeq'),$$

given that function f only depends on the EO via its arguments, as well as

$$\hat{P}(y', \succeq) = \frac{1}{n(y')} \sum_{i=1}^{n(y')} f\left(e^m_{\succeq}(y'_i, \overline{y}'), z^m, z^a\right) = \frac{1}{n(y')} \sum_{i=1}^{n(y')} f\left(e^m_{\succeq'}(y'_i, \overline{y}'), z^{m'}, z^{a'}\right) = \hat{P}(y', \succeq'),$$

which shows that (37) holds, hence *Line relativism independence in poor societies* holds.

7.7 Proof of Theorem 5

I derive the lower-bound on function f given in (11). To do so, I extend the proof of Ebert and Moyes (2002) in order to show that

$$f\left(e^m_{\succeq}(y_i, \overline{y}), z(\overline{y}^m), 0\right) = \left(1 - \frac{e^m_{\succeq}(y_i, \overline{y})}{z(\overline{y}^m)}\right)^{\alpha_0}.$$

Consider the subset $\mathcal{R}^f_0$ of EOs in $\mathcal{R}$ whose poverty line is flat and absolute threshold is zero. Formally,

$$\mathcal{R}^f_0 := \{\succeq \in \mathcal{R} | Z^a(\succeq) = 0 \text{ and } Z(\succeq) \text{ is flat}\}.$$

Any $\succeq \in \mathcal{R}^f_0$ with $z = Z(\succeq)$ is such that

- $z(\overline{y}) = z^* \in \mathbb{R}_{++}$ for all $\overline{y} \geq 0$ and
- $Z^a(\succeq) = 0$.

In particular, any $\succeq \in \mathcal{R}^f_0$ is such that for all $(y_i, \overline{y}) \in X_p(\succeq)$ we have

$$e^m_{\succeq}(y_i, \overline{y}) = y_i.$$

Take any $n \geq 3$. Let $Y^n(\succeq) \subset Y(\succeq)$ be the subset of income distributions of size n. Formally, I define

$$Y^n(\succeq) := \{y \in Y(\succeq) | n(y) = n\}.$$

By assumption, index P is ordinally equivalent to (10), which becomes for all $\succeq \in \mathcal{R}^f_0$ and all $y \in Y^n(\succeq)$

$$\hat{P}(y, \succeq) := \frac{1}{n} \sum_{i=1}^{n} f(y_i, z^*, 0),$$

where $f(y_i, z^*, 0) = 0$ for all $y_i > z^*$.

Ignoring the constant 0 in the definition of f, the previous equation becomes

$$\hat{P}(y, z^*) := \frac{1}{n} \sum_{i=1}^{n} f(y_i, z^*), \tag{38}$$

and holds for all $\succeq \in \mathcal{R}^f_0$, where $z^* = z(\overline{y}^m)$.

In the next steps of the proof, I define a particular transformation o of index $\hat{P}$ on the domain $\mathcal{R}^f_0$ and derives the properties of representation o when $\hat{P}$ satisfies *Scale consistency* and *Translation consistency*.

Definition of particular representation

Given $\succeq \in \mathcal{R}_0^f$, let distribution y^a be defined such that all but the richest individual earn $a \in [0, z^*]$ and the richest individual's income is such that mean income is equal to $\overline{y}^B = z^*$:

$$y^a := \left(a, \ldots, a, n\overline{y}^B - (n-1)a\right).$$

Let $o : \cup_{\succeq \in \mathcal{R}_0^f} Y^n(\succeq) \times \mathbb{R}_{++} \to [0, z^*]$ be an increasing transformation of the poverty index $\hat{P}$ implicitely defined by[38]

$$\hat{P}(y^o, z^*) = \hat{P}(y, z^*),$$

for all $\succeq \in \mathcal{R}_0^f$ and all $y \in Y^n(\succeq)$. Given the definition of $\hat{P}$, the implicit definition of $o(y, z^*)$ is such that

$$\frac{n-1}{n} f\left(o(y, z^*), z^*\right) = \frac{1}{n} \sum_{i=1}^{n} f\left(y_i, z^*\right),$$

or yet, since individual n is non-poor for all $y \in Y^n(\succeq)$ and $f\left(z^*, z^*\right) = 0$

$$o(y, z^*) = f^{-1}\left(\frac{1}{n-1} \sum_{i=1}^{n-1} f\left(y_i, z^*\right), z^*\right). \tag{39}$$

Function o is a well-defined numerical representation since for all $\succeq \in \mathcal{R}_0^f$ and all $y \in Y^n(\succeq)$

1. there exists a value $o(y, z^*) \in [0, z^*]$ such that
$$\hat{P}(y^o, z^*) = \hat{P}(y, z^*)$$
2. we have that $y^o \in Y^n(\succeq)$.

First, I prove property 2 assuming that property 1 holds. By Flat Line in Poor Societies, the income distribution y^{z^*}, which is

$$\left(z^*, \ldots, z^*, n\overline{y}^B - (n-1)z^*\right),$$

belongs to $Y^n(\succeq)$ since individual n is non-poor: $n\overline{y}^B - (n-1)z^* = z^*$. As a result, the income distribution y^o belongs to $Y^n(\succeq)$ for all $o < z^*$ and hence all $o \in [0, z^*]$.

Second, I prove property 1. Given the properties of function f specified in the statement of Theorem 4, For all $\succeq \in \mathcal{R}$ and all $y \in Y^n(\succeq)$ we have that

$$\hat{P}(y, \succeq) \in \left[0, \frac{n-1}{n}\right].$$

By definition of $\hat{P}$, we have for all $\succeq \in \mathcal{R}_0^f$ that

$$\begin{aligned} \hat{P}(y^o, \succeq) &= 0 && \text{if } o = z^*, \\ \hat{P}(y^o, \succeq) &= \frac{n-1}{n} && \text{if } o = 0. \end{aligned}$$

By continuity of function f and equation (39), there exists a value of $o \in [0, z^*]$ such that $P(y, \succeq) = P(y^o, \succeq)$ for all $y \in Y^n(\succeq)$.

Since f is strictly increasing in its first argument and $f(b, z^*) = 0$ for all $b \geq z^*$, we have that

$$y_1 \leq o(y, z^*) \leq z^* \leq y_n.$$

[38] Representation o is not a numerical representation as defined above, since, given $\succeq$ its image set is $[0, z^*] \neq [0, 1]$.

Properties of representation o

Lemmas 4 and 5 derive the properties of $o(y, z^*)$ when its associated index P satisfies *Translation consistency* and *Scale consistency*.

Lemma 4. *Let P be and index ordinally equivalent to (10) that satisfies Scale consistency. For all $\succeq \in \mathcal{R}_0^f$, its associated representation $o(y, z^*)$ is such that*

$$o(y^{\times\lambda}, \lambda z^*) = \lambda o(y, z^*),$$

for all $y \in Y^n(\succeq)$ and all $\lambda > 0$ such that $\succeq^{\times\lambda} \in \mathcal{R}$, $y^{\times\lambda} \in Y^n(\succeq^{\times\lambda})$ and $y^{\lambda o} \in Y^n(\succeq^{\times\lambda})$.[39]

Proof. Take any $\succeq \in \mathcal{R}_0^f$, $y \in Y^n(\succeq)$ and $\lambda > 0$. By the definition of $o(y, z^*)$ we have that

$$\hat{P}(y, \succeq) = \hat{P}(y^o, \succeq), \tag{40}$$

if $o = o(y, z^*)$.

Let the transformation $y^{\lambda o}$ of y^o be defined as:

$$y^{\lambda o} := \left(\lambda o, \ldots, \lambda o, n\overline{y}^B - (n-1)\lambda o\right).$$

Equation (40) and the previous definition imply by *Scale consistency* that

$$\hat{P}(y^{\times\lambda}, \succeq^{\times\lambda}) = \hat{P}(y^{\lambda o}, \succeq^{\times\lambda}). \tag{41}$$

By definition of $o(y, z^*)$ we have that

$$\hat{P}(y^{\times\lambda}, \succeq^{\times\lambda}) = \hat{P}(y^{\times\lambda^o}, \succeq^{\times\lambda}), \tag{42}$$

where $y^{\times\lambda^o}$ corresponds to the income distribution for which all but the richest individual earn $o(y^{\times\lambda}, \lambda z^*)$.

By transitivity we have that

$$\hat{P}(y^{\times\lambda^o}, \succeq^{\times\lambda}) = \hat{P}(y^{\lambda o}, \succeq^{\times\lambda}).$$

By monotonicity, this implies that

$$o(y^{\times\lambda}, \lambda z^*) = \lambda o(y, z^*).$$

■

Importantly, for all $\succeq \in \mathcal{R}$ and all $y \in Y^n(\succeq)$, if $\lambda \in (0, 1]$, then we have that $\succeq^{\times\lambda} \in \mathcal{R}$, $y^{\times\lambda} \in Y^n(\succeq^{\times\lambda})$ and $y^{\lambda o} \in Y^n(\succeq^{\times\lambda})$. Lemma 5 derives a parallel property for poverty indices satisfying *Translation consistency*.

Lemma 5. *Let P be and index ordinally equivalent to (10) that satisfies Translation consistency. For all $\succeq \in \mathcal{R}_0^f$, its associated representation $o(y, z^*)$ is such that*

$$o(y^{+\delta}, z^* + \delta) = o(y, z^*) + \delta,$$

for all $y \in Y^n(\succeq)$ and all $\delta \in \mathbb{R}$ such that $\succeq^{+\delta} \in \mathcal{R}$, $y^{+\delta} \in Y^n(\succeq^{+\delta})$ and $y^{o+\delta} \in Y^n(\succeq^{+\delta})$.[40]

Proof. Similar to the proof of Lemma 4 and hence omitted. ■

[39] As defined in the proof, the income distribution $y^{\lambda o}$ is

$$y^{\lambda o} := \left(\lambda o, \ldots, \lambda o, n\overline{y}^B - (n-1)\lambda o\right).$$

[40] The income distribution $y^{o+\delta}$ is defined as

$$y^{o+\delta} := \left(o + \delta, \ldots, o + \delta, n\overline{y}^B - (n-1)(o + \delta)\right).$$

Reproduce the reasoning of Ebert and Moyes (2002)

In this section, I show that if P is and index ordinally equivalent to (10) that satisfies *Translation consistency* and *Scale consistency*, then for all $\succeq \in \mathcal{R}_0^f$ and all $y \in Y^n(\succeq)$ its function f is such that

$$f(t,z) = \begin{cases} \left(\frac{z-t}{z}\right)^{\alpha} & \text{for all } t \leq z \\ 0 & \text{for all } t > z \end{cases} \tag{43}$$

for some $\alpha > 0$.'

The proof is lengthy and based on functional equation arguments exploiting the properties derived in Lemmas 4 and 5.

Consider any $\succeq \in \mathcal{R}_0^f$ (hence, any $z = z^* \in \mathbb{R}_{++}$). Choose ϵ with $0 < \epsilon < z$ and pick $y \in Y^n(\succeq)$ such that $z - \epsilon \leq y_i \leq z$ for all $i \leq n-1$ and $\overline{y} \geq z$. By Lemma 4, we have that

$$o(y^{\times\lambda}, \lambda z) = \lambda o(y, z),$$

which by (39) is equivalent to

$$f^{-1}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} f\left(\lambda y_i, \lambda z\right), \lambda z\right) = \lambda f^{-1}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} f\left(y_i, z\right), z\right),$$

provided $\succeq^{\times\lambda} \in \mathcal{R}$, $y^{\times\lambda} \in Y^n(\succeq^{\times\lambda})$ and $y^{\lambda o} \in Y^n(\succeq^{\times\lambda})$. Set $\lambda := \frac{\epsilon}{z} < 1$, which guarantees that $\succeq^{\times\lambda} \in \mathcal{R}$, $y^{\times\lambda} \in Y^n(\succeq^{\times\lambda})$ and $y^{\lambda o} \in Y^n(\succeq^{\times\lambda})$ when $\succeq \in \mathcal{R}_0^f$ and $y \in Y^n(\succeq)$ such that $z - \epsilon \leq y_i \leq z$ for all $i \leq n-1$ and $\overline{y} \geq z$.[41]

Replacing λ in the previous equation by its new expression yields

$$\frac{z}{\epsilon} f^{-1}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} f\left(\frac{\epsilon}{z} y_i, \epsilon\right), \epsilon\right) = f^{-1}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} f\left(y_i, z\right), z\right). \tag{44}$$

Similarly, by Lemma 5 we have that

$$o(y^{+\delta}, z+\delta) = o(y, z) + \delta,$$

which by (39) is equivalent to

$$f^{-1}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} f\left(y_i + \delta, z + \delta\right), z + \delta\right) = f^{-1}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} f\left(y_i, z\right), z\right) + \delta,$$

provided that $\succeq^{+\delta} \in \mathcal{R}$, $y^{+\delta} \in Y^n(\succeq^{+\delta})$ and $y^{o+\delta} \in Y^n(\succeq^{+\delta})$. Set $\delta := \epsilon - z$, which guarantees that $\succeq^{+\delta} \in \mathcal{R}$, $y^{+\delta} \in Y^n(\succeq^{+\delta})$ and $y^{o+\delta} \in Y^n(\succeq^{+\delta})$ when $\succeq \in \mathcal{R}_0^f$, $\delta \in (-z, 0)$ and $y \in Y^n(\succeq)$ such that $z - \epsilon \leq y_i \leq z$ for all $i \leq n-1$ and $\overline{y} \geq z$.[42]

Replacing δ in the previous equation by its new expression yields

$$f^{-1}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} f\left(y_i + \epsilon - z, \epsilon\right), \epsilon\right) + z - \epsilon = f^{-1}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} f\left(y_i, z\right), z\right). \tag{45}$$

Combining (44) and (45) yields

$$\frac{z}{\epsilon} f^{-1}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} f\left(\frac{\epsilon}{z} y_i, \epsilon\right), \epsilon\right) = f^{-1}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} f\left(y_i + \epsilon - z, \epsilon\right), \epsilon\right) + z - \epsilon. \tag{46}$$

Now I introduce

$$F_{\epsilon,z}(t) := f\left(\frac{\epsilon}{z} t, \epsilon\right),$$
$$G_{\epsilon,z}(t) := f(t + \epsilon - z, \epsilon),$$

[41] In particular, we have that $\succeq^{\times\lambda} \in \mathcal{R}_0^f$ with $z^* = \lambda z < z$ and $Z^a(\succeq^{\times\lambda}) = Z^a(\succeq) = 0$.

[42] In particular, we have that $\succeq^{+\delta} \in \mathcal{R}_0^f$ with $z^* = z + \delta < z$ and $Z^a(\succeq^{+\delta}) = Z^a(\succeq) = 0$.

for $t \in [z-\epsilon, z]$.

From these definitions

$$F_{\epsilon,z}^{-1}(u) = \frac{z}{\epsilon} f^{-1}(u,\epsilon) \qquad \text{for } u \in [h,k] = Imagef\left(\left[\frac{\epsilon}{z}(z-\epsilon), \epsilon\right]\right),$$
$$G_{\epsilon,z}^{-1}(u) = f^{-1}(u,\epsilon) + z - \epsilon \qquad \text{for } u \in [h_1,k_1] = Imagef\left([0,\epsilon],\epsilon\right).$$

By definition $[h,k] \subset [h_1,k_1]$ since $\frac{\epsilon}{z}(z-\epsilon) > 0$ and from (46) and these definitions we have

$$F_{\epsilon,z}^{-1}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} F_{\epsilon,z}(y_i)\right) = G_{\epsilon,z}^{-1}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} G_{\epsilon,z}(y_i)\right).$$

I apply Theorem 2 in (Aczel (1966), p.290) and obtain

$$F_{\epsilon,z}^{-1}(u) = G_{\epsilon,z}^{-1}(ru+s) \qquad \text{for } u \in [h,k],$$

where $r = r(\epsilon,z)$ and $s = s(\epsilon,z)$ and therefore

$$\frac{z}{\epsilon} f^{-1}(u,\epsilon) = f^{-1}(ru+s,\epsilon) + z - \epsilon,$$

which can be rearranged to

$$\frac{z}{\epsilon}\left(f^{-1}(u,\epsilon) - \epsilon\right) = f^{-1}(ru+s,\epsilon) - \epsilon,$$

or, introducing $g^{-1}(u,\epsilon) = f^{-1}(u,\epsilon) - \epsilon$ we have

$$\frac{z}{\epsilon}\left(g^{-1}(u,\epsilon)\right) = g^{-1}(ru+s,\epsilon).$$

Replacing u by $g(v,\epsilon)$ yields

$$\frac{z}{\epsilon} v = g^{-1}(rg(v,\epsilon)+s,\epsilon),$$

or

$$g\left(\frac{z}{\epsilon}v,\epsilon\right) = rg(v,\epsilon) + s \qquad \text{for all } v \in \left[\frac{\epsilon}{z}(z-\epsilon)-\epsilon, 0\right].$$

Now I insert $v = 0$ and get $g(0,\epsilon) = rg(0,\epsilon)$, i.e.

$$s = (1-r)g(0,\epsilon).$$

Therefore,

$$g\left(\frac{z}{\epsilon}v,\epsilon\right) = rg(v,\epsilon) + (1-r)g(0,\epsilon),$$

and hence

$$g\left(\frac{z}{\epsilon}v,\epsilon\right) - g(0,\epsilon) = r(\epsilon,z)(g(v,\epsilon) - g(0,\epsilon)). \tag{47}$$

The previous equation "factors" a function of a product between z and v into a product of two functions, one of which depends on z and the other on v. I define three new functions H, F and G, some of which depend on the new variable $w = -v$ in the following way

$$H(w) := g\left(-\frac{w}{\epsilon},\epsilon\right) - g(0,\epsilon) \qquad \text{for } w \in \left[0, -\frac{\epsilon}{z}(z-\epsilon)+\epsilon\right], \tag{48}$$
$$F(z) := r(\epsilon,z), \tag{49}$$
$$G(w) := g(-w,\epsilon) - g(0,\epsilon). \tag{50}$$

With these definitions, (47) becomes

$$H(zw) = F(z)G(w).$$

The solution to the previous equation is given by Theorem 4 (Aczel (1966), p.144):

$$H(w) = abw^{\alpha}, \tag{51}$$
$$F(z) = bz^{\alpha}, \tag{52}$$
$$G(w) = aw^{\alpha}. \tag{53}$$

Using (48), (50), (51) and (53) we get $b = \epsilon^{-\alpha}$ and therefore[43]

$$r(\epsilon, z) = \epsilon^{-\alpha} z^{\alpha}.$$

In summary, using (50), (53) and the fact that $w = -v$ we get

$$g(v, \epsilon) = a(\epsilon)(-v)^{\alpha(\epsilon)} + b(\epsilon),$$
$$r(\epsilon, z) = \epsilon^{-\alpha(\epsilon)} z^{\alpha(\epsilon)},$$

and given that $g^{-1}(u, \epsilon) = f^{-1}(u, \epsilon) - \epsilon$ we have

$$f(t, \epsilon) = g(t - \epsilon, \epsilon) = a(\epsilon)(\epsilon - t)^{\alpha(\epsilon)} + b(\epsilon),$$

where $b(\epsilon) = g(0, \epsilon)$ for all $t \in \left[\frac{\epsilon}{z}(z - \epsilon), \epsilon\right]$, and $\alpha(\epsilon) > 0$ and $a(\epsilon) > 0$ since f has to be well-defined and strictly decreasing in t.

These considerations can be made for every ϵ with $0 < \epsilon < z$ and every $z \in \mathbb{R}_{++}$ (hence every $\succeq \in \mathcal{R}_0^f$). Given that[44]

$$f^{-1}(u, \epsilon) = \epsilon + g^{-1}(u, \epsilon) = \epsilon - \left(\frac{u - b(\epsilon)}{a(\epsilon)}\right)^{\frac{1}{\alpha(\epsilon)}},$$

and remembering that ϵ is a transformation of z yields

$$o(y, z) = f^{-1}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} f\left(y_i, z\right), z\right) = z - \left(\frac{1}{n-1}\sum_{i=1}^{n-1}(z - y_i)^{\alpha(z)}\right)^{\frac{1}{\alpha(z)}}$$

From Lemma 4, we have that $o(y^{\times\lambda}, \lambda z) = \lambda o(y, z)$ and thus

$$\lambda z - \left(\frac{1}{n-1}\sum_{i=1}^{n-1}(\lambda z - \lambda y_i)^{\alpha(\lambda z)}\right)^{\frac{1}{\alpha(\lambda z)}} = \lambda\left[z - \left(\frac{1}{n-1}\sum_{i=1}^{n-1}(z - y_i)^{\alpha(z)}\right)^{\frac{1}{\alpha(z)}}\right].$$

This implies that $\alpha(\lambda z) = \alpha(z) = \alpha$ and therefore

$$f(y_i, z) = a(z)(z - y_i)^{\alpha} - b(z).$$

For the case $y_i = z$ the previous equation becomes $f(z, z) = b(z)$, which implies that $b(z) = 0$ since $f(z, z) = 0$ (from (10)). Furthermore, for the case $y_i = 0$, we have $f(0, z) = a(z)z^{\alpha}$, which implies that $a(z) = z^{-\alpha}$ since $f(0, z) = 1$ (from (10) again). Therefore, we have

$$f(y_i, z) = \left(\frac{z - y_i}{z}\right)^{\alpha}.$$

Bounds in Theorem 5

The exponential expression of (43) is obtained for all $\succeq \in \mathcal{R}_0^f$. By (10), this expression is valid for all $\succeq \in \mathcal{R}_0$, where

$$\mathcal{R}_0 := \{\succeq \in \mathcal{R} | Z^a(\succeq) = 0\},$$

given that function f does not depend on the particular EO considered beyond its arguments. This proves the lower-bound on function f.

The exponential upper-bound on f is obtained using a parallel argument. The lower-bound is associated to $\alpha_0 > 0$ and the upper-bound to $\alpha_1 > 0$. Given that expression (10) requires that f is non-decreasing in z^a, this implies that $\alpha_0 \geq \alpha_1 > 0$.

[43] The definitions imply that $H(w) = G\left(\frac{w}{\epsilon}\right)$ and hence $abw^{\alpha} = a\left(\frac{w}{\epsilon}\right)^{\alpha}$.
[44] The following expression is obtained by inversing $g(v, \epsilon) = u$.

8 Index satisfying *Scale consistency* and *Translation consistency*

Assume that index P is defined as

$$P(y, \succeq) := \frac{1}{n(y)} \sum_{i=1}^{q(y)} \left(1 - \frac{e_{\succeq}^{m}(y_i, \overline{y})}{z^m}\right)^{\alpha},$$

with $\alpha > 0$. I prove that P satisfies *Translation consistency*.

Take any $\succeq \in \mathcal{R}$ and any two $x, y \in Y(\succeq)$ with $\overline{x} = \overline{y}$. Assume that

$$P(x, \succeq) \geq P(y, \succeq),$$

which is

$$\frac{1}{n(x)} \sum_{i=1}^{q(x)} \left(1 - \frac{e_{\succeq}^{m}(x_i, \overline{x})}{z^m}\right)^{\alpha} \geq \frac{1}{n(y)} \sum_{i=1}^{q(y)} \left(1 - \frac{e_{\succeq}^{m}(y_i, \overline{y})}{z^m}\right)^{\alpha},$$

or

$$\frac{1}{n(x)(z^m)^{\alpha}} \sum_{i=1}^{q(x)} \left(z^m - e_{\succeq}^{m}(x_i, \overline{x})\right)^{\alpha} \geq \frac{1}{n(y)(z^m)^{\alpha}} \sum_{i=1}^{q(y)} \left(z^m - e_{\succeq}^{m}(y_i, \overline{y})\right)^{\alpha},$$

or still

$$\frac{1}{n(x)} \sum_{i=1}^{q(x)} \left(z^m - e_{\succeq}^{m}(x_i, \overline{x})\right)^{\alpha} \geq \frac{1}{n(y)} \sum_{i=1}^{q(y)} \left(z^m - e_{\succeq}^{m}(y_i, \overline{y})\right)^{\alpha}. \tag{54}$$

I show that

$$P(x^{+\delta}, \succeq^{+\delta}) \geq P(y^{+\delta}, \succeq^{+\delta}),$$

which is

$$\frac{1}{n(x)} \sum_{i=1}^{q(x)} \left(1 - \frac{e_{\succeq^{+\delta}}^{m}(x_i + \delta, \overline{x})}{z^m + \delta}\right)^{\alpha} \geq \frac{1}{n(y)} \sum_{i=1}^{q(y)} \left(1 - \frac{e_{\succeq^{+\delta}}^{m}(y_i + \delta, \overline{y})}{z^m + \delta}\right)^{\alpha}.$$

Given that the transformation $\succeq^{+\delta}$ is such that

$$e_{\succeq^{+\delta}}^{m}(y_i + \delta, \overline{y}) = e_{\succeq}^{m}(y_i, \overline{y}) + \delta,$$

the previous equation is rewritten

$$\frac{1}{n(x)(z^m + \delta)^{\alpha}} \sum_{i=1}^{q(x)} \left(z^m - e_{\succeq}^{m}(x_i, \overline{x})\right)^{\alpha} \geq \frac{1}{n(y)(z^m + \delta)^{\alpha}} \sum_{i=1}^{q(y)} \left(z^m - e_{\succeq}^{m}(y_i, \overline{y})\right)^{\alpha},$$

which, by (54), yields the desired result. *Scale consistency* is proven using a parallel argument.

9 Proof of Theorem 6

This proof is in two steps. In step 1, I show that $f_0 := f\left(e_{\succeq}^{m}(y_i, \overline{y}), z(\overline{y}^m), 0\right)$ satisfies *Monotonicity in Income* only if $\alpha_0 = 1$. In step 2, I show using *Transfer* that f has the same expression as that of f_0 for all other values of z^a. In step 3, I show that the index proposed satisfies both *Monotonicity in Income* and *Transfer*.

Step 1: *Monotonicity in Income* only for linear expression for f_0

By Theorem 5, any index P ordinally equivalent to (10) satisfies *Scale consistency* and *Translation consistency* only if for all $(y_i, \overline{y}) \in X$ we have

$$f_0\left(e^m_{\succeq}(y_i,\overline{y}), z^m\right) := f\left(e^m_{\succeq}(y_i,\overline{y}), z^m, 0\right) = \left(1 - \frac{e^m_{\succeq}(y_i,\overline{y})}{z^m}\right)^{\alpha}$$

where $\alpha > 0$. Appendix 8 shows that this condition is sufficient for P to satisfy *Scale consistency* and *Translation consistency* at least on the restricted domain $\mathcal{R}_0$, where

$$\mathcal{R}_0 := \{\succeq \in \mathcal{R} | Z^a(\succeq) = 0\}.$$

Consider any $\succeq \in \mathcal{R}_0$ for which $z = Z(\succeq)$ is a weakly relative line of slope $s > 0$.

Theorem 2 shows that any additive P whose numerical representation is almost everywhere differentiable satisfies *Monotonicity in Income* on $\mathcal{R}_0$ if and only if for all $\overline{y} \geq \overline{y}^m$, $a \in [0, z(\overline{y}))$ and $b \in (z^a, z(\overline{y})]$, we have:

$$\frac{\partial f_0\left(e^m_{\succeq}(b,\overline{y}), z^m\right)}{\partial \overline{y}} \leq -\frac{\partial f_0\left(e^m_{\succeq}(a,\overline{y}), z^m\right)}{\partial y_i} \tag{55}$$

Given its exponential form, for all $y \in Y(\succeq)$ and all $i \leq q(y)$ the partial derivative of f_0 with respect to own income is:

$$\frac{\partial f_0\left(e^m_{\succeq}(y_i,\overline{y}), z^m\right)}{\partial y_i} = -\frac{\alpha}{z^m}\left(1 - \frac{e^m_{\succeq}(y_i,\overline{y})}{z^m}\right)^{\alpha-1}\frac{z^m - z^a}{z(\overline{y}) - z^a}. \tag{56}$$

For all $y \in Y(\succeq)$ and all $i \leq q(y)$, the partial derivative of f_0 with respect to mean income is:

$$\frac{\partial f_0\left(e^m_{\succeq}(y_i,\overline{y}), z^m\right)}{\partial \overline{y}} = -\frac{\partial z(\overline{y})}{\partial \overline{y}}\frac{y_i - z^a}{z(\overline{y}) - z^a}\frac{\partial f_0\left(e^m_{\succeq}(y_i,\overline{y}), z^m\right)}{\partial y_i}, \tag{57}$$

where $\frac{\partial z(\overline{y})}{\partial \overline{y}} = s$ given that $\overline{y} \geq \overline{y}^m$ and the definition of a weakly relative line. Observe that the previous expression holds as well for any non-poor individual i who earns an income $y_i = z(\overline{y})$, because $s > 0$.[45]

Given (56) and (57), condition (55) becomes

$$s\frac{b}{z(\overline{y})} \leq \left(\frac{z^m - e^m_{\succeq}(a,\overline{y})}{z^m - e^m_{\succeq}(b,\overline{y})}\right)^{\alpha-1}. \tag{58}$$

I show that *Monotonicity in Income* is violated on $\mathcal{R}_0$ if $\alpha \neq 1$. Two cases must be considered.

- Case 1: $0 < \alpha < 1$.
  For any $a \in [0, z(\overline{y}))$ there exists a value $\underline{b} \in [a, z(\overline{y}))$ such that if $b \geq \underline{b}$, the necessary condition for *Monotonicity in Income* expressed in (58) is violated.

  As b tends to $z(\overline{y})$ we have that the left-hand-side of (58) tends to s and the right-hand-side of (58) tends to 0 as $e^m_{\succeq}(b,\overline{y})$ tends to z^m and $\alpha < 1$.

- Case 2: $\alpha > 1$.
  For any $b \in (z^a, z(\overline{y}))$ there exists a value $\underline{a} \in [b, z(\overline{y}))$ such that if $a \geq \underline{a}$, the necessary condition for *Monotonicity in Income* expressed in (58) is violated.

  As a tends to $z(\overline{y})$ we have that the right-hand-side of (58) tends to 0 as $e^m_{\succeq}(a,\overline{y})$ tends to z^m and $\alpha > 1$.

Step 2: *Transfer* only if $f = f_0$ for all z^a

In step 1, I show that any index P ordinally equivalent to (10) that satisfies *Scale consistency* and *Translation consistency* violates *Monotonicity in Income* on $\mathcal{R}_0$ except if

$$f\left(e^m_{\succeq}(y_i,\overline{y}), z^m, 0\right) = \left(1 - \frac{e^m_{\succeq}(y_i,\overline{y})}{z^m}\right)$$

[45] Its expression is defined in equation (57) as $\lim_{x\to\overline{y}^-}\partial_1 f_0\left(e^m_{\succeq}(y_i,x), z^m\right)\partial_2 e^m_{\succeq}(y_i,x)$.

Theorem 4 shows that for all $z^a \in [0, z^m]$ and all $e \in [0, z^m]$ we have

$$f(e, z^m, 0) = \left(1 - \frac{e}{z^m}\right) \leq f(e, z^m, z^a). \tag{59}$$

A direct corollary of the condition for *Transfer* derived in Theorem 3 is that for all $z^a \in [0, z^m]$ and all $e \in [0, z^m]$ we have

$$\left(1 - \frac{e}{z^m}\right) \geq f(e, z^m, z^a). \tag{60}$$

Together, (59) and (60) implies for all $z^a \in [0, z^m]$ that

$$f\left(e^m_{\succeq}(y_i, \overline{y}), z^m, z^a\right) = \left(1 - \frac{e^m_{\succeq}(y_i, \overline{y})}{z^m}\right). \tag{61}$$

Step 3: The index satisfies *Monotonicity in Income* and *Transfer*

There remain to show that the index defined by (10) and (61) satisfies both *Monotonicity in Income* and *Transfer*. I show that the necessary and sufficient conditions derived in Theorems 2 and 3 are met. These conditions are based on the partial derivatives fo this index (whose numerical representation is almost everywhere differentiable). For all $\succeq \in \mathcal{R}$, $y \in Y(\succeq)$ and all $i \leq q(y)$ the partial derivative with respect to own income is:

$$\frac{\partial f\left(e^m_{\succeq}(y_i, \overline{y}), z^m, z^a\right)}{\partial y_i} = -\frac{1}{z^m} \qquad \text{if } y_i < z^a, \tag{62}$$

and

$$\frac{\partial f\left(e^m_{\succeq}(y_i, \overline{y}), z^m, z^a\right)}{\partial y_i} = -\frac{1}{z^m}\frac{z^m - z^a}{z(\overline{y}) - z^a} \qquad \text{if } z^a \leq y_i \leq z(\overline{y}). \tag{63}$$

Regarding the partial derivative with respect to mean income, we have

$$\frac{\partial f_0\left(e^m_{\succeq}(y_i, \overline{y}), z^m\right)}{\partial \overline{y}} = 0 \qquad \text{if } y_i \leq z^a, \tag{64}$$

and

$$\frac{\partial f_0\left(e^m_{\succeq}(y_i, \overline{y}), z^m\right)}{\partial \overline{y}} = \frac{\partial z(\overline{y})}{\partial \overline{y}} \frac{y_i - z^a}{z(\overline{y}) - z^a} \frac{1}{z^m} \frac{z^m - z^a}{z(\overline{y}) - z^a} \qquad \text{if } z^a < y_i \leq z(\overline{y}). \tag{65}$$

The condition for *Transfer* derived in 3 amounts to requiring that for all a, b with $0 \leq a < b < z^m$ we have that

$$\partial_1 f(a, z^m, z^a) \leq \partial_1 f(b, z^m, z^a),$$

which is guaranteed by (62) and (63) since given that $a < b$ we have either

$$\partial_1 f(a, z^m, z^a) = \partial_1 f(b, z^m, z^a),$$

if $a < b \leq z^a$ or if $z^a \leq a < b$, or we have

$$\partial_1 f(a, z^m, z^a) = \frac{z^m - z^a}{z(\overline{y}) - z^a}\partial_1 f(b, z^m, z^a),$$

if $a < z^a < b$.

The condition for *Monotonicity in Income* derived in 2 amounts to requiring that for all $\succeq \in \mathcal{R}$, all $\overline{y} > \overline{y}^m$, all $a \in [0, z(\overline{y}))$ and all $b \in (z^a, < z(\overline{y})$ we have that

$$\partial_2 f\left(e^m_{\succeq}(b, \overline{y}), z^m, z^a\right) \leq -\partial_1 f\left(e^m_{\succeq}(a, \overline{y}), z^m, z^a\right).$$

Given that $z(\overline{y}) \geq z^m$, (62) and (63) show that

$$\frac{1}{z^m}\frac{z^m - z^a}{z(\overline{y}) - z^a} \leq -\partial_1 f\left(e^m_{\succeq}(a, \overline{y}), z^m, z^a\right).$$

Equation (65) show that

$$\partial_2 f\left(e^m_{\succeq}(b,\overline{y}), z^m, z^a\right) \leq \frac{\partial z(\overline{y})}{\partial \overline{y}} \frac{b-z^a}{z(\overline{y})-z^a} \frac{1}{z^m} \frac{z^m-z^a}{z(\overline{y})-z^a}.$$

Together, the sufficient condition for *Monotonicity in Income* holds if

$$\frac{\partial z(\overline{y})}{\partial \overline{y}} \frac{b-z^a}{z(\overline{y})-z^a} \leq 1,$$

which is the case as $b \leq z(\overline{y})$ and $\frac{\partial z(\overline{y})}{\partial \overline{y}} \leq 1$ by Positive Slope less than One.

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