Measurement of equality of opportunity: A normative approach*

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Abstract. We develop a normative approach to the measurement of equality of opportunity. That is, we measure inequality of opportunity by the welfare gain obtained in moving from the actual income distribution to the optimal income distribution of the total available income. Our study brings together the main approaches in the literature: we axiomatically characterize social welfare functions, we obtain prominent allocation rules as their optima, and we derive familiar classes of inequality of opportunity measures. Our analysis captures moreover the key philosophical distinctions in the literature: ex post versus ex ante compensation, and liberal versus utilitarian reward.

Keywords. Equality of opportunity · Income inequality · Responsibility · Social welfare

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1 Introduction

The ideal of equality of opportunity rests on two principles. The compensation principle condemns the inequalities that arise from individual circumstances (e.g., parental background). The reward principle, by contrast, condones or even justifies the inequalities due to the exercise of individual responsibility (e.g., work effort). Differentiating between circumstance and responsibility characteristics allows for a broad ethical range, from leftist positions that would put most individual characteristics in the circumstance basket, to rightist positions that would put most characteristics in the responsibility basket.

We develop a normative approach to the measurement of inequality of opportunity. In this approach, measures of inequality of opportunity are derived from social welfare functions.[1] We first characterize social welfare functions on the basis of axioms that express elementary ethical values. To derive inequality of opportunity measures, we define inequality of opportunity to be equal to the social welfare gain obtained in moving from the actual income distribution to the optimal income distribution of the total available income. The resulting inequality of opportunity measures inherit a normative foundation from the social welfare functions on which they are based.

The literature on equality of opportunity has grown rapidly since the pioneering contributions by Roemer (1993), Van de gaer (1993), Fleurbaey (1994) and Bossert (1995).[2] Unfortunately, this literature lacks a unified framework within which contributions can be easily connected and compared. First, different contributions have adopted sharply distinct philosophical views on how to define the compensation and reward principles. As a result, there is no single agreed upon conceptualization of equality of opportunity, but rather several competing ones. Second, contributions have focused on a variety of criteria as the object of study. Some study social welfare functions, others study allocation rules and yet others study measures of inequality of opportunity. We argue that the normative approach is particularly useful to work towards a framework that encompasses the philosophical and methodological diversity of the literature.

First, the normative approach is an axiomatic approach, which makes it well suited to study the implications of different philosophical views. Our

[1]The normative approach was originally proposed in the simpler setting of income inequality measurement by Kolm (1969), Atkinson (1970) and Sen (1973).

[2]For surveys, see Fleurbaey (2008), Fleurbaey and Maniquet (2011), Fleurbaey and Schokkaert (2011), Pignataro (2012), Ramos and Van de gaer (2016), Roemer and Trannoy (2014, 2016) and Ferreira and Peragine (2016). The economic literature builds on earlier work in political philosophy: see Rawls (1971), Dworkin (1981a,b), Arneson (1989, 1990) and Cohen (1989).

key axioms embody the different views in the literature on compensation and reward. The compensation principle comes in two versions. Ex post compensation (Roemer, 1993; Fleurbaey, 1994) says that the incomes of individuals exercising the same responsibility should be equalized, whereas ex ante compensation (Van de gaer, 1993) says that the incomes of groups with different circumstances should be equalized.[3] The reward principle also comes in two versions. Liberal reward (Fleurbaey, 1994; Bossert and Fleurbaey, 1996) says that income differences due to the exercise of responsibility should respect the market returns to responsibility, whereas utilitarian reward (Roemer, 1993; Van de gaer, 1993) says that all income differences due to responsibility are irrelevant. We have axioms for the ex post and ex ante versions of compensation, for the liberal and utilitarian versions of reward, as well as for several variants of these. This allows the study of all these different philosophical outlooks within a single framework.

Second, the normative approach produces a natural connection between the different criteria that have been used to formalize the idea of equality of opportunity. The connection between the characterized social welfare functions and the derived inequality of opportunity measures is immediate. Moreover, we are naturally led to examine the allocation rules corresponding to the characterized social welfare functions because the optimal distribution of a given amount of income plays a crucial role in the normative approach. Our social welfare functions, allocation rules and inequality of opportunity measures all take forms that cover and generalize prominent classes in the literature. Our approach thus links together these different criteria in a unified framework.

We proceed as follows. The next section introduces notation and some basic axioms that we always impose on the social welfare functions. Section 3 defines and discusses the axioms that capture the ex post and ex ante versions of compensation and the liberal and utilitarian versions of reward. Section 4 characterizes classes of social welfare functions using the basic axioms and different combinations of compensation and reward principles. Section 5 derives optimal distributions from the obtained social welfare functions. We link the induced allocation rules to those proposed in the literature. Section 6 explains the normative approach to the measurement of inequality of opportunity and derives the classes of inequality measures corresponding to the previously characterized classes of social welfare functions. The obtained inequality of opportunity measures are discussed in the light of the literature. Section 7 concludes.

[3]The distinction between the ex post and ex ante perspectives is due to Ooghe, Schokkaert and Van de gaer (2007). They link the ex ante perspective to the literature on equalizing opportunity sets (e.g., Kranich, 1996; Ok, 1997; Ok and Kranich, 1998 and Weymark, 2003.)

2 Preliminaries

Each individual is characterized by his circumstance and responsibility characteristics. The set of all circumstance characteristics is $C = \{1, 2, \ldots, c\}$ and the set of all responsibility characteristics is $R = \{1, 2, \ldots, r\}$. For simplicity, we assume that each combination (i, k) in $C \times R$ occurs exactly once.[4] We refer to each (i, k) as an individual.

We use a $c \times r$ real-valued matrix X to represent an income distribution. The ikth entry of X, denoted by x_{ik}, is the income of individual (i, k). The ith row of X is denoted by $x_{i\cdot}$ and the kth column is denoted by $x_{\cdot k}$. We denote the average income in X by $\bar{X}$. Similarly, we denote the average incomes in $x_{i\cdot}$ and $x_{\cdot k}$ by $\bar{x}_{i\cdot}$ and $\bar{x}_{\cdot k}$. We write $1_{c\times r}$ for the $c \times r$ matrix with 1 at each entry and 1_r for the r-dimensional vector with 1 at each entry.

The hypothetical laissez-faire market incomes are ethically significant according to the liberal reward principle. We denote this income distribution by M in $\mathbb{R}^{c\times r}$ and refer to it simply as the market income distribution. We assume that M is fixed.[5]

We use a social welfare function to compare income distributions. A social welfare function $W : \mathbb{R}^{c\times r} \to \mathbb{R}$ assigns to each income distribution X in $\mathbb{R}^{c\times r}$ a real number $W(X)$. The function W depends on M, but we suppress this dependency in the notation.

We impose axioms on the social welfare function to make concrete its normative properties. In the remainder of this section, we formulate three basic axioms. The next section discusses more substantive axioms representing the ideas of compensation and reward.

Monotonicity says that increasing the income of an individual is socially desirable provided that no other individual's income decreases.

Monotonicity. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if $x_{ik} \geq x'_{ik}$ for each individual (i, k) in $C \times R$ and $x_{jl} > x'_{jl}$ for some individual (j, l) in $C \times R$, then $W(X) > W(X')$.

Continuity ensures that social welfare comparisons are not overly sensitive to small changes in the income distributions.

Continuity. The function W is continuous.

Translation invariance demands that the social welfare ranking of two income distributions does not change if the same amount is added to each income

[4] The extension to the general case where some combinations occur more than once or do not occur at all is possible, but would require considerably heavier notation without adding real substance.

[5] In Section 7, we discuss the repercussions of allowing M to vary.

in both income distributions.[6]

Translation invariance. For all income distributions X and X' in $\mathbb{R}^{c\times r}$ and for each real number λ, we have $W(X) \geq W(X')$ if and only if $W(X+\lambda 1_{c\times r}) \geq W(X' + \lambda 1_{c\times r})$.

Translation invariance ensures that the inequality of opportunity measures we derive later are absolute. That is, adding the same amount to each income does not change the level of inequality of opportunity.

3 Compensation and reward

3.1 Compensation axioms

The compensation principle says that income inequalities due to differences in circumstances ought to be redressed. There are two versions of compensation, ex post compensation and ex ante compensation. To understand the terminology, imagine that circumstance characteristics are determined prior to responsibility characteristics. Ex ante compensation is defined in terms of the income possibilities of circumstance groups before responsibility characteristics are determined, whereas ex post compensation is defined in terms of the actual incomes that arise after responsibility characteristics are also determined.

Ex post compensation comprises two components, a Pigou-Dalton transfer principle and a symmetry principle. Together, these components express the idea that individuals who exercise the same responsibility should be treated equally. Ex post Pigou-Dalton requires that an income transfer that widens the income gap between two individuals in the same responsibility group reduces social welfare.

Ex post Pigou-Dalton. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two individuals (i,k) and (j,k) in $C \times R$ such that $x_{ik} \geq x_{jk}$ and a positive real number δ such that $x'_{ik} = x_{ik} + \delta$ and $x'_{jk} = x_{jk} - \delta$ with X and X' coinciding everywhere else, then $W(X) > W(X')$.

Ex post symmetry demands that switching the incomes of two individuals in the same responsibility group does not change social welfare.

Ex post symmetry. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two individuals (i,k) and (j,k) in $C\times R$ such that $x_{ik} = x'_{jk}$ and $x_{jk} = x'_{ik}$ with X and X' coinciding everywhere else, then $W(X) = W(X')$.

[6] In Section 7, we discuss how our results change if translation invariance is replaced by scale invariance. Scale invariance says that the social welfare ranking of two income distributions should not change if we multiply each income with the same factor in both income distributions.

We refer to the combination of ex post Pigou-Dalton and ex post symmetry as ex post compensation.

Ex post compensation. Both ex post Pigou-Dalton and ex post symmetry hold.

Next, we define the ex ante version of compensation. To understand ex ante compensation, interpret row i of an income distribution as the (income) opportunities of an individual with circumstance characteristic i. Ex ante compensation says that differences in circumstances do not justify differences in these opportunities. The axiom consists of, again, a Pigou-Dalton transfer and a symmetry component.

Ex ante Pigou-Dalton requires that increasing the gap between opportunities decreases social welfare. Assume that the minimum income in circumstance group i is greater than the maximum income in circumstance group j. We can then conclude that group i is unambiguously better off than group j. Now, imagine a transfer from each individual in j to each individual in i. Ex ante Pigou-Dalton requires that such a transfer reduces social welfare.

Ex ante Pigou-Dalton. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two circumstance groups i and j in C such that $\min_{k\in R} x_{ik} \geq \max_{k\in R} x_{jk}$ and a positive real number δ such that $x'_{i\cdot} = x_{i\cdot} + \delta 1_r$ and $x'_{j\cdot} = x_{j\cdot} - \delta 1_r$ with X and X' coinciding everywhere else, then $W(X) > W(X')$.

Ex ante symmetry requires that switching two rows of an income distribution—one row again unambiguously better than the other as defined above—does not change social welfare.

Ex ante symmetry. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two circumstance groups i and j in C such that $\min_{k\in R} x_{ik} \geq \max_{k\in R} x_{jk}$ and $x_{i\cdot} = x'_{j\cdot}$ and $x_{j\cdot} = x'_{i\cdot}$ with X and X' coinciding everywhere else, then $W(X) = W(X')$.

We refer to the combination of ex ante Pigou-Dalton and ex ante symmetry as ex ante compensation.

Ex ante compensation. Both ex ante Pigou-Dalton and ex ante symmetry hold.

Obviously, ex post Pigou-Dalton implies ex ante Pigou-Dalton and ex post symmetry implies ex ante symmetry. By consequence, ex post compensation implies ex ante compensation.[7]

[7]Consider a stronger version of ex ante compensation, which regards group i as better off than group j if the income vector of group i first order stochastically dominates the income

3.2 *Reward axioms*

The reward principle complements the compensation principle. Whereas compensation aims to neutralize differences in circumstances, reward tells us how to respect differences in responsibility. The literature considers two versions of reward, liberal reward and utilitarian reward.

Liberal reward states that differences in the market incomes of individuals in the same circumstance group should be respected. A useful restatement of this idea is that each individual in the same circumstance class should receive the same subsidy, where a subsidy is defined as the actual income minus the market income.

Liberal reward consists of two components, a Pigou-Dalton transfer principle and a symmetry principle. Consider two individuals in the same circumstance group i. The subsidies received by (i, k) and (i, l) in income distribution X are $x_{ik} - m_{ik}$ and $x_{il} - m_{il}$. Assume that the subsidy received by (i, k) is greater than the subsidy received by (i, l). Liberal Pigou-Dalton requires that transferring income from (i, l) to (i, k) reduces social welfare, as such an income transfer further widens the gap between the subsidies received by the two individuals.

Liberal Pigou-Dalton. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two individuals (i, k) and (i, l) in $C \times R$ such that $x_{ik} - m_{ik} \geq x_{il} - m_{il}$ and a positive real number δ such that $x'_{ik} = x_{ik} + \delta$ and $x'_{il} = x_{il} - \delta$ with X and X' coinciding everywhere else, then $W(X) > W(X')$.

We illustrate the axiom with an example. Imagine a society with one circumstance group and three responsibility groups. Consider the income distributions $X = (9, 9, 15)$ and $X' = (10, 8, 15)$. The market income distribution is $M = (7, 9, 14)$. The distributions of subsidies in X and X' are $X - M = (2, 0, 1)$ and $X' - M = (3, -1, 1)$. The gap between the subsidies received by the first two individuals is smaller in X than in X'. Thus, liberal Pigou-Dalton says that X is better than X'.[8]

Liberal symmetry demands that switching the subsidies of two individuals in the same circumstance group leaves social welfare unchanged. Note that such a switch does not alter the total income of the circumstance group.

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vector of group j. The Pigou-Dalton component then requires that a transfer from a member of group j to a member of group i reduces social welfare, and the symmetry component requires that switching the income vectors of groups i and j does not change social welfare. This stronger version of ex ante compensation clashes with ex post compensation: see Fleurbaey and Peragine (2013). It is easy to show that the stronger version is implied by the combination of ex ante compensation and utilitarian reward. Hence, the social welfare functions in Theorem 5 also satisfy the stronger version of ex ante compensation.

[8] Repeated use of liberal Pigou-Dalton yields the optimal income distribution $(8, 10, 15)$. In this income distribution, each individual receives a subsidy of 1.

Liberal symmetry. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two individuals (i,k) and (i,l) in $C \times R$ such that $x_{ik} - m_{ik} = x'_{il} - m_{il}$ and $x_{il} - m_{il} = x'_{ik} - m_{ik}$ with X and X' coinciding everywhere else, then $W(X) = W(X')$.

To illustrate the axiom, let $X'' = (7, 11, 15)$ and $M = (7, 9, 14)$. Liberal symmetry says that X'' and $X = (9, 9, 15)$ are equally good since $X'' - M = (0, 2, 1)$ is obtained from $X - M = (2, 0, 1)$ by switching the subsidies received by the first two individuals.

We refer to the combination of liberal Pigou-Dalton and liberal symmetry as liberal reward.

Liberal reward. Both liberal Pigou-Dalton and liberal symmetry hold.

Liberal reward says that, for an individual with circumstances i, the move from responsibility k to l should be rewarded as it is rewarded by the market, that is, by an income change of $m_{il} - m_{ik}$. We stress that the same axiom can capture alternative reward principles by letting M be, instead of the market income distribution, an income distribution featuring alternative ideal income differences, e.g., based on an independent concept of desert.[9]

Next, we define utilitarian reward. Utilitarian reward takes the agnostic view that equality of opportunity should be silent on how to reward differences in responsibility. Accordingly, utilitarian reward requires the social welfare function to be neutral with respect to transfers within a circumstance group. There is no need to separately define Pigou-Dalton transfer and symmetry components for utilitarian reward since the axiom as stated includes both ideas.

Utilitarian reward. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two individuals (i,k) and (i,l) in $C \times R$ and a positive real number δ such that $x'_{ik} = x_{ik} + \delta$ and $x'_{il} = x_{il} - \delta$ with X and X' coinciding everywhere else, then $W(X) = W(X')$.

4 Social welfare functions

4.1 Compensation and liberal reward

We first focus on the combination of ex post compensation and liberal reward. As the following example shows, these two axioms clash. Assume that there are two circumstance groups and two responsibility groups. Consider

$$X = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}, \quad X' = \begin{bmatrix} 1 & 7 \\ 5 & 3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 4 & 10 \\ 2 & 0 \end{bmatrix}.$$

[9] See Roemer (2012, pp. 178-179) for a discussion of alternative reward principles.

We have

$$X - M = \begin{bmatrix} -1 & -5 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad X' - M = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}.$$

Ex post Pigou-Dalton implies $W(X) > W(X')$, whereas liberal Pigou-Dalton implies $W(X') > W(X)$.

Ex post compensation and liberal reward can be combined only if market income can be written as an additively separable function of circumstance and responsibility characteristics.[10] We say that market incomes are additively separable if $m_{ik} - m_{il} = m_{jk} - m_{jl}$ for all circumstance groups i and j and all responsibility groups k and l.[11]

Proposition 1. *A social welfare function that satisfies ex post Pigou-Dalton and liberal Pigou-Dalton exists only if market incomes are additively separable.*

We will in two ways deal with the incompatibility between ex post Pigou-Dalton and liberal Pigou-Dalton. First, we combine the axioms under the restriction of additively separable market incomes (Theorem 1). Second, we consider compromise versions of liberal reward and ex post compensation (Theorems 2 and 3).

Theorem 1 restricts market incomes to be additively separable, and characterizes social welfare functions that satisfy ex post compensation and liberal reward in addition to the three basic axioms monotonicity, continuity and translation invariance. As we will see in Section 5, the social welfare functions in Theorem 1 extend the "natural" allocation rule. We denote the set of cr-dimensional real valued vectors by $\mathbb{R}^{cr}$.

Theorem 1. *Let market incomes in M be additively separable. A social welfare function W satisfies monotonicity, continuity, translation invariance, ex post compensation and liberal reward if and only if there exists a strictly increasing, continuous, translatable*[12] *and strictly Schur-concave*[13] *function*

[10]This is a well known result in the setting of allocation rules. See, for example, Bossert (1995) and Bossert and Fleurbaey (1996).

[11]Previous studies have considered a function f that assigns to each individual (i, k) a market income $f(i, k)$. The market income function f is additively separable if there exist functions g and h such that, for each (i, k) in $C \times R$, we have $f(i, k) = g(i) + h(k)$. This is equivalent to the condition that, for all i and j in C and k and l in R, we have $f(i, k) - f(i, l) = f(j, k) - f(j, l)$. This clearly corresponds to our definition of additive separability of market incomes.

[12]A function $f : \mathbb{R}^n \to \mathbb{R}$ is translatable if we have $f(x) \geq f(y)$ if and only if $f(x + \delta 1_n) \geq f(y + \delta 1_n)$ for all x and y in $\mathbb{R}^n$ and each real number δ.

[13]A bistochastic matrix is a nonnegative square matrix of which each row sums to 1 and each column sums to 1. A permutation matrix is a bistochastic matrix of which each component is either 0 or 1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is Schur concave if $f(Bx) \geq f(x)$ for

$f : \mathbb{R}^{cr} \to \mathbb{R}$ *such that, for each* X *in* $\mathbb{R}^{c \times r}$,

$$W(X) = f(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot}). \quad (1)$$

We explain why the social welfare function in Theorem 1 satisfies ex post compensation and liberal reward. (The intuition for Theorems 2 and 3 is similar.) First, for any two individuals in the same responsibility group, the same number is subtracted from their incomes. Indeed, additive separability of market incomes implies $m_{ik} - \bar{m}_{i\cdot} = m_{jk} - \bar{m}_{j\cdot}$ for all circumstance groups i and j and each responsibility group k. Strict Schur-concavity of f then ensures that ex post compensation is satisfied. Second, for any two individuals in the same circumstance group, what goes into f is their subsidies plus a uniform constant. Again, strict Schur-concavity of f guarantees that liberal reward is satisfied.

Next, we drop the restriction that market incomes are additively separable, and consider compromise versions of the liberal reward and ex post compensation axioms.[14]

We start with the compromise version of liberal reward. The idea is to use for each circumstance group the market incomes of a predetermined circumstance group $\hat{c}$ instead of the group's actual market incomes. Liberal reward is then guaranteed only with respect to the reference circumstance group $\hat{c}$. The compromise version of liberal Pigou-Dalton is as follows.

Liberal Pigou-Dalton for $\hat{c}$. Let $\hat{c}$ be a circumstance characteristic in C. For all income distributions X and X' in $\mathbb{R}^{c \times r}$, if there exist two individuals (i, k) and (i, l) in $C \times R$ such that $x_{ik} - m_{\hat{c}k} \geq x_{il} - m_{\hat{c}l}$ and a positive real number δ such that $x'_{ik} = x_{ik} + \delta$ and $x'_{il} = x_{il} - \delta$ with X and X' coinciding everywhere else, then $W(X) > W(X')$.

The corresponding version of the liberal symmetry axiom is as follows.

Liberal symmetry for $\hat{c}$. Let $\hat{c}$ be a circumstance characteristic in C. For all income distributions X and X' in $\mathbb{R}^{c \times r}$, if there exist two individuals (i, k) and (i, l) in $C \times R$ such that $x_{ik} - m_{\hat{c}k} = x'_{il} - m_{\hat{c}l}$ and $x_{il} - m_{\hat{c}l} = x'_{ik} - m_{\hat{c}k}$ with X and X' coinciding everywhere else, then $W(X) = W(X')$.

Liberal reward for $\hat{c}$ combines liberal Pigou-Dalton for $\hat{c}$ and liberal symmetry for $\hat{c}$. Note that if market incomes are additively separable, then liberal reward for $\hat{c}$ is equivalent to liberal reward. If market incomes are not additively separable, then the two axioms clash.

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each x in $\mathbb{R}^n$ and each $n \times n$ bistochastic matrix B. If, in addition, $f(Bx) > f(x)$ whenever B is not a permutation matrix, then f is strictly Schur-concave. Note that (strict) Schur-concavity of f implies symmetry of f.

[14] These versions are inspired by Bossert and Fleurbaey (1996).

Liberal reward for $\hat{c}$. Both liberal Pigou-Dalton for $\hat{c}$ and liberal symmetry for $\hat{c}$ hold.

Theorem 2 characterizes the social welfare functions that satisfy ex post compensation and liberal reward for $\hat{c}$ in addition to the three basic axioms. The result follows easily from Theorem 1[15] and we state it without proof. In Section 5 we will see that the social welfare functions in the theorem extend the egalitarian-equivalent allocation rule.

Theorem 2. *A social welfare function W satisfies monotonicity, continuity, translation invariance, ex post compensation and liberal reward for $\hat{c}$ if and only if there exists a strictly increasing, continuous, translatable and strictly Schur-concave function $f : \mathbb{R}^{cr} \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c\times r}$,*

$$W(X) = f(x_{11} - m_{\hat{c}1}, \ldots, x_{ik} - m_{\hat{c}k}, \ldots, x_{cr} - m_{\hat{c}r}). \tag{2}$$

Next, we turn to the compromise version of ex post compensation. This version guarantees ex post Pigou-Dalton only with respect to a chosen responsibility group $\hat{r}$. The compromise version of ex post Pigou-Dalton is as follows.

Ex post Pigou-Dalton for $\hat{r}$. Let $\hat{r}$ be a responsibility characteristic in R. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two individuals (i, k) and (j, k) in $C \times R$ such that $x_{ik} - m_{ik} + m_{i\hat{r}} \geq x_{jk} - m_{jk} + m_{j\hat{r}}$ and a positive real number δ such that $x'_{ik} = x_{ik} + \delta$ and $x'_{jk} = x_{jk} - \delta$ with X and X' coinciding everywhere else, then $W(X) > W(X')$.

The corresponding version of ex post symmetry is as follows.

Ex post symmetry for $\hat{r}$. Let $\hat{r}$ be a responsibility characteristic in R. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two individuals (i, k) and (j, k) in $C \times R$ such that $x_{ik} - m_{ik} + m_{i\hat{r}} = x'_{jk} - m_{jk} + m_{j\hat{r}}$ and $x_{jk} - m_{jk} + m_{j\hat{r}} = x'_{ik} - m_{ik} + m_{i\hat{r}}$ with X and X' coinciding everywhere else, then $W(X) = W(X')$.

Ex post compensation for $\hat{r}$ combines ex post Pigou-Dalton for $\hat{r}$ and ex post symmetry for $\hat{r}$. Note that if market incomes are additively separable, then ex post compensation for $\hat{r}$ coincides with ex post compensation. If not, then the two axioms clash.

Ex post compensation for $\hat{r}$. Both ex post Pigou-Dalton for $\hat{r}$ and ex post symmetry for $\hat{r}$ hold.

[15] Replace M by the income distribution of which each row equals $m_{\hat{c}\cdot}$ and apply Theorem 1.

Theorem 3 characterizes the social welfare functions that satisfy ex post compensation for $\hat{r}$ and liberal reward in addition to the three basic axioms. The proof is similar to that of Theorem 1 and is therefore omitted. Section 5 will show that the social welfare functions in the theorem extend the conditional equality allocation rule.

Theorem 3. *A social welfare function W satisfies monotonicity, continuity, translation invariance, ex post compensation for $\hat{r}$ and liberal reward if and only if there exists a strictly increasing, continuous, translatable and strictly Schur-concave function $f : \mathbb{R}^{cr} \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c\times r}$,*

$$W(X) = f(x_{11} - m_{11} + m_{1\hat{r}}, \ldots, x_{ik} - m_{ik} + m_{i\hat{r}}, \ldots, x_{cr} - m_{cr} + m_{c\hat{r}}). \quad (3)$$

We now move on to the combination of ex ante compensation and liberal reward. As the following example shows, ex ante symmetry and liberal Pigou-Dalton clash. Consider the income distributions

$$X = \begin{bmatrix} 10 & 14 \\ 1 & 3 \end{bmatrix}, \; X' = \begin{bmatrix} 1 & 3 \\ 10 & 14 \end{bmatrix}, \; X'' = \begin{bmatrix} 0 & 4 \\ 11 & 13 \end{bmatrix} \text{ and } \; X''' = \begin{bmatrix} 11 & 13 \\ 0 & 4 \end{bmatrix}$$

and the market income distribution

$$M = \begin{bmatrix} 10 & 14 \\ 1 & 3 \end{bmatrix}.$$

We have $W(X) = W(X')$ by ex ante symmetry, $W(X') < W(X'')$ by liberal Pigou-Dalton and $W(X'') = W(X''')$ by ex ante symmetry. Hence, $W(X) < W(X''')$. However, we have $W(X''') < W(X)$ by liberal Pigou-Dalton.

We again obtain that a necessary condition to avoid the clash is that market incomes are additively separable.

Proposition 2. *A social welfare function W that satisfies ex ante symmetry and liberal Pigou-Dalton exists only if market incomes are additively separable.*

For the case of ex ante compensation, we will not explore domain restrictions and compromise axioms. We suffice instead by remarking that all the social welfare functions in Theorems 1 and 2 satisfy ex ante compensation, as the latter is implied by ex post compensation.

4.2 Compensation and utilitarian reward

We begin with the combination of ex post compensation and utilitarian reward. The following example shows that the two axioms clash.[16] Consider the income distributions

[16]This example is essentially the same as that used by Ramos and Van de gaer (2016) to show a related incompatibility.

$$X = \begin{bmatrix} 8 & 7 \\ 6 & 9 \end{bmatrix} \quad \text{and} \quad X' = \begin{bmatrix} 7 & 8 \\ 7 & 8 \end{bmatrix}.$$

Ex post Pigou-Dalton implies $W(X') > W(X)$, whereas utilitarian reward implies $W(X) = W(X')$.

We consider a weakening of utilitarian reward and combine it with ex post compensation.[17] Uniform utilitarian reward says that transferring the same amount δ from each individual in a responsibility group to each individual in another responsibility group should not alter social welfare.

Uniform utilitarian reward. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two responsibility groups k and l in R and a positive real number δ such that $x'_{\cdot k} = x_{\cdot k} + \delta 1_c$ and $x'_{\cdot l} = x_{\cdot l} - \delta 1_c$ with X and X' coinciding everywhere else, then $W(X) = W(X')$.

We impose an additional axiom that puts structure on social welfare comparisons. The axiom requires that the social welfare function first aggregates the incomes of each responsibility group and second aggregates the obtained values across circumstance groups. Because this order of aggregation requires knowledge of individuals' responsibility characteristics, we refer to the axiom as ex post aggregation.

Ex post aggregation. There exist a function $\phi : \mathbb{R}^r \to \mathbb{R}$ and functions $\gamma_1, \ldots, \gamma_r : \mathbb{R}^c \to \mathbb{R}$ such that, for each income distribution X in $\mathbb{R}^{c\times r}$, we have $W(X) = \phi(\gamma_1(x_{\cdot 1}), \ldots, \gamma_r(x_{\cdot r}))$.

Theorem 4 characterizes social welfare functions that satisfy ex post compensation and uniform utilitarian reward in addition to ex post aggregation and the three basic axioms monotonicity, continuity and translation invariance.[18]

Theorem 4. *A social welfare function W satisfies monotonicity, continuity, translation invariance, ex post aggregation, ex post compensation and uniform utilitarian reward if and only if there exist a strictly increasing and continuous function $F : \mathbb{R} \to \mathbb{R}$ and a strictly increasing, continuous, unit-translatable*[19] *and strictly Schur-concave function $f : \mathbb{R}^c \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c\times r}$,*

$$W(X) = F\Big(\frac{1}{r}\sum_{k\in R} f(x_{\cdot k})\Big). \tag{4}$$

[17] Since utilitarian reward does not take market incomes into account, a restriction on the domain of market income distributions is not an option in this case.

[18] Ooghe et al. (2007) characterize classes of social welfare functions that overlap with those in Theorems 4 and 5.

[19] A function $f : \mathbb{R}^n \to \mathbb{R}$ is unit-translatable if $f(x + \delta 1_n) = f(x) + \delta$ for each real number δ.

The social welfare functions in the theorem first aggregate the incomes in each responsibility group using the function f. Strict Schur-concavity of f ensures that ex post compensation is satisfied. The obtained values are then aggregated by averaging, which ensures satisfaction of uniform utilitarian reward.

The social welfare function underlying Roemer's (1993) mean-of-mins allocation rule, $\frac{1}{r}\sum_{k\in R}\min_{i\in C} x_{ik}$, is not a member of the class in Theorem 4, but can be approached arbitrarily closely by choosing f sufficiently concave.

We now turn to the combination of ex ante compensation and utilitarian reward. Theorem 5 characterizes social welfare functions that satisfy ex ante compensation and utilitarian reward in addition to the three basic axioms.

Theorem 5. *A social welfare function W satisfies monotonicity, continuity, translation invariance, ex ante compensation and utilitarian reward if and only if there exists a strictly increasing, continuous, translatable and strictly Schur-concave function $f : \mathbb{R}^c \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c\times r}$,*

$$W(X) = f\Big(\frac{1}{r}\sum_{k\in R} x_{1k}, \frac{1}{r}\sum_{k\in R} x_{2k}, \ldots, \frac{1}{r}\sum_{k\in R} x_{ck}\Big). \tag{5}$$

The social welfare functions in the theorem first average the incomes of each circumstance group, thus ensuring satisfaction of utilitarian reward. These averages are then aggregated using the strictly Schur-concave function f, which ensures satisfaction of ex ante compensation.

The social welfare function underlying Van de gaer's (1993) min-of-means allocation rule, $\min_{i\in C}\frac{1}{r}\sum_{k\in R} x_{ik}$, while not a member of the class in Theorem 5, can be approached arbitrarily closely by choosing f sufficiently concave.

5 Allocation rules

A crucial step in the normative approach to inequality of opportunity measurement is to determine how a given amount of total income should be distributed in order to maximize social welfare. We now present the optimal income distributions for the social welfare functions characterized in the previous section. As we will see, these optima correspond to established allocation rules.

Proposition 3 presents the optimal distributions for the classes of social welfare functions in Theorems 1 to 5. In each case, the whole class settles on the same optimal distributions. The proof of the proposition is straightforward and is therefore omitted. We say that X^* is an optimal distribution of the total income amount $cr\mu$ if $\bar{X}^* = \mu$ and $W(X^*) \geq W(X)$ for each X in $\{X \in \mathbb{R}^{c\times r} : \bar{X} = \mu\}$.

Proposition 3. *Let W be a social welfare function that satisfies monotonicity, continuity and translation invariance.*

(i) Let W satisfy, in addition, ex post compensation and liberal reward, and let market incomes in M be additively separable (Theorem 1). The unique optimal distribution X^* in $\mathbb{R}^{c \times r}$ of the total income amount $cr\mu$ is such that

$$x^*_{ik} = m_{ik} - \bar{m}_{i\cdot} + \mu \qquad \text{for each } (i,k) \text{ in } C \times R.$$

(ii) Let W satisfy, in addition, ex post compensation and liberal reward for $\hat{c}$ (Theorem 2). The unique optimal distribution X^* in $\mathbb{R}^{c \times r}$ of the total income amount $cr\mu$ is such that

$$x^*_{ik} = m_{\hat{c}k} - \bar{m}_{\hat{c}\cdot} + \mu \qquad \text{for each } (i,k) \text{ in } C \times R.$$

(iii) Let W satisfy, in addition, ex post compensation for $\hat{r}$ and liberal reward (Theorem 3). The unique optimal distribution X^* in $\mathbb{R}^{c \times r}$ of the total income amount $cr\mu$ is such that

$$x^*_{ik} = m_{ik} - m_{i\hat{r}} + \bar{m}_{\cdot\hat{r}} - \bar{M} + \mu \qquad \text{for each } (i,k) \text{ in } C \times R.$$

(iv) Let W satisfy, in addition, ex post aggregation, ex post compensation and uniform utilitarian reward (Theorem 4). Each optimal distribution X^* in $\mathbb{R}^{c \times r}$ of the total income amount $cr\mu$ is such that

$$x^*_{ik} = x^*_{jk} \qquad \text{for all } i \text{ and } j \text{ in } C \text{ and each } k \text{ in } R.$$

(v) Let W satisfy, in addition, ex ante compensation and utilitarian reward (Theorem 5). Each optimal distribution X^* in $\mathbb{R}^{c \times r}$ of the total income amount $cr\mu$ is such that

$$\bar{x}^*_{i\cdot} = \bar{x}^*_{j\cdot} \qquad \text{for all } i \text{ and } j \text{ in } C.$$

Assume that $\mu = \bar{M}$. The optima in Proposition 3(i), 3(ii) to 3(iii) coincide, respectively, with the "natural", egalitarian-equivalent and conditional equality allocation rules introduced by Bossert (1995) and Bossert and Fleurbaey (1996).[20] The "natural" allocation rule applies only if market incomes in M are additively separable. Additive separability implies that there exist a real number a_i for each circumstance group i in C and a real number b_k for each responsibility group k in R such that $m_{ik} = a_i + b_k$. The "natural"

[20] If $\mu \neq \bar{M}$, then each individual receives what he would have received under these allocation rules plus the difference $\mu - \bar{M}$.

allocation rule assigns to each individual (i,k) the income $b_k+\bar{a}$, i.e., the part of his market income determined by responsibility plus the average of the part determined by circumstances. The allocation rule in Proposition 3(i) indeed coincides with the "natural" allocation rule since $m_{ik}=a_i+b_k$, $\bar{m}_{i\cdot}=a_i+\bar{b}$ and $\mu=\bar{M}=\bar{a}+\bar{b}$. The egalitarian-equivalent allocation rule, obtained in Proposition 3(ii), assigns to each individual (i,k) the income $m_{\hat{c}k}-\bar{m}_{\hat{c}\cdot}+\bar{M}$, i.e., the market income she would have received if her circumstance were $\hat{c}$ plus a uniform amount. The conditional equality allocation rule, obtained in Proposition 3(iii), assigns to each individual (i,k) the income $m_{ik}-m_{i\hat{r}}+\bar{m}_{\cdot\hat{r}}$, i.e., the average market income of the responsibility group $\hat{r}$ plus the amount by which the individual's market income deviates from the market income he would have had were k equal to $\hat{r}$.

The optima in Proposition 3(iv) and 3(v) coincide with the allocation rules proposed by Roemer (1993) and Van de gaer (1993). Indeed, the income distributions in Proposition 3(iv) maximize Roemer's mean-of-mins, $\frac{1}{r}\sum_{k\in R}\min_{i\in C}x_{ik}$, and those in 3(v) maximize Van de gaer's min-of-means, $\min_{i\in C}\frac{1}{r}\sum_{k\in R}x_{ik}$.

6 Inequality of opportunity measures

The normative approach to inequality measurement identifies inequality with the welfare loss incurred by having the actual rather than the optimal distribution of the available income. First, we review the procedure proposed by Kolm (1969), Atkinson (1970) and Sen (1973) to derive measures of income inequality. Next, we extend this procedure to our setting. As we will see later, the Kolm-Atkinson-Sen (KAS) income inequality measure constitutes the basic building block of our measures of inequality of opportunity. Note that we consider absolute measures of income inequality and inequality of opportunity, that is, adding the same amount to each income leaves the level of inequality unaltered.

Consider the income inequality setting in which all individuals are identical. Let x in $\mathbb{R}^n$ be an income distribution for n individuals, and let $w:\mathbb{R}^n\to\mathbb{R}$ be a strictly increasing, continuous, translatable and strictly Schur-concave social welfare function. The equally distributed equivalent income $\xi(x)$ associated with w is the income that, if received by each individual, would yield the same welfare level as x. Formally, $\xi(x)$ is the real number such that $w(\xi(x)1_n)=w(x)$. The function $J:\mathbb{R}^n\to\mathbb{R}$ is said to be the KAS inequality measure associated with w if

$$J(x)=\bar{x}-\xi(x)\qquad\text{for each } x \text{ in } \mathbb{R}^n. \tag{6}$$

The KAS measure has an intuitive interpretation. For each x in $\mathbb{R}^n$, $J(x)$

is the per capita income that could be destroyed if incomes are equalized while maintaining the same level of welfare. It is a measure of waste due to inequality.

Next, we extend the above procedure to our setting. The difference with the income inequality setting is that the equal distribution is not necessarily optimal. Let X in $\mathbb{R}^{c\times r}$ be an income distribution, and let $W : \mathbb{R}^{c\times r} \to \mathbb{R}$ be a strictly increasing and continuous social welfare function. The optimally distributed equivalent average income $\Xi(X)$ is the average income that, if distributed optimally among the individuals, would yield the same welfare level as X. Formally, $\Xi(X) = \bar{Y}$ with Y such that $W(Y) = W(X)$ and $W(Y) \geq W(Z)$ for each Z in $\mathbb{R}^{c\times r}$ for which $\bar{Z} = \bar{Y}$. The function $I : \mathbb{R}^{c\times r} \to \mathbb{R}$ is said to be the inequality of opportunity measure associated with W if

$$I(X) = \bar{X} - \Xi(X) \qquad \text{for each } X \text{ in } \mathbb{R}^{c\times r}.$$

For each X in $\mathbb{R}^{c\times r}$, $I(X)$ is the per capita income that could be destroyed if income is optimally distributed while maintaining the same level of welfare.

Proposition 4 presents the inequality of opportunity measures corresponding to the social welfare functions described in the five theorems in Section 4.

Proposition 4. *Let W be a social welfare function that satisfies monotonicity, continuity and translation invariance.*

(i) Let W satisfy, in addition, ex post compensation and liberal reward, and let market incomes in M be additively separable (Theorem 1). For each income distribution X in $\mathbb{R}^{c\times r}$, we have

$$I(X) = J(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot}),$$

where $J : \mathbb{R}^{cr} \to \mathbb{R}$ is the KAS inequality measure associated with f in equation (1).

(ii) Let W satisfy, in addition, ex post compensation and liberal reward for $\hat{c}$ (Theorem 2). For each income distribution X in $\mathbb{R}^{c\times r}$, we have

$$I(X) = J(x_{11} - m_{\hat{c}1}, \ldots, x_{ik} - m_{\hat{c}k}, \ldots, x_{cr} - m_{\hat{c}r}),$$

where $J : \mathbb{R}^{cr} \to \mathbb{R}$ is the KAS inequality measure associated with f in equation (2).

(iii) Let W satisfy, in addition, ex post compensation for $\hat{r}$ and liberal reward (Theorem 3). For each income distribution X in $\mathbb{R}^{c\times r}$, we have

$$I(X) = J(x_{11} - m_{11} + m_{1\hat{r}}, \ldots, x_{ik} - m_{ik} + m_{i\hat{r}}, \ldots, x_{cr} - m_{cr} + m_{c\hat{r}}),$$

where $J : \mathbb{R}^{cr} \to \mathbb{R}$ is the KAS inequality measure associated with f in equation (3).

(iv) Let W satisfy, in addition, ex post aggregation, ex post compensation and uniform utilitarian reward (Theorem 4). For each income distribution X in $\mathbb{R}^{c\times r}$, we have

$$I(X) = \frac{1}{r}\sum_{k\in R} J(x_{\cdot k}),$$

where $J : \mathbb{R}^c \to \mathbb{R}$ is the KAS inequality measure associated with f in equation (4).

(v) Let W satisfy, in addition, ex ante compensation and utilitarian reward (Theorem 5). For each income distribution X in $\mathbb{R}^{c\times r}$, we have

$$I(X) = J\Big(\frac{1}{r}\sum_{k\in R} x_{1k}, \frac{1}{r}\sum_{k\in R} x_{2k}, \dots, \frac{1}{r}\sum_{k\in R} x_{ck}\Big),$$

where $J : \mathbb{R}^c \to \mathbb{R}$ is the KAS inequality measure associated with f in equation (5).

Proposition 4 reveals that inequality of opportunity measurement reduces to the application of an income inequality measure to an appropriately adjusted income distribution. Our approach singles out the absolute KAS inequality measure as the income inequality measure to be employed.[21] This use of income inequality measures as a basic building block is ubiquitous in the equality of opportunity literature, not surprisingly without the restriction to absolute KAS inequality measures.[22] We now discuss the five parts of Proposition 4 in connection with the previous literature.

The measures in Proposition 4(i) to 4(iii) apply an income inequality measure to a distribution of corrected incomes where the correction term is determined by the market income distribution. The measures in Proposition 4(ii) and 4(iii) correspond, respectively, to the fairness gap and direct unfairness measures proposed by Fleurbaey and Schokkaert (2009).

Alternatively, the measures in Proposition 4(i) to 4(iii) can be written as measures of distance between the vector of actual incomes and the vector of optimal incomes. Indeed, these measures are equivalent to the application of J to the vector $(x_{11} - x^*_{11}, \dots, x_{cr} - x^*_{cr})$, where x^*_{ik} is the optimal income

[21] We provide an example of how the choice of f fixes the choice of J. Assume that f is the constant inequality aversion social welfare function, i.e., for each x in $\mathbb{R}^n$, $f(x) = \ln[(1/n)\sum_{i=1}^n -e^{-\alpha x_i}]$ with $\alpha > 0$. Note that this function satisfies all the properties imposed on f in Theorems 1 to 5. The equally distributed equivalent income is $\xi(x) = -(1/\alpha)\ln[(1/n)\sum_{i=1}^n e^{\alpha x_i}]$. The KAS inequality measure associated with f is $J(x) = (1/\alpha)\ln[(1/n)\sum_{i=1}^n e^{-\alpha(x_i-\bar{x})}]$, which is Kolm's (1976) income inequality measure.

[22] In Section 7, we discuss variations of our approach that would warrant the use of a wider class of income inequality measures.

as given in Proposition 3(i) to 3(iii).[23] The measures proposed by Devooght (2008) and Almås, Cappelen, Lind, Sørensen and Tungodden (2011) use this idea of distance between the actual and the optimal. Devooght (2008) uses the optimum corresponding to the egalitarian-equivalent rule as in Proposition 4(ii), but employs Cowell's (1985) measure of distributional change as a measure of distance. Almås et al. (2011) use the optimum corresponding to the so-called generalized proportionality rule (Cappelen and Tungodden, 2017) and adopt the relative Gini index as a measure of distance. The advantage of our approach is that both the optimum and the distance measure follow from the axioms imposed on the social welfare function.

The measure in Proposition 4(iv) measures inequality of opportunity by the sum of the inequality levels of the responsibility groups. Aaberge, Mogstad and Peragine (2011) propose a measure in this form with J a rank-dependent inequality measure. The measure in Proposition 4(iv) can be interpreted as measuring the inequality within responsibility groups while disregarding the (unproblematic) inequality between responsibility groups. This interpretation has been exploited by Checchi and Peragine (2010), who propose, among others, the within responsibility group component of the mean logarithmic deviation as a measure of inequality of opportunity.

The measure in Proposition 4(v) applies an inequality measure to the vector of the average incomes of the circumstance groups. This measure is dual to the measure in Proposition 4(iv) in the sense that it measures the inequality between circumstance groups while disregarding the (unproblematic) inequality within circumstance groups (Checchi and Peragine, 2010). From the ex ante perspective, the average income of circumstance group i represents the value of the opportunities of an individual with circumstance characteristic i. In this interpretation, the measure in Proposition 4(v) directly gauges the inequality between the opportunities of the circumstance groups. Measures of this ilk have been used extensively (with various choices of J) in empirical analysis. See, for example, Bourguignon, Ferreira and Menéndez (2007), Cogneau and Mesplé-Somps (2008), Lefranc, Pistolesi and Trannoy (2008), Aaberge et al. (2011), Ferreira and Gignoux (2011), Ferreira, Gignoux and Aran (2011), Belhaj Hassine (2012), Singh (2012) and Piraino (2015).

7 Concluding remarks

We conclude with three remarks. First, we discuss how to obtain relative instead of absolute measures of inequality of opportunity. This requires replac-

[23] Consider, for example, the measure in Proposition 4(iii). Because J is absolute, we have $J(x_{11} - m_{11} + m_{1\hat{r}}, \ldots, x_{cr} - m_{cr} + m_{c\hat{r}}) = J(x_{11} - x^*_{11}, \ldots, x_{cr} - x^*_{cr})$, where $x^*_{ik} = m_{ik} - m_{i\hat{r}} - (\bar{M} - \bar{m}_{\cdot\hat{r}}) + \bar{X}$.

ing translation invariance by scale invariance, according to which the social welfare ranking of two income distributions does not change if each income in both income distributions is multiplied by the same factor. This leads to a straightforward change in Theorem 5: the function f in equation (5) becomes homogeneous instead of translatable. For the other theorems, the required changes are less straightforward. Scale invariance clashes with liberal reward and liberal reward for $\hat{c}$.[24] Hence, to obtain relative variants of Theorems 1, 2 and 3, we would need to replace liberal reward and liberal reward for $\hat{c}$ with versions that require to respect relative rather than absolute differences in market incomes within circumstance groups. Uniform utilitarian reward and scale invariance do not clash.[25] However, uniform utilitarian reward is not in the spirit of relative inequality as the transfers are in equal absolute amounts. Therefore, to obtain a relative variant of Theorem 4, uniform utilitarian reward would have to be replaced by a version in which the transfers are proportional to income. We leave these variations for future research.

Second, we outline the direct approach to inequality of opportunity measurement as an alternative to the normative approach. In the direct approach, axioms are directly imposed on measures of inequality of opportunity. In our axioms, statements on social welfare would have to be replaced by reverse statements on inequality of opportunity (e.g., $W(X) > W(X')$ becomes $I(X) < I(X')$). The only substantial changes needed would be to omit monotonicity and to strengthen translation invariance so that it requires inequality of opportunity to remain unaltered if the same amount is added to each income. The advantage of such a direct approach is that it allows a wider class of income inequality measures—not only the normative ones—to serve as building blocks for the measures of inequality of opportunity. The disadvantage is that the connection between inequality of opportunity measures and social welfare functions would be lost.

Third, we discuss a more challenging extension, relevant only for the classes that satisfy some version of liberal reward, namely, to allow for a variable market income distribution. This would be needed for comparisons between countries or over time. This extension suggests a stronger version of monotonicity that requires the pair (X, M) to yield a higher level of social welfare than the pair (X', M') if $x_{ik} \geq x'_{ik}$ for each individual (i, k) and $x_{jl} > x'_{jl}$ for

[24] Assume there is one circumstance group and two responsibility groups. Let $M = (0, 4)$, $X = (2, 6)$ and $X' = (3, 5)$. Liberal Pigou-Dalton implies $W(X) > W(X')$. Next, let $Y = 4X = (8, 24)$ and $Y' = 4X' = (12, 20)$. Scale invariance implies $W(Y) > W(Y')$, but liberal Pigou-Dalton implies $W(Y) < W(Y')$.

[25] For each x in $\mathbb{R}^c$, let $\hat{x}$ be a rearrangement of x such that $\hat{x}_1 \leq \hat{x}_2 \leq \cdots \leq \hat{x}_c$. Consider a social welfare function in the form of equation (4), where, for each x in $\mathbb{R}^c$, $f(x) = \sum_{i=1}^{c} a_i \hat{x}_i$ with $a_1 > a_2 > \cdots > a_c$ and $\sum_{i=1}^{c} a_i = 1$. This social welfare function satisfies all the axioms in Theorem 4 and, in addition, scale invariance.

some individual (j, l), regardless of the change from M to M'. However, as we have shown in a simpler setting (Bosmans and Öztürk, 2015), this stronger monotonicity axiom clashes with liberal reward and continuity. We leave the question of how to incorporate variable market income distributions in the measurement of inequality of opportunity for future research.

Appendix

Proof of Proposition 1. Let W be a social welfare function that satisfies ex post Pigou-Dalton and liberal Pigou-Dalton.

Assume to the contrary that market incomes are not additively separable. That is, there exist i and j in C and k and l in R such that $m_{ik} - m_{il} \neq m_{jk} - m_{jl}$. Let X be an income distribution such that $x_{ik} = x_{jk} = (m_{ik} + m_{jk})/2$ and $x_{il} = x_{jl} = (m_{il} + m_{jl})/2$. Let X' be an income distribution such that $x'_{ik} + x'_{il} = x_{ik} + x_{il}$, $x'_{ik} - x'_{il} = m_{ik} - m_{il}$, $x'_{jk} + x'_{jl} = x_{jk} + x_{jl}$ and $x'_{jk} - x'_{jl} = m_{jk} - m_{jl}$ with X' and X coinciding everywhere else. Ex post Pigou-Dalton implies $W(X) > W(X')$, whereas liberal Pigou-Dalton implies $W(X') > W(X)$. We have a contradiction. □

The following two lemmas are used throughout the proofs. A progressive transfer is a transfer of income from a richer to a poorer individual such that the one that starts out with less money does not end up with more than the other. We say that a function is Pigou-Dalton consistent if its value increases as a result of a progressive transfer. See Olkin and Marshall (1979, pp. 10-12) for the first lemma and Dasgupta, Sen and Starrett (1973, p. 183) for the second lemma.

Lemma 1. *For all vectors a and b in $\mathbb{R}^n$, a is obtained from b by a finite sequence of progressive transfers and permutations if and only if $a = bB$ for some $n \times n$ bistochastic matrix B.*

Lemma 2. *Each symmetric and Pigou-Dalton consistent function $f : \mathbb{R}^n \to \mathbb{R}$ is Schur-concave.*

Proof of Theorem 1. It is easy to verify that the specified social welfare function satisfies the axioms in the case of additively separable market incomes. We focus on the reverse implication.

Let W be a social welfare function that satisfies the axioms.

By monotonicity and continuity, there exists a strictly increasing and continuous function $\hat{f} : \mathbb{R}^{cr} \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c \times r}$, we have $W(X) = \hat{f}((x_{ik})_{(i,k) \in C \times R})$. Translation invariance implies that, for all x and x' in $\mathbb{R}^{cr}$ and each real number λ, we have $\hat{f}(x) \geq \hat{f}(x')$ if and only if $\hat{f}(x + \lambda 1_{cr}) \geq \hat{f}(x' + \lambda 1_{cr})$, i.e., $\hat{f}$ is a translatable function. Let f be the function $f : \mathbb{R}^{cr} \to \mathbb{R}$ such that, for each vector $(x_{ik})_{(i,k) \in C \times R}$, we have $f((x_{ik} - m_{ik} + \bar{m}_{i\cdot})_{(i,k) \in C \times R}) = \hat{f}((x_{ik})_{(i,k) \in C \times R})$. It follows that, for each X in $\mathbb{R}^{c \times r}$, we have $W(X) = f(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot})$. The function f is strictly increasing, continuous and translatable since $\hat{f}$ is strictly increasing, continuous and translatable.

Next, we show that f is symmetric. Let X and X' be income distributions such that the vector $(x'_{11}-m_{11}+\bar{m}_{1\cdot}, \ldots, x'_{ik}-m_{ik}+\bar{m}_{i\cdot}, \ldots, x'_{cr}-m_{cr}+\bar{m}_{c\cdot})$ is obtained from the vector $(x_{11}-m_{11}+\bar{m}_{1\cdot}, \ldots, x_{ik}-m_{ik}+\bar{m}_{i\cdot}, \ldots, x_{cr}-m_{cr}+\bar{m}_{c\cdot})$ by a switch of two components. First, assume the switch is between the components corresponding to individuals (i,k) and (j,k). Note that $m_{ik}-\bar{m}_{i\cdot} = m_{jk}-\bar{m}_{j\cdot}$ by additive separability of M. Because the same value $(m_{ik}-\bar{m}_{i\cdot} = m_{jk}-\bar{m}_{j\cdot})$ is subtracted from the incomes x_{ik} and x_{jk}, the switch is equivalent to a switch of these incomes. Hence, $W(X) = W(X')$ by ex post symmetry. Second, assume the switch is between the components corresponding to individuals (i,k) and (i,l). This switch is equivalent to a switch of the subsidies $x_{ik}-m_{ik}$ and $x_{il}-m_{il}$. Hence, we have $W(X) = W(X')$ by liberal symmetry. Third, assume the switch is between the components corresponding to individuals (i,k) and (j,l). Let Y be the income distribution such that $(y_{11}-m_{11}+\bar{m}_{1\cdot}, \ldots, y_{ik}-m_{ik}+\bar{m}_{i\cdot}, \ldots, y_{cr}-m_{cr}+\bar{m}_{c\cdot})$ is obtained from the vector $(x_{11}-m_{11}+\bar{m}_{1\cdot}, \ldots, x_{ik}-m_{ik}+\bar{m}_{i\cdot}, \ldots, x_{cr}-m_{cr}+\bar{m}_{c\cdot})$ by a switch of the components corresponding to individuals (i,l) and (j,l). Using the same reasoning as above, by ex post symmetry, we have $W(X) = W(Y)$. Let Y' be the income distribution such that $(y'_{11}-m_{11}+\bar{m}_{1\cdot}, \ldots, y'_{ik}-m_{ik}+\bar{m}_{i\cdot}, \ldots, y'_{cr}-m_{cr}+\bar{m}_{c\cdot})$ is obtained from the vector $(y_{11}-m_{11}+\bar{m}_{1\cdot}, \ldots, y_{ik}-m_{ik}+\bar{m}_{i\cdot}, \ldots, y_{cr}-m_{cr}+\bar{m}_{c\cdot})$ by a switch of the components corresponding to individuals (i,k) and (i,l). Using the same reasoning as above, by liberal symmetry, we have $W(Y) = W(Y')$. The vector $(x'_{11}-m_{11}+\bar{m}_{1\cdot}, \ldots, x'_{ik}-m_{ik}+\bar{m}_{i\cdot}, \ldots, x'_{cr}-m_{cr}+\bar{m}_{c\cdot})$ is obtained from the vector $(y'_{11}-m_{11}+\bar{m}_{1\cdot}, \ldots, y'_{ik}-m_{ik}+\bar{m}_{i\cdot}, \ldots, y'_{cr}-m_{cr}+\bar{m}_{c\cdot})$ by a switch of the components corresponding to individuals (i,l) and (j,l). Using the same reasoning as above, by ex post symmetry, we have $W(X') = W(Y')$. Thus, we obtain $W(X) = W(X')$.

Finally, we show that f is strictly Schur-concave. Since f is symmetric, it suffices to show that f is Pigou-Dalton consistent. Let X and X' be income distributions such that the vector $(x'_{11}-m_{11}+\bar{m}_{1\cdot}, \ldots, x'_{ik}-m_{ik}+\bar{m}_{i\cdot}, \ldots, x'_{cr}-m_{cr}+\bar{m}_{c\cdot})$ is obtained from the vector $(x_{11}-m_{11}+\bar{m}_{1\cdot}, \ldots, x_{ik}-m_{ik}+\bar{m}_{i\cdot}, \ldots, x_{cr}-m_{cr}+\bar{m}_{c\cdot})$ by a single progressive transfer. First, assume the transfer is from the component corresponding to individual (i,k) to the component corresponding to individual (j,k). Because the same value is subtracted from the incomes x_{ik} and x_{jk}, this transfer is equivalent to a progressive transfer of income between (i,k) and (j,k). Hence, $W(X) > W(X')$ by ex post Pigou-Dalton. Second, assume the transfer is from the component corresponding to individual (i,k) to the component corresponding to individual (i,l). This transfer is equivalent to a progressive transfer from the subsidy $x_{ik}-m_{ik}$ to the subsidy $x_{il}-m_{il}$. Hence, we have $W(X) > W(X')$ by liberal Pigou-Dalton. Third, assume the transfer is from the component corresponding to individual (i, k) to the component corresponding to (j, l). Let Y be the income distribution such that $(y_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, y_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, y_{cr} - m_{cr} + \bar{m}_{c\cdot})$ is obtained from the vector $(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot})$ by a switch of the components corresponding to individuals (i, l) and (j, l). Using the same reasoning as above, by ex post symmetry, we have $W(X) = W(Y)$. Let Y' be the income distribution such that $(y'_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, y'_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, y'_{cr} - m_{cr} + \bar{m}_{c\cdot})$ is obtained from the vector $(y_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, y_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, y_{cr} - m_{cr} + \bar{m}_{c\cdot})$ by a transfer from the component corresponding to (i, k) to the component corresponding to (i, l). Using the same reasoning as above, by liberal Pigou-Dalton, we have $W(Y') > W(Y)$. The vector $(x'_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x'_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x'_{cr} - m_{cr} + \bar{m}_{c\cdot})$ is obtained from the vector $(y'_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, y'_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, y'_{cr} - m_{cr} + \bar{m}_{c\cdot})$ by a switch of the components corresponding to individuals (i, l) and (j, l). Using the same reasoning as above, by ex post symmetry, we have $W(X') = W(Y')$. Thus, we obtain $W(X') > W(X)$. □

Proof of Proposition 2. Let W be a social welfare function that satisfies liberal Pigou-Dalton and ex ante symmetry.

Assume to the contrary that market incomes are not additively separable. That is, there exist i and j in C and k and l in R such that $m_{ik} - m_{il} \neq m_{jk} - m_{jl}$. Let X be an income distribution such that there exist positive real numbers α and β such that $x_{i\cdot} = m_{i\cdot} + \alpha 1_r$ and $x_{j\cdot} = m_{j\cdot} + \beta 1_r$ with $\min_{k \in R} x_{ik} > \max_{k \in R} x_{jk}$. Let X' be the income distribution obtained from X by switching the ith and the jth rows of X. Let X'' be the income distribution obtained from X' by an income transfer of an amount ϵ between individuals (i, k) and (i, l) that corresponds to a progressive transfer in their subsidies. Moreover, let $\epsilon < \min_{k \in R} x_{ik} - \max_{k \in R} x_{jk}$, which implies that $\min_{k \in R} x''_{jk} > \max_{k \in R} x''_{ik}$. Let X''' be the income distribution obtained from X'' by switching the ith and the jth rows of X''.

We have $W(X) = W(X')$ by ex ante symmetry, $W(X') < W(X'')$ by liberal Pigou-Dalton and $W(X'') = W(X''')$ by ex ante symmetry. Hence, $W(X) < W(X''')$. However, we have $W(X) > W(X''')$ by liberal Pigou-Dalton. We have a contradiction. □

Proof of Theorem 4. It is easy to verify that the specified social welfare function satisfies the axioms. We focus on the reverse implication.

Let W be a social welfare function that satisfies the axioms. By monotonicity, continuity and ex post aggregation, there exist a strictly increasing and continuous function $h : \mathbb{R}^r \to \mathbb{R}$ and strictly increasing and continuous functions $f_1, f_2, \ldots, f_r : \mathbb{R}^c \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c \times r}$, we have

$W(X) = h(f_1(x_{\cdot 1}), f_2(x_{\cdot 2}), \dots, f_r(x_{\cdot r}))$. Using the symmetry imposed by uniform utilitarian reward, we can define strictly increasing and continuous functions $\hat{g} : \mathbb{R}^r \to \mathbb{R}$ and $\hat{f} : \mathbb{R}^c \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c\times r}$, we have $W(X) = \hat{g}(\hat{f}(x_{\cdot 1}), \hat{f}(x_{\cdot 2}), \dots, \hat{f}(x_{\cdot r}))$.

Translation invariance implies that, for all x and x' in $\mathbb{R}^c$ and each real number λ, we have $\hat{f}(x) \geq \hat{f}(x')$ if and only if $\hat{f}(x + \lambda 1_c) \geq \hat{f}(x' + \lambda 1_c)$, i.e., $\hat{f}$ is a translatable function. Hence, there exist a strictly increasing and continuous function $\psi : \mathbb{R} \to \mathbb{R}$ and a unit-translatable function $f : \mathbb{R}^c \to \mathbb{R}$ such that $\hat{f} = \psi \circ f$. Define the strictly increasing and continuous function $g : \mathbb{R}^c \to \mathbb{R}$ such that, for each $(t_1, t_2, \dots, t_r)$ in $\mathbb{R}^r$, we have $g(t_1, t_2, \dots, t_r) = \hat{g}(\psi(t_1), \psi(t_2), \dots, \psi(t_r))$. It follows that, for each X in $\mathbb{R}^{c\times r}$, we have $W(X) = g(f(x_{\cdot 1}), f(x_{\cdot 2}), \dots, f(x_{\cdot r}))$. Note that by translation invariance, g is translatable. The function f is strictly increasing and continuous because g and $\hat{f}$ are strictly increasing and continuous. Moreover, f is symmetric by ex post symmetry, Pigou-Dalton consistent by ex post Pigou-dalton, hence strictly Schur-concave.

Next we show that W is a strictly increasing function of $\frac{1}{r}\sum_{k\in R} f(x_{\cdot k})$. Let X and X' in $\mathbb{R}^{c\times r}$ be such that $\frac{1}{r}\sum_{k\in R} f(x_{\cdot k}) \geq \frac{1}{r}\sum_{k\in R} f(x'_{\cdot k})$. Let Y and Y' in $\mathbb{R}^{c\times r}$ be income distributions such that $y_{ik} = f(x_{\cdot k})$ and $y'_{ik} = f(x'_{\cdot k})$ for each (i, k) in $C \times R$. We have $W(Y) = g(f(f(x_{\cdot 1})1_c), f(f(x_{\cdot 2})1_c), \dots, f(f(x_{\cdot r})1_c))$. Since f is unit-translatable, there exists a real number α such that $W(Y) = g(f(x_{\cdot 1}) + \alpha, f(x_{\cdot 2}) + \alpha, \dots, f(x_{\cdot r}) + \alpha)$. Similarly, there exists a real number α' such that $W(Y') = g(f(x'_{\cdot 1}) + \alpha', f(x'_{\cdot 2}) + \alpha', \dots, f(x'_{\cdot r}) + \alpha')$. Thus, by translatability of g, we have $W(X) \geq W(X')$ if and only if $W(Y) \geq W(Y')$. Next, let Z and Z' be income distributions such that $z_{ik} = \frac{1}{r}\sum_{k\in R} f(x_{\cdot k})$ and $z'_{ik} = \frac{1}{r}\sum_{k\in R} f(x'_{\cdot k})$ for each (i, k) in $C \times R$. By monotonicity, $W(Z) \geq W(Z')$ if and only if $\frac{1}{r}\sum_{k\in R} f(x_{\cdot k}) \geq \frac{1}{r}\sum_{k\in R} f(x'_{\cdot k})$. By uniform utilitarian reward, $W(Y) = W(Z)$ and $W(Y') = W(Z')$. Hence, $W(Y) \geq W(Y')$ if and only if $\frac{1}{r}\sum_{k\in R} f(x_{\cdot k}) \geq \frac{1}{r}\sum_{k\in R} f(x'_{\cdot k})$. We have already established that $W(X) \geq W(X')$ if and only if $W(Y) \geq W(Y')$. Therefore, $W(X) \geq W(X')$ if and only if $\frac{1}{r}\sum_{k\in R} f(x_{\cdot k}) \geq \frac{1}{r}\sum_{k\in R} f(x'_{\cdot k})$. It follows that there exists a strictly increasing and continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $W(X) = F(\frac{1}{r}\sum_{k\in R} f(x_{\cdot k}))$ for each X in $\mathbb{R}^{c\times r}$. □

Proof of Theorem 5. It is easy to verify that the specified social welfare function satisfies the axioms. We focus on the reverse implication.

Let W be a social welfare function that satisfies the axioms. First, we show that, for each X and X' in $\mathbb{R}^{c\times r}$, if $\sum_{k\in R} x_{ik} = \sum_{k\in R} x'_{ik}$ for each i in C, then we have $W(X) = W(X')$. Let Y be the income distribution obtained from X such that for each individual (i, k), we have $y_{ik} = \bar{x}_{i\cdot}$ and let Y' be the income distribution obtained from X' such that for each individual

(i,k), we have $y'_{ik} = \bar{x}'_{i\cdot}$. By utilitarian reward, we have $W(X) = W(Y)$ and $W(X') = W(Y')$. By construction, $Y = Y'$ and hence $W(X) = W(X')$.

Furthermore, if $\sum_{k\in R} x_{ik} \geq \sum_{k\in R} x'_{ik}$ for each i in C with at least one inequality holding strictly, then we have $W(X) > W(X')$. This follows using monotonicity and the reasoning above. It follows that there exists a strictly increasing function $f : \mathbb{R}^c \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c\times R}$, we have $W(X) = f(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \ldots, \bar{x}_{c\cdot})$. The function f is continuous by continuity, symmetric by ex ante symmetry and translatable by translation invariance.

Next, we show that f is strictly Schur-concave. Let X and X' in $\mathbb{R}^{c\times r}$ be such that the vector $(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \ldots, \bar{x}_{c\cdot})$ is obtained from the vector $(\bar{x}'_{1\cdot}, \bar{x}'_{2\cdot}, \ldots, \bar{x}'_{c\cdot})$ by a progressive transfer. Let Y be an income distribution such that $y_{i\cdot} = \bar{x}_{i\cdot} 1_r$ for each i in C, and let Y' be an income distribution such that $y'_{i\cdot} = \bar{x}'_{i\cdot} 1_r$ for each i in C. Utilitarian reward implies that $W(X) = W(Y)$ and $W(X') = W(Y')$. Ex ante Pigou-Dalton implies that $W(Y) > W(Y')$. Hence, we have $W(X) > W(X')$. That is, f is a Pigou-Dalton consistent function. Since it is also symmetric, by lemma 2, f is Schur-concave. □

Proof of Proposition 4. Let W be a social welfare function that satisfies monotonicity, continuity and translation invariance.

(i) Let W satisfy, in addition, ex post compensation and liberal reward, and let market incomes in M be additively separable (Theorem 1). Let X be an income distribution in $\mathbb{R}^{c\times r}$. First, to find $\Xi(X)$, we look for the optimal income distribution Y such that $W(Y) = W(X)$. Since Y is optimal, we have $y_{ik} = m_{ik} - \bar{m}_{i\cdot} + \bar{Y}$ for each (i,k) in $C \times R$ by Proposition 3(i). By Theorem 1, $W(Y) = f(\bar{Y}, \ldots, \bar{Y})$ and $W(X) = f(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot})$. Since $W(Y) = W(X)$, we have $f(\bar{Y}, \ldots, \bar{Y}) = f(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot})$. Hence, $\bar{Y} = \xi(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot})$, where ξ is the equally distributed equivalent income associated with f. Since $\bar{Y} = \Xi(X)$, we obtain $I(X) = \bar{X} - \xi(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot})$, i.e., $I(X) = J(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot})$, where J is the KAS inequality measure associated with f in equation (1). The proofs of (ii) and (iii) are similar and are therefore omitted.

(iv) Let W satisfy, in addition, ex post compensation and uniform utilitarian reward (Theorem 4). Let X be an income distribution in $\mathbb{R}^{c\times r}$. Let Y be an income distribution in $\mathbb{R}^{c\times r}$ such that $y_{ik} = \alpha_k 1_c$ and $f(\alpha_k 1_c) = f(x_{\cdot k})$ for each k in R. By Proposition 3(iv), Y is optimal and by Theorem 4, $W(X) = W(Y)$. For each k in R, we have $\alpha_k = \xi(x_{\cdot k})$, where ξ is the equally distributed equivalent income associated with f. Using $\Xi(X) = \bar{Y} = \sum_{k\in R} \alpha_k / r = \sum_{k\in R} \xi(x_{\cdot k})/r$ and $X = \sum_{k\in R} \bar{x}_{x\cdot k}/r$, we obtain $I(X) = \bar{X} - \Xi(X) = \sum_{k\in R} [\bar{x}_{\cdot k} - \xi(\bar{x}_{\cdot k})]/r$, i.e., $I(X) = \sum_{k\in R} J(x_{\cdot k})/r$, where J is the

KAS inequality measure associated with f in equation (4).

(v) Let W satisfy, in addition, ex ante compensation and utilitarian reward (Theorem 5). Let X be an income distribution in $\mathbb{R}^{c\times r}$. Again, we look for an optimal distribution Y such that $W(Y) = W(X)$. Since Y is optimal, we have $\bar{y}_{i\cdot} = \bar{y}_{j\cdot}$ for all circumstance groups i and j in C by Proposition 3(v). By Theorem 5, $W(Y) = f(\bar{Y}, \ldots, \bar{Y})$ and $W(X) = f(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \ldots, \bar{x}_{c\cdot})$. Since $W(Y) = W(X)$, we have $f(\bar{Y}, \ldots, \bar{Y}) = f(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \ldots, \bar{x}_{c\cdot})$. Hence, $\bar{Y} = \xi(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \ldots, \bar{x}_{c\cdot})$, where ξ is the equally distributed equivalent income associated with f. We obtain that $I(X) = \bar{X} - \xi(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \ldots, \bar{x}_{c\cdot})$, i.e., $I(X) = J(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \ldots, \bar{x}_{c\cdot})$, where J is the KAS inequality measure associated with f in equation (5). □

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