Monetary Policy With Diverse Private Expectations *

Mordecai Kurz[†], Maurizio Motolese[‡], Giulia Piccillo[§], Howei Wu[¶‖]

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Abstract

We study the impact of diverse beliefs on the efficacy of monetary policy. Individual belief is modeled by a state variable that defines the laws of motion perceived by an agent. Belief dynamics is deduced from theoretical considerations and from empirical evidence. Correlation across agents renders belief distribution a state variable. Our main findings are that central results under Rational Expectations are misleading in failing to account for the effect of diverse expectations, which is natural in real markets. Although the model has two standard shocks of technology and costpush, the Rational Expectations results of a smooth trade-off between inflation and output volatility fail to hold. Under diverse private expectations (i) inflation and output volatility are non-monotonic in policy parameters; (ii) trade-off between these two volatility measures is typically not smooth: changed policy parameters may induce unexpected sharp changes and even discontinuity in market performance; (iii) higher optimism about the future state of the economy increases inflation and may lower or increase aggregate output; (iv) higher dispersion of individual beliefs lowers inflation. In some recent work, researchers measure aggregate risk by an index of "market disagreement" but offer no justification. We find that increased disagreement is not an expression of aggregate risk. A rise in aggregate risk could cause increased disagreement but the causality does not work the other way around.

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†Department of Economics, Serra Street at Galvez, Stanford University, Stanford, CA. 94305-6072, USA, mordecai@stanford.edu

‡Department of Economics and Finance, Università Cattolica del Sacro Cuore, Via Necchi 5, 20123, Milano, Italy, maurizio.motolese@unicatt.it

§Department of Economics - AE2, Maastricht University, School of Business and Economics, Maastricht, The Netherlands, g.piccillo@maastrichtuniversity.nl

¶Institute for Advanced Research, Shanghai University of Finance and Economics, No.777 Guoding Rd, Yangpu District, Shanghai, 200433 China, wu.houwei@mail.shufe.edu.cn

‖Corresponding author: Maurizio Motolese

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1 Introduction

It is well recognized private expectations are very important for the conduct of monetary policy and are often cited by central banks as a reason for taking one action or another. Persistence of inflationary expectations is a force central banks struggle with when aggressive anti-inflationary policy is advocated and pessimistic private expectations about future returns are often given as a reason for actions to boost investment demand. Milani (2011, 2013) show that dynamics of private market sentiments explain about 40%-50% of the business cycle in the U.S. economy. Empirical evidence reveals there exist wide fluctuations of individual private expectations as well as a significant cross-sectional diversity among them (see Kurz et al. (2003) and Falck et al. (2017)). Falck et al. (2017) show different levels of disagreement across forecasters impact the transmission of Monetary Policy. We give a graphical representation of such a cross-sectional diversity in Figures 1 and 2, where we report a sample of the distribution of private forecasts of GDP growth rate and CPI inflation rate. The forecast data is from the *Survey of Professional Forecasters* from 2010:Q1 to 2017:Q4 and the forecasts were made in each quarter for the annualized quarter-over-quarter percentage change over the next quarter. Figures 1 and 2 also clearly show almost all forecasters often over- or under-estimate future rates exhibiting waves of *optimism* and *pessimism*.

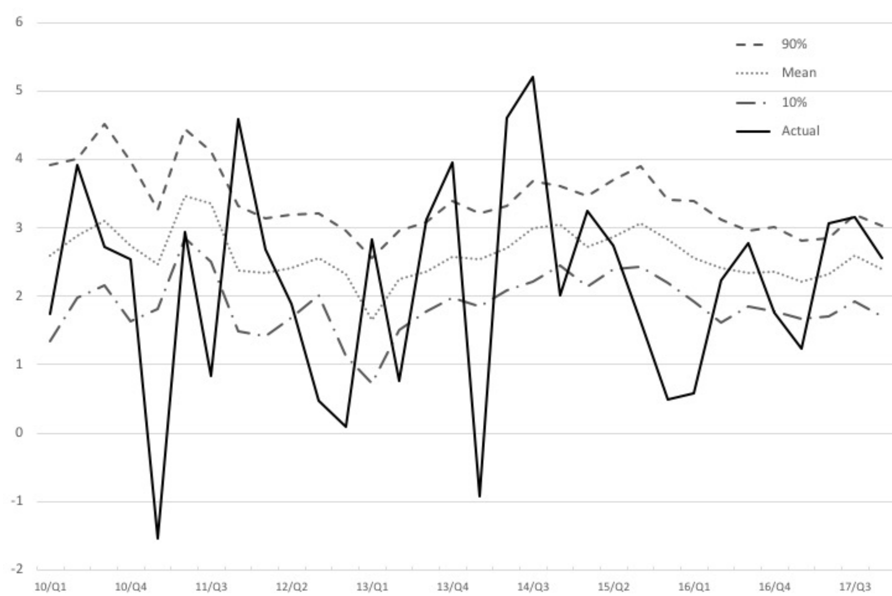


Figure 1: Distribution of private forecasts of GDP growth.

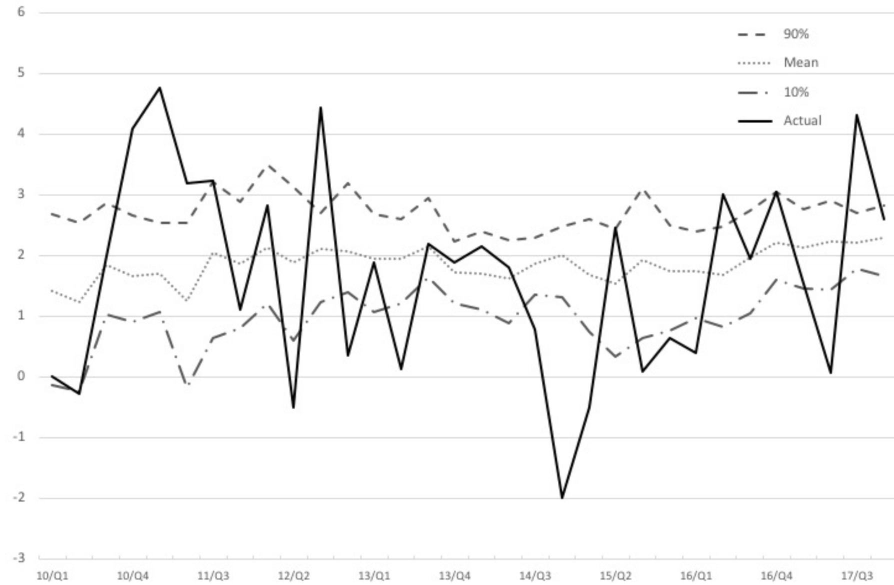


Figure 2: Distribution of private forecasts of CPI Inflation.

Expectations matter in an economy with random shocks and a discussion of expectations entails an assessment of the private response to these shocks. Yet, most work on monetary policy is based on Rational Expectations (in short RE) under which expectations as such have no independent effect. The term *independent effect* identifies effect of changes in the distribution of private expectations which are *different* from effects of changes in the shocks' true mathematical expectations. Aiming for more realism, we follow Blanchard's (2016) suggestion and introduce plausible diverse private expectations. Deviations from RE true expectations may be irrational but our view is that the economy is an evolving sequence of environments with changing shocks' true distributions. Therefore, true mathematical distributions are not known and cannot be learned with precision because of the short duration of each environment. As a result, agents form their best subjective private expectations with perfectly sensible (i.e. rational) but non-converging deviations. In Section 2 we outline our approach to the question of diverse beliefs which is at the foundations of this paper.

Other treatments of subjective beliefs have been used in the literature. Some view individuals as Bayesian agents with asymmetric private information but with the same prior belief. Alternatively, Branch and Evans (2006, 2011) and Branch and McGough (2009, 2011) use an axiomatic approach motivated by bounded rationality in which axioms are directly on the expectations operators. Some formulations are based on direct learning. Bullard and Mitra (2002) and Evans and Honkapohja (2003, 2006) assume agents are boundedly rational who do not know the equilibrium map and base their forecasts on a learning model. It is often the case, however, that with time the learning process converges to RE beliefs.

Under belief diversity three new channels emerge that alter monetary policy's efficacy. First, diverse beliefs imply diverse choice functions. This is a standard channel, causing individual endogenous variables to be different from their corresponding aggregates. The second arises from a recognition by rational agents that endogenous variables depend upon market belief. Hence, to forecast endogenous variables one must forecast market belief. This channel is central to our work here. The third channel arises from market expectations about future actions of the central bank. Virtually all research on this issue either assumes bank's credibility and individuals' belief in commitment to a monetary rule or no credibility and bank's discretion. Here we adopt the first assumption although the importance of the problem is obvious. It surfaced in the debate about policy to extricate the economy from the Zero Lower Bound (ZLB) by the central bank *managing* market belief so as to generate inflation expectations (e.g. Krugman (1998), Eggertsson and Woodford (2004), Eggertsson (2006) and Woodford (2007a,b, 2012)). We do not believe a central bank can manage market belief, particularly when a large segment of private agents disagrees with the central bank. The fact is that when the rate was at zero the Fed failed to generate inflationary expectations in spite of several QE programs and promises to keep the zero rate beyond the formally implied exit date from the ZLB.

Our economic environment is a New Keynesian Model (in short, NKM) developed along standard lines (e.g. Clarida et al. (1999), Woodford (2003), Galí (2008), Walsh (2010)) which is adapted to an economy with diverse beliefs. Private expectations are rational in the sense that they are compatible with past data and cannot be contradicted by the evidence. We thus study Rational Belief Equilibria (in short, RBE) defined in Kurz (1994) and developed for NKM in Kurz (2012) and Kurz et al. (2013) whose focus was the problem of aggregation in a model under log linear approximation. The present paper focuses on the policy implications of the model of Kurz (2012) solved with a second order approximation to quantify and analyze also the effect of belief heterogeneity on market performance and policy. Such approximation is a natural setting to study the effects of changes in the cross-sectional standard deviations of agents' characteristics on economic performance and policy. We explain in Section 3 and Appendix B the method of approximation and solution approach.

The initial NKM with a single technological shock was criticized for exhibiting no trade-off between volatilities of output and inflation (see Blanchard and Galí (2010)), labeling this phenomenon *divine coincidence*. They resolved it by adding a second shock to the Phillips Curve (for discussion see Galí (2008), Chapter 5). We thus study monetary economies with two shocks, therefore the response surface, which is a collection of output and inflation volatilities $(\sigma_{\hat{y}}, \sigma_\pi)$ in response to policy parameters (ξ_π, ξ_y) of inflation and output gap, exhibit two key properties under RE:

(i) trade-off between volatilities of output and of inflation;

(ii) monotonic effect of changes in each of ξ_π and of ξ_y on $(\sigma_{\hat{y}}, \sigma_\pi)$.

These enable a prediction of the effect of changing the policy rule: higher weight on the gap lowers output volatility and increases inflation volatility whereas a higher weight on inflation increases output volatility and lowers inflation volatility. They permit a central bank to forecast the effect of any policy experimentation that may be carried out before a final policy is selected.

Policy trade-off is not new to the policy debate and results that support the presence of trade-off and its characterization are extensive (e.g. Taylor (1979, 1993a,b), Fuhrer (1997), Svensson (1997), Ball (1999), Rotemberg and Woodford (1999), Rudebusch and Svensson (1999) and others). These results face two fundamental difficulties. First (which applies to RE as well), feasibility of trade-off is no proof that *efficient policies* are not boundary policies, questioning the Blanchard and Galí's (2010) conclusion. Second (which is related to belief diversity), this large research program is model-dependent where RE is universally assumed. It then follows from the Bayesian procedure employed that compatibility of the hypotheses with the data is not a compelling proof that better explanations of the evidence are unavailable. Enlarging the private expectation framework changes the model and reopens the question of trade-off.

To understand why these results do not hold under diverse beliefs and why the RE results are rather special recall that policy is effective since it alters agents' consumption choices by altering the cost of inter-temporal substitutions. Under RE all opportunities for such future choices are entirely determined by exogenous forces which are common knowledge. All diverse belief models - Rational Beliefs (in short, RB) or any model of bounded rationality - imply diverse perception of future inter-temporal choices. Belief diversity can have strong effects since different forecasts of future states lead to different perceived income and inter-temporal substitution effects associated with expected output, wages and inflation. Diverse perceived gains from choices over time face a policy that can select only the cost of such choices. Hence, private perceptions may enhance policy objectives by being aligned with the policy or they may negate the policy objectives by agents who make decisions that contradict these objectives. Such interaction between private expectations and policy takes three forms. First, private expectations may strengthen or weaken the effect of policy on $(\sigma_{\hat{y}}, \sigma_\pi)$ relative to its effect under RE. Second, also the level of disagreement across private expectations alters the effect of policy on $(\sigma_{\hat{y}}, \sigma_\pi)$. Third, it alters the shape of the response surface to policy so as to negate the two key properties under RE. Indeed, since different policy instruments result in different configurations of equilibrium income and substitution effects, each being dominant for some values of the policy parameters (ξ_π, ξ_y), market belief inserts into the response surface an additional non-linear force that is not present under RE. As a result, under *almost all diverse belief parameter configurations* the response surfaces deviate from the surface under RE in four possible ways:

(i) the effect of policy instruments (ξ_π, ξ_y) is non-monotonic;

(ii) there is typically no smooth trade-off between $\sigma_{\hat{y}}$ and σ_π;

(iii) under the non-linearity of the surface $[(\sigma_{\hat{y}}, \sigma_\pi), (\xi_\pi, \xi_y)]$ it may exhibit points of singularity at which $\sigma_{\hat{y}}$ or σ_π become unbounded;

(iv) cross-sectional diversity of private expectations further alters the effect of policy instruments (ξ_π, ξ_y).

From a formal perspective Kurz (2012) and Kurz et al. (2013) show that with diverse beliefs, two random terms are added to the aggregated NKM: one to the IS curve and one to the Phillips Curve and this, by itself, shows the existence of diverse beliefs changes the nature of the trade-off. Indeed, Kurz et al. (2005b) have anticipated this result by showing diverse beliefs by themselves and *complete price flexibility* are sufficient to render monetary policy effective with some trade-off between inflation and output volatility.

Sections 3 and 4 are then devoted to exploring the characteristic of private expectations that give rise to each one of the outcomes listed above. This work is done by simulating the model under alternative configurations of belief parameters and interpreting these belief structures. Apart from the positive aspects of our results they have important implications to the formation and evaluation of policy. They show that:

(i) private expectations put in question the existence of smooth trade-off between inflation and output volatility;

(ii) such trade-off may not take place on a smooth concave surface but may take the form of a choice between regions of the policy space, each representing efficient central bank policy and explaining why different central banks may employ sharply different policies;

(iii) non-monotonic effect of policy instruments imply the existence of singularities which are difficult for any bank to compute with precision. Consequently, changes of policy rule may induce unexpected market results unleashed by the effect of private expectations on policy outcomes;

(iv) a selection of an efficient policy requires study and extensive information on private expectations. This ambiguity explains that aggressive policies can have unexpected effects.

In Section 5 we study the macroeconomic effects of mean market belief and of the cross-sectional diversity of beliefs. Aggregate output and inflation response to shifts in mean market belief varies in intensity and sign with the beliefs environment and policy. Increased belief dispersion, measured by cross-sectional standard deviation of beliefs, is a market externality which does not necessarily cause an increase in macroeconomic uncertainty and a decline in real activity, contradicting claims by Bloom (2009). Section 6 concludes.

2 A NKM with diverse private expectations

2.1 The monopolistic competitors

The economy is a standard NKM with a continuum of agents and products. Agents are consumer-producers and we keep track of their functions: as households we index them by superscripts $j \in [0, 1]$

but as firms with distinct products we index them by subscripts $i \in [0,1]$. Household j manages firm j hence in equations that involve a household and a firm, the index j is in the subscript and superscript.

Firm i produces an intermediate good i sold at price p_{it}. The firms are monopolistic competitors who select optimal prices of their intermediate goods given demand, wage rate and production technology which uses only labor, defined by

$$Y_{it} = \zeta_t N_{it}, \quad \zeta_t > 0 \text{ is a random variable with } \mathbb{E}^m(\zeta_t) = 1. \tag{1}$$

With belief diversity we have different model probabilities and note that m in (1) is the stationary *empirical* probability deduced from past data hence is common knowledge. A belief of agent j is a model specifying how his subjective probability *differs from m*. These are explained later.

To generate final consumption household j purchases intermediate goods from all firms in the economy and produces its own final consumption via the transformation

$$C_t^j = \left[\int_0^1 (C_{it}^j)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}, \ \theta > 1$$

P_t is the price of final consumption, which is also *The Price Level*, defined in equilibrium by

$$P_t \equiv \left[\int_0^1 p_{it}^{1-\theta} di \right]^{\frac{1}{1-\theta}}$$

2.2 Households

Each household j maximizes an objective

$$\mathbb{E}_t^j \sum_{\tau=0}^{\infty} \beta^{t+\tau} \left[\frac{\left(C_{t+\tau}^j\right)^{1-\sigma}}{1-\sigma} - \frac{\left(L_{t+\tau}^j\right)^{1+\eta}}{1+\eta} - \frac{\tilde{\tau}_b}{2} \left(\frac{B_{t+\tau}^j}{P_{t+\tau}} \right)^2 \right], \tag{2}$$

where $0 < \beta < 1$, $\sigma > 0$, $\eta > 0$. A small penalty on excessive borrowing is a substitute for transversality conditions. Kurz et al. (2013) show it is used to avoid a Ponzi equilibrium with unbounded borrowing. The budget constraint is defined by

$$C_t^j + \frac{B_t^j}{P_t} + \frac{T_t^j}{P_t} = \frac{W_t^N}{P_t} L_t^j + \frac{B_{t-1}^j(1+r_{t-1})}{P_t} + \frac{1}{P_t}\left[p_{jt} Y_{jt} - W_t^N N_{jt}\right] \tag{3}$$

where C is consumption, L is labor supplied, T are transfers, W^N is nominal wage, B is one period debt held and r is a nominal interest rate[1]. Initial debt $B_0^j = 0$ is given for all j. Markets for

[1]We consider a cashless economy. The central bank enforces a nominal interest rate by a policy rule specified later.

bonds and for labor are competitive and wages are flexible. We derive Euler equations and compute a second-order approximation to equilibria of this economy. Let $b_t^j = B_t^j/P_t$, the real wage rate $W_t = W_t^N/P_t$ and the rate of inflation $\pi_t = P_t/P_{t-1}$. Agent j's optimality conditions are:

$$\tilde{\tau}_b b_t^j + \left(C_t^j\right)^{-\sigma} \frac{1}{1+r_t} = \mathbb{E}_t^j \left[\beta \left(C_{t+1}^j\right)^{-\sigma} \frac{1}{\pi_{t+1}}\right] \tag{4}$$

$$\left(C_t^j\right)^{-\sigma} W_t = \left(L_t^j\right)^{\eta} \tag{5}$$

2.3 Optimal pricing with cost push shocks

Optimal pricing by firms is based on Calvo (1983). We know that the demand function of i is

$$Y_{it} = \left(\frac{p_{it}}{P_t}\right)^{-\theta} Y_t$$

Maximizing over labor input is the same as maximizing over p_{it}. The profit function is then

$$\Pi_{it} = \frac{1}{P_t}\left[p_{it}Y_{it} - W_t^N N_{it}\right] = \left[\left(\frac{p_{it}}{P_t}\right) Y_{it} - \frac{W_t^N}{P_t}\left(\frac{Y_{it}}{\zeta_t}\right)\right]$$

Nominal marginal cost is W_t^N/ζ_t and real marginal cost is $\varphi_t = W_t^N/(\zeta_t P_t)$.

Assumption 1 *In a Calvo (1983) pricing process the distribution of beliefs among firm-agents is the same for those who adjust prices as those who do not adjust prices*[2].

Assumption 2 *An agent-firm chooses an optimal price so as to maximize discounted future profits given his own belief and considers transfer a lump sum. Actual transfers made ensure all households receive the same real profits, avoiding income effect of Calvo pricing. Transfers to agent j then equal*

$$\frac{T_t^j}{P_t} = \Pi_t - \Pi_t^j, \quad \Pi_t = \int_0^1 \Pi_{it} di.$$

Let p_{it}^* be the optimal price of i, $q_{it} = p_{it}^*/P_t$ and ω the probability a firm cannot adjust its price at date t. We assume firms price with the additional uncertainty of a cost push shock[3] ν_t which follows a Markov process specified later in Section 2.6. Optimal pricing is therefore given by

$$q_{it} = \frac{\theta}{\theta-1} \frac{\mathbb{E}_t^i \sum_{\iota=0}^{\infty} \beta^{\iota}\omega^{\iota} \left(C_{t+\iota}^i\right)^{-\sigma} Y_{t+\iota}\varphi_{it}\left(\frac{P_{t+\iota}}{P_t}\right)^{\theta}}{\mathbb{E}_t^i \sum_{\iota=0}^{\infty} \beta^{\iota}\omega^{\iota} \left(C_{t+\iota}^i\right)^{-\sigma} Y_{t+\iota}\left(\frac{P_{t+\iota}}{P_t}\right)^{\theta-1}} e^{\nu_t}. \tag{6}$$

[2]Assumption 1 ensures the belief distribution is orthogonal to the Calvo lottery. For more details the interested reader is referred to the discussion of Assumptions 1 and 2 in Kurz et al. (2013).

[3]For a detailed micro-foundation of the cost push shock in the presence of diverse beliefs, the reader is referred to Piccillo (2013).

It is then straightforward to show that (6) can be written as follows

$$q_{it} = \frac{\theta}{\theta - 1}\frac{S_{i,t}}{V_{i,t}}e^{\nu_t}, \text{ where } S_{i,t} = (C_t^i)^{-\sigma}Y_t\varphi_t + \beta\omega\,\mathbb{E}_t^i\left[S_{i,t+1}\pi_{t+1}^{\theta}\right], \qquad (6')$$
$$V_{i,t} = (C_t^i)^{-\sigma}Y_t + \beta\omega\,\mathbb{E}_t^i\left[V_{i,t+1}\pi_{t+1}^{\theta-1}\right]$$

2.4 Monetary policy

We assume the Central Bank inflation target is zero, hence the steady state inflation is $\bar{\pi} = 1$. Steady state interest rate is $\bar{r} = (1-\beta)/\beta$. The Central Bank sets the nominal interest rate according to a Taylor type policy rule[4] of the form

$$r_t = \bar{r} + \left(\frac{\pi_t}{\bar{\pi}}\right)^{\xi_\pi}\left(\frac{Y_t}{Y_t^F}\right)^{\xi_y} - 1 \qquad (7)$$

where Y_t^F is output under flexible prices taken to be *potential* output[5]. The weights ($\xi_\pi > 1, \xi_y \geq 0$) measure policy intensity. Larger values of ξ_π are taken to be *more aggressive inflation stabilization policy* and larger values of ξ_y as *more aggressive output stabilizing policy.*

2.5 Market clearing

To close the model and solve for equilibria we define the market clearing conditions

$$\int_0^1 C_t^j dj = C_t = Y_t, \quad \int_0^1 N_{jt}dj = N_t = L_t = \int_0^1 L_t^j dj, \quad \int_0^1 B_t^j dj = 0 \qquad (8)$$

where C_t, Y_t, N_t and L_t are respectively the aggregate levels of consumption, output, labor demand and labor supply.

We also need to specify the equilibrium relationship between aggregated optimal price and inflation which, due to Assumption 1 as it is shown in Appendix B, is written as

$$1 = (1-\omega)\int_0^1 q_{jt}^{(1-\theta)}dj + \omega(\pi_t)^{-(1-\theta)} \qquad (9)$$

2.6 The structure of beliefs

We follow the RB approach due to Kurz (see Kurz (1994, 1997, 2009)). It is founded on the fact agents cannot know something about the economy's stochastic dynamics. The *lack of knowledge about any observable leads to the crucial deduction that agents will form belief about its future.* It

[4] We assume there is no room for the Central Bank to deviate from a fixed rule. For an analysis of the effects of deviations from the policy rule via an exogenous monetary policy shock see Kurz et al. (2013) and Wu (2014).

[5] The flexible prices output level we consider is well known (see Walsh (2010) p. 335) and function of the productivity shock only: $Y_t^F = ((\theta-1)/\theta)^{1/(\sigma+\eta)}\zeta_t^{(1+\eta)/(\sigma+\eta)}$.

leads to three probabilities that play central role in the theory, and we keep track of them. First the true probability of the process of the model's shocks $\chi_t = (\hat{\zeta}_t, \nu_t)$; it is *not known to the agents* and assumed here to have Markov transition

$$\chi_{t+1} = \lambda_\chi \chi_t + \lambda^s_\chi s_t + \tilde{\rho}^\chi_{t+1},\ \tilde{\rho}^\chi_{t+1} \sim N(0, \tilde{\sigma}^2_\chi) \tag{10}$$

with time varying parameters, expressed by zero mean s_t, common to all shocks, justifying later a single belief state about s_t. Agents cannot learn (10) since the economy experiences on-going structural changes due to technological and institutional changes, each different from all prior ones. The parameters $(\lambda_\chi, \lambda^s_\chi)$ may also be time varying. The RB theory requires that (10) has an empirical distribution, hence frequencies of finite events converge, a property most economic time series have. One then proves that this property enables all agents to use past data to compute the joint *unique stationary empirical probability* of χ_t (the second probability) denoted by m which is common knowledge. In this paper we assume m has Markov transition

$$\chi_{t+1} = \lambda_\chi \chi_t + \rho^\chi_{t+1},\ \rho^\chi_{t+1} \sim N(0, \sigma^2_\chi),\ \ \text{i.i.d} \tag{11}$$

(11) is , broadly speaking, the long run average of (10). Three crucial facts matter here:

(i) *we do not know* (10), *hence simulations are made in accord with* m;

(ii) *the economy's long run performance is guided by* m *in* (11);

(iii) *agents may not believe* (11) *is the truth.*

The RB rationality principle is then entirely natural: a belief is considered rational if it induces economic dynamics with an empirical distribution (11). It replaces the RE condition that agents know (10) by the requirement that a belief should not contradict the empirical evidence (11). Since (10) is unknown, we relax RE by assuming it requires a common belief that m is the truth.

Agents' belief are subjective conditional expectations of s_t, described by state variables g^j_t. We assume, for simplicity, a single belief variable although multiple states may be used. The notation $(\hat{\zeta}^j_{t+1}, \nu^j_{t+1})$ expresses j's *perception* of $t+1$ shocks before observing them. By convention $(\mathbb{E}^j_t\,\hat{\zeta}_{t+1},\ \mathbb{E}^j_t\,\nu_{t+1})$[6] is the same as $(\mathbb{E}^j_t\,\hat{\zeta}^j_{t+1},\ \mathbb{E}^j_t\,\nu^j_{t+1})$ since j's expectations are taken only with respect to perceptions. Therefore, agent j perceives $(\hat{\zeta}^j_{t+1}, \nu^j_{t+1})$ at date t to be distributed as

$$\begin{aligned} \hat{\zeta}^j_{t+1} &= \lambda_\zeta \hat{\zeta}_t + \lambda^g_\zeta g^j_t + \rho^{j\zeta}_{t+1} \\ \nu^j_{t+1} &= \lambda_\nu \nu_t + \lambda^g_\nu g^j_t + \rho^{j\nu}_{t+1} \end{aligned} \qquad \begin{pmatrix} \rho^{j\zeta}_{t+1} \\ \rho^{j\nu}_{t+1} \end{pmatrix} \sim N\left(\mathbf{0}, \begin{bmatrix} \hat{\sigma}^2_\zeta, & 0 \\ 0, & \hat{\sigma}^2_\nu \end{bmatrix} = \mathbf{\Sigma}^{\mathbf{j}}_\chi\right) \tag{12}$$

[6]The notation $(\hat{\zeta}^j_{t+1}, \nu^j_{t+1})$ is used to highlight *perception* of the variables $(\hat{\zeta}_{t+1}, \nu_{t+1})$ by agent j before they are observed. In general, for an aggregate variable x_{t+1}, there is no difference between $\mathbb{E}^j_t\,x^j_{t+1}$ and $\mathbb{E}^j_t\,x_{t+1}$ since j's expectations can be taken only with respect to j's perception. However, it is important to keep in mind the context. If in a discussion the variable x_{t+1} is assumed to be observed at $t+1$, then it cannot be perceived at that date. Hence, the notation x^j_{t+1} expresses *perception* of x_{t+1} by agent j before the variable is observed and $\mathbb{E}^j_t\,x_{t+1}$ expresses the expectations of x_{t+1} by j, *in accordance with his perception.* This procedure does not apply to j-specific variables such as $\mathbb{E}^j_t\,\hat{c}^j_{t+1}$ which has a natural interpretation.

(12) shows g_t^j specifies the *difference* between j's date t forecasts and the forecasts under m. The RB principle requires (12) to reproduce (11) over time, which restricts the variance-covariance matrix and the dynamics of g_t^j which we define by

$$g_{t+1}^j = \lambda_Z g_t^j + \lambda_Z^\chi[\chi_{t+1} - \lambda_\chi \chi_t] + \rho_{t+1}^{jg},\ \rho_t^{jg} \sim N\left(0, \sigma_g^2\right), \quad \rho_t^{jg} \text{ are correlated across } j \tag{13}$$

This transition is central to the development and is taken as an agent's primitive characteristic. How general is (13)? If we first ignore the two learning feedbacks the equation is reduced to

$$g_{t+1}^j = \lambda_Z g_t^j + \rho_{t+1}^{jg},\ \rho_t^{jg} \sim N\left(0, \sigma_g^2\right), \quad \rho_t^{jg} \text{ are correlated across } j \tag{14}$$

Combining (14) and (12) one shows that RB restrictions on Σ_χ^j make the time average of (12) reproduce (11). We have tested the validity of (13) or (14) empirically and analitically. In Kurz and Motolese (2011, 2012) the data reveal the Markov property with high persistence $\lambda_Z \geq 0.7$ and in Kurz (2008, 2009, 2012), Kurz and Motolese (2011, 2012) and Kurz et al. (2013) a Bayesian learning model is used to prove (14) as a conclusion. Hence, given the Markov property of (11) condition (14) is as general as one can have, except for allowable burst of short term deviations from (14) that average to 0 over time and cannot be explained by any systematic theory; but is this not a common occurrence in markets?

To address j's short term surprises of forecasting $(\hat{\zeta}_{t+1}^j, \nu_{t+1}^j)$, we allow in (13) learning feedback terms $\lambda_Z^\zeta[\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \lambda_Z^\nu[\nu_{t+1} - \lambda_\nu \nu_t]$ that permit placing more weight on recent data. Agents know that any real time learning violates the RB principle since it puts into forecasts information which is not in the market. Since learning feedbacks average zero over time, (13) actually agree, on average, with (11). Learning feedbacks do introduce correlation and variances which are not in (11) but agents (and researchers) can easily adjust the RB restrictions to account for them.

Several natural assumptions are made. First, each agent is anonymous in assuming his belief has no effect on the market. Second, although each g_t^j is not publicly observed, the distribution of all g_t^j is observed hence mean market belief $Z_t = \int_0^1 g_t^j dj$ and the cross-sectional variance $\sigma_{cs}^2(g)_t$ are observed. Like all observed variables, they have an empirical distribution and induce diverse beliefs about their future. The RB approach shows that *agents forming beliefs about mean market belief* do expand their state spaces but do not trigger an infinite regress since Z_t is common knowledge. Leaving $\sigma_{cs}^2(g)_t$ aside, for the moment, the empirical distribution of mean market belief is deduced from (13) with stationary transition

$$Z_{t+1} = \lambda_Z Z_t + \lambda_Z^\chi[\chi_{t+1} - \lambda_\chi \chi_t] + \rho_{t+1}^Z, \quad \rho_{t+1}^Z = \int_0^1 \rho_{t+1}^{jg} dj \tag{15}$$

Note $\rho_{t+1}^Z \neq 0$ since, due to correlation across agents, the law of large numbers does not hold. Since correlation is not determined by individual rationality it is a *belief externality.* Agents' uncertainty about future market belief Z_{t+1} is central to our approach; hence, the stationary transition in (11) are supplemented, for completeness, with stationary transition (15).

Agents may not believe (15) is the true transition of Z_t and form their own belief about its future according to

$$Z^j_{t+1} = \lambda_Z Z_t + \lambda^{\chi}_Z[\chi_{t+1} - \lambda_\chi \chi_t] + \lambda^g_Z g^j_t + \rho^{jZ}_{t+1} \tag{16}$$

The full perception model $(\hat{\zeta}^j_{t+1}, \nu^j_{t+1}, Z^j_{t+1}, g^j_{t+1})$ is then the third probability that plays a central role in the theory. It is described by transition functions

$$\begin{array}{rl} \hat{\zeta}^j_{t+1} = & \lambda_\zeta \hat{\zeta}_t + \lambda^g_\zeta g^j_t + \rho^{j\zeta}_{t+1} \\ \nu^j_{t+1} = & \lambda_\nu \nu_t + \lambda^g_\nu g^j_t + \rho^{j\nu}_{t+1} \\ Z^j_{t+1} = & \lambda_Z Z_t + \lambda^{\chi}_Z[\chi_{t+1} - \lambda_\chi \chi_t] + \lambda^g_Z g^j_t + \rho^{jZ}_{t+1} \\ g^j_{t+1} = & \lambda_Z g^j_t + \lambda^{\chi}_Z[\chi_{t+1} - \lambda_\chi \chi_t] + \rho^{jg}_{t+1} \end{array} \quad \begin{pmatrix} \rho^{j\zeta}_{t+1} \\ \rho^{j\nu}_{t+1} \\ \rho^{jZ}_{t+1} \\ \rho^{jg}_{t+1} \end{pmatrix} \sim N\left(\mathbf{0}, \mathbf{\Sigma^j}\right) \tag{17}$$

where $\mathbf{\Sigma^j} = \left[\begin{array}{c|cc} \mathbf{\Sigma^j_\chi} & \multicolumn{2}{c}{\mathbf{0}} \\ \hline \mathbf{0} & \hat{\sigma}^2_Z, & \hat{\sigma}_{Zg} \\ & \hat{\sigma}_{Zg}, & \sigma^2_g \end{array}\right]$. We set $\lambda^g_\zeta = 1$ which is a normalization[7].

Kurz et al. (2013) prove that by aggregating the linear approximations of (4) and (6), the aggregated IS and Philips curves are not functions of aggregates only: the mean market belief Z_t has an amplification effect on the dynamics of the economy. Here we go beyond the simple world of a linear approximation and address the role of cross-sectional diversity and study its policy implications *under a quadratic approximation.* It is only under orders of approximation higher than the linear one that the issue of diversity of expectations can be meaningfully addressed. This raises new aggregation issues as explained later in Section 3.

Before deriving the transition function of the cross-sectional variance of beliefs $\sigma^2_{cs}(g)_t$ we need to introduce the following assumption to specify the stochastic structure of the random term ρ^{jg}_{t+1} that was unspecified in the transition function of g^j_{t+1} in (13).

ASSUMPTION **3** $\rho^{jg}_{t+1} = \Upsilon_{t+1}(1 + \varepsilon^{jg}_{t+1}) \Rightarrow \rho^Z_{t+1} = \Upsilon_{t+1}$ *where the common component* Υ_{t+1} *is a sequence of i.i.d. random variables with mean* 0 *and variance* σ^2_Z. *In addition,* ε^{jg}_{t+1} *are i.i.d. with mean* 0 *and variance* σ^2_ε *and are uncorrelated across agents, independent of* Υ_{t+1}. *Both are uncorrelated with* (g^j_t, Z_t).

To use Assumption 3 apply (21) below and compute

$$\int_0^1 (g^j_t)^2 dj = \sigma^2_{cs}(g)_t + Z^2_t$$

[7] The three probability measures discussed up to now are central to the RB theory. Since we want to explain why and how the RBE approach departs from RE, we stress it is important to fully understand the nature of these three probabilities and the differences among them. The reader is urged to pause and ensure these differences are clear, since the rest of the paper cannot be understood without them.

To deduce $\sigma^2_{cs}(g)_t$ consider the relationship between ρ^{jg}_{t+1} and $\rho^Z_{t+1} = \int_0^1 \rho^{jg}_{t+1} dj$. Use (13) and (15) to conclude

$$(g^j_{t+1} - Z_{t+1}) = \lambda_Z (g^j_t - Z_t) + (\rho^{jg}_{t+1} - \rho^Z_{t+1})$$

By Assumption 3

$$(g^j_{t+1} - Z_{t+1}) = \lambda_Z (g^j_t - Z_t) + \Upsilon_{t+1} \varepsilon^{jg}_{t+1}. \tag{18}$$

Take squares and integrate. With notation $\sigma^2_{cs}(g)_t$, this variance is a Markov process with transition:

$$\sigma^2_{cs}(g)_{t+1} = \lambda^2_Z \sigma^2_{cs}(g)_t + \Upsilon^2_{t+1} \sigma^2_\varepsilon \tag{19}$$

and this is the desired transition function. Note also that here $\sigma^2_{cs}(g)_t$ fluctuates with a known transition but *it is not normally distributed*![8]

3 Solving the aggregation problem for a macroeconomic heterogeneous agents model

An important property of a linear model which is central to solving the aggregation problem is that the set of state variables is closed under aggregation. To see how aggregation depends upon the structure of beliefs we briefly recall the equilibrium in the log linearized economy (see Kurz et al. (2013)) contains the optimal individual decision functions of the form

$$\hat{c}^j_t = A^Z_y Z_t + A^\zeta_y \hat{\zeta}_t + A^\nu_y \nu_t + A^b_y b^j_{t-1} + A^g_y g^j_t \equiv A_y \cdot (Z_t, \hat{\zeta}_t, \nu_t, b^j_{t-1}, g^j_t) \tag{20a}$$
$$\hat{q}_{jt} = A^Z_q Z_t + A^\zeta_q \hat{\zeta}_t + A^\nu_q \nu_t + A^b_q b^j_{t-1} + A^g_q g^j_t \equiv A_q \cdot (Z_t, \hat{\zeta}_t, \nu_t, b^j_{t-1}, g^j_t) \tag{20b}$$
$$b^j_t = A^Z_b Z_t + A^\zeta_b \hat{\zeta}_t + A^\nu_b \nu_t + A^b_b b^j_{t-1} + A^g_b g^j_t \equiv A_b \cdot (Z_t, \hat{\zeta}_t, \nu_t, b^j_{t-1}, g^j_t) \tag{20c}$$

The linear market clearing conditions (8) and (9) imply the aggregates are

$$\hat{y}_t = A^Z_y Z_t + A^\zeta_y \hat{\zeta}_t + A^\nu_y \nu_t + A^b_y 0 + A^g_y Z_t \equiv A_y \cdot (Z_t, \hat{\zeta}_t, \nu_t, 0, Z_t) \tag{20d}$$
$$\pi_t = A^Z_\pi Z_t + A^\zeta_\pi \hat{\zeta}_t + A^\nu_\pi \nu_t + A^b_\pi 0 + A^g_\pi Z_t \equiv A_\pi \cdot (Z_t, \hat{\zeta}_t, \nu_t, 0, Z_t) \tag{20e}$$
$$\hat{q}_t = A^Z_q Z_t + A^\zeta_q \hat{\zeta}_t + A^\nu_q \nu_t + A^b_q 0 + A^g_q Z_t \equiv A_q \cdot (Z_t, \hat{\zeta}_t, \nu_t, 0, Z_t) \tag{20f}$$

where $A_q = \omega/(1-\omega)A_\pi$. Aggregates are functions of $(Z_t, \hat{\zeta}_t, \nu_t)$ where Z_t is the new factor added to the NKM. Agents take market belief as an observed variable, like prices, and forecast its value like any other variable.

This last property is lost for higher order approximations due to proliferation of moments. To understand how proliferation occurs consider an agent specific state variable x^j_t that aggregates to

[8] To compute σ^2_ε in (19) keep in mind that by Assumption 3 the variance of ρ^{jg}_{t+1} is $\sigma^2_g = Var(\Upsilon_{t+1}(1+\varepsilon_{t+1})) = \sigma^2_Z(1+\sigma^2_\varepsilon)$ while the covariance between individual and mean market belief is $Cov(\rho^{jg}_{t+1}, \rho^Z_{t+1}) = \mathbb{E}((1+\varepsilon^{jg}_{t+1})\Upsilon^2_{t+1}) = \sigma^2_Z$. Given $\rho = \mathrm{Corr}(\rho^{jg}_{t+1}, \rho^Z_{t+1})$, it follows that $\rho = Cov(\rho^{jg}_{t+1}, \rho^Z_{t+1})/(\sigma_g \sigma_Z) = 1/\sqrt{1+\sigma^2_\varepsilon} \Rightarrow \sigma^2_\varepsilon = 1/\rho^2 - 1$

x_t. Agent j's decisions are functions of x_t^j and in a second order approximation are also functions of $(x_t^j)^2$ and $(x_t)^2$ which are now state variables. But if we now aggregate $(x_t^j)^2$ we conclude that

$$\int_0^1 (x_t^j)^2 dj = \int_0^1 (x_t^j - x_t + x_t)^2 dj = \sigma_{cs}^2(x)_t + (x_t)^2 \tag{21}$$

and *a new state variable* $\sigma_{cs}^2(x)_t$ *emerges* – the cross-sectional variance of x_t^j. Equation (21) explains why under a second order approximation our focus shifts to studying the effect of variability of the cross-sectional dispersion of variables. That is, while the linear case examines only the aggregate effect of the mean of each individual variable, now we look at a more accurate index of diversity, which is the cross-sectional dispersion of such variables that emerges out of aggregation.

Dispersion of distributions are particularly important for quantifying the effect of belief diversity on market performance and policy. With a linear approximation the effect is measured with one coefficient. For example, A_y^g in (20a) is the effect of g^j on individual consumption. But individual choices are subject to income and substitution effects leading one to expect a non-linear effect rather than a single signed effect. In fact, in the linearized version of the model equilibrium parameters of belief variables change signs in response to changed policy parameters as later seen in (34) and Figures 8-9 where singularity reflects a change in net weight of income vs. substitution effect induced by changed policy parameters. A quadratic approximation enables a richer examination of these effects, in addition to a study of the manner in which policy alters them.

3.1 Selecting a set of state variables closed under aggregation: the role of cross-sectional variances

The previous Section shows that with a higher order of approximation the only set of state variables closed under aggregation is the set of all infinite moments. Being infeasible it follows that any higher order approximation of finite order needs an ad-hoc decision to disregard higher moments at a degree which is appropriate for each model. Our approach is based on this same inevitable principle.

To explain our approach suppose D^j is a vector of agent j specific state variables and D is a vector of economy-wide state variables. Suppose decisions are quadratic in state variables then all decisions are linear functions of $(D, D^2, DD^j, D^j, (D^j)^2)$ where squares and cross products are standard vector operations. To aggregate these functions one averages them over j, concluding:

(i) averaging over (D, D^2) has no effect;

(ii) averaging over (DD^j, D^j) leads to $(D\int_0^1 D^j dj, D)$ which are in (i);

(iii) averaging over $(D^j)^2$ introduces the cross-sectional variances of these variables as in (21). If these moments are included in the vector D then aggregation does not add any new state variables. If aggregation adds new moments, they are of higher order and excluded as a modeling choice, suitable for the problem which the model intends to study.

Our selection of state variables begins by noting that utility functions and beliefs are symmetric in our model, hence, the effects of asset holdings and beliefs are also symmetric in the sense that only the *distribution* of assets and beliefs impacts the equilibrium, not the identity of an agent who holds a given combination of assets and beliefs. Next, we identify the two individual specific state variables $D^j = (b^j_{t-1}, g^j_t)$ the theory defined as causal in the sense they *cause and explain* the behavior of endogenous variables. They are essential to the theory and to attain consistent aggregation with truncation of higher moments we must include all first and second moments of the joint distribution of (b^j_{t-1}, g^j_t). This is done by identifying the implied second moments that must be included under aggregation in the set of economy-wide state variables:

$$\int_0^1 (g^j_t)^2 dj = \sigma^2_{cs}(g)_t + Z^2_t \tag{22}$$

$$\int_0^1 (b^j_{t-1})^2 dj = \sigma^2_{cs}(b)_t \tag{23}$$

$$\int_0^1 (b^j_{t-1} g^j_t) dj = Cov(b, g)_t \tag{24}$$

Hence the economy wide state variables we need for the joint distribution of (b^j_{t-1}, g^j_t) are $(Z_t, (Z_t)^2, \sigma^2_{cs}(g)_t, \sigma^2_{cs}(b)_t, Cov(b, g)_t)$. Since we study the effects of asset holdings and beliefs, our economy wide state variables will then be

$$D = (Z_t, \hat{\zeta}_t, \nu_t, \sigma^2_{cs}(g)_t, \sigma^2_{cs}(b)_t, Cov(b, g)_t) \tag{25}$$

Note that, even after including $(\sigma^2_{cs}(g)_t, \sigma^2_{cs}(b)_t, Cov(b, g)_t)$, the set of state variables is still not closed under aggregation since proliferation of moments continues for (25). For example, evaluating D^2 implies that $\sigma^2_{cs}(g)_t$ generates a new variable $(\sigma^2_{cs}(g)_t)^2$ as in (21). To define a set of state variables closed under aggregation with truncation of higher order moments suitable for a second order approximation we follow Preston and Roca (2007) and Den Haan and Rendahl (2010), and assume:

Assumption 4 *In addition to state variables (D, D^j), moments of state variables included are*

$$D^2 = ((Z_t)^2, (\hat{\zeta}_t)^2, (\nu_t)^2, Z_t\hat{\zeta}_t, Z_t\nu_t, \hat{\zeta}_t\nu_t) \tag{26a}$$

$$(D^j)^2 = ((b^j_{t-1})^2, (g^j_t)^2, b^j_{t-1}g^j_t) \tag{26b}$$

$$DD^j = (b^j_{t-1}Z_t, b^j_{t-1}\hat{\zeta}_t, b^j_{t-1}\nu_t, g^j_t Z_t, g^j_t\hat{\zeta}_t, g^j_t\nu_t) \tag{26c}$$

The set is closed under aggregation with truncated moments. All higher order moments are ignored.

Assumption 4 ensures decision functions and aggregates are approximated by strictly second order polynomials. For instance, aggregate income is approximated by the polynomial

$$\hat{y}_t = P^y(\hat{\zeta}_t, \nu_t, Z_t, \sigma^2_{cs}(g)_t, \sigma^2_{cs}(b)_t, Cov(b, g)_t, \hat{\zeta}^2_t, \nu^2_t, Z^2_t, \hat{\zeta}_t\nu_t, \hat{\zeta}_t Z_t, \nu_t Z_t; A^y)$$

where A^y is the polynomial's vector of coefficients. In P^y terms like $(\hat{\zeta}_t \sigma^2_{cs}(g)_t,\ (\sigma^2_{cs}(b)_t)^2)$ are ignored since they are of order higher than 2.

Assumption 4 also determines ways of computing higher moments and cross-sectional variances. For example, to compute cross-sectional variance of individual consumption, recall that it satisfies

$$\sigma^2_{cs}(c)_t = \int_0^1 (\hat{c}^j_t)^2 dj - (\hat{y}_t)^2 \tag{27}$$

hence the problem is how to compute the integral of squared consumption. Squaring the full second order polynomial that describes individual consumption means including in (27) terms of order higher than 2 which, by Assumption 4, are ignored.

A second order approximation amounts to using second order polynomials to define individual decision functions and aggregating them to impose market clearing. Hence, an equilibrium is a set of polynomial parameters that satisfy individual optimum conditions and market clearing. The principle of decomposition - a simple consequence of Taylor's theorem - says the solution of the non-linear model consists of a linear and a non-linear part but *the linear part is exactly the solution of the corresponding NKM linear model.* Hence, in developing a second order approximation the linear model is a vital first step which is a reference to all that we do. This is particularly important when we implement an iterative procedure to solve for the aggregates and impose market clearing. The decomposition principle has other implications. Since the exogenous shocks $\chi_t \in \{\hat{\zeta}_t, \nu_t\}$ are assumed to be independent, decomposition means we can solve for parameters of endogenous variables as functions of $(\hat{\zeta}_t, \nu_t)$, *one at a time.* Decomposition does not hold with respect to belief state variables as they are correlated, and this fact is important for understanding the role of diverse beliefs in the model.

By Assumption 4 the terms squared in (27) are only the linear terms of the consumption function and by the principle of decomposition it is equivalent to taking the solution of the linear model, squaring it and integrating as in (27).

Given a set of state variables a consistent aggregation also requires that all decision functions are functions of state variables whose laws of motion are specified. Hence, we now need to show how to deduce or construct transition functions for all state variables. This is our next task.

3.2 Formulating transition functions for $(\sigma^2_{cs}(b)_t, \mathrm{Cov}(b,g)_t)$

Transition functions of state variables are deduced either analytically from the stochastic properties assumed or via an approximation which, by Assumption 4, is carried out by using the linear solution and deducing from it the needed transition. Our discussion exhibits examples of both cases.

The transition function for $\sigma^2_{cs}(g)_t$ was discussed in Section 2.6 already, therefore we turn first to consider the transition function of $\sigma^2_{cs}(b)_t$ which is approximated from the linear solution as follows.

Square (20c) and integrate over agents to deduce

$$\int_0^1 (b_t^j)^2 dj = \int_0^1 \Big[(A_b^b)^2(b_{t-1}^j)^2 + 2A_b^bA_b^g(b_{t-1}^jg_t^j) + 2A_b^bA_b^\nu(b_{t-1}^j\nu_t)$$
$$+ 2A_b^bA_b^Z(b_{t-1}^jZ_t) + 2A_b^bA_b^\zeta(b_{t-1}^j\hat{\zeta}_t) + (A_b^g)^2(g_t^j)^2 + 2A_b^gA_b^\nu(g_t^j\nu_t)$$
$$+ 2A_b^gA_b^Z(g_t^jZ_t) + 2A_b^gA_b^\zeta(g_t^j\hat{\zeta}_t) + (A_b^\nu)^2(\nu_t)^2 + 2A_b^\nu A_b^Z(\nu_tZ_t)$$
$$+2A_b^\nu A_b^\zeta(\nu_t\hat{\zeta}_t) + (A_b^Z)^2(Z_t)^2 + 2A_b^ZA_b^\zeta(Z_t\hat{\zeta}_t) + (A_b^\zeta)^2(\hat{\zeta}_t)^2\Big] dj$$

Now apply the market clearing condition $\int_0^1 b_t^j dj = 0$ and (22)-(24) to conclude

$$\sigma_{cs}^2(b)_t = (A_b^b)^2\sigma_{cs}^2(b)_{t-1} + 2A_b^bA_b^gCov(b,g)_t + (A_b^g)^2\sigma_{cs}(g)_t^2$$
$$+ \left[(A_b^g)^2 + 2A_b^gA_b^Z + (A_b^Z)^2\right](Z_t)^2 + 2A_b^\zeta\left[A_b^g + A_b^Z\right](Z_t\hat{\zeta}_t)$$
$$+ 2A_b^\nu\left[A_b^g + A_b^Z\right](Z_t\nu_t) + (A_b^\zeta)^2(\hat{\zeta}_t)^2 + (A_b^\nu)^2(\nu_t)^2 + 2A_b^\zeta A_b^\nu(\hat{\zeta}_t\nu_t) \tag{28}$$

The transition of $Cov(b,g)_t$ is also approximated with the linear solution for bond holdings specified in (20c). Multiply (20c) by (18) and integrate over j to have

$$\int_0^1 b_t^j(g_{t+1}^j - Z_{t+1})dj = \int_0^1 \left[A_b^bb_{t-1}^j\lambda_Z(g_t^j - Z_t) + A_b^bb_{t-1}^j\Upsilon_{t+1}\varepsilon_{t+1}^{jg}\right] dj$$
$$+ \int_0^1 \left[A_b^ZZ_t\lambda_Z(g_t^j - Z_t) + A_b^ZZ_t\Upsilon_{t+1}\varepsilon_{t+1}^{jg}\right] dj + \int_0^1 \Big[A_b^\zeta\hat{\zeta}_t\lambda_Z(g_t^j - Z_t)$$
$$+ A_b^\zeta\hat{\zeta}_t\Upsilon_{t+1}\varepsilon_{t+1}^{jg}\Big] dj + \int_0^1 \left[A_b^\nu\nu_t\lambda_Z(g_t^j - Z_t) + A_b^\nu\nu_t^j\Upsilon_{t+1}\varepsilon_{t+1}^{jg}\right] dj$$
$$+ \int_0^1 \left[A_b^gg_t^j\lambda_Z(g_t^j - Z_t) + A_b^gg_t^j\Upsilon_{t+1}\varepsilon_{t+1}^{jg}\right] dj$$

Therefore, by Assumption 3, (22) and (24)

$$\mathrm{Cov}(b,g)_{t+1} = \lambda_ZA_b^b\,\mathrm{Cov}(b,g)_t + \lambda_ZA_b^g\sigma_{cs}(g)_t^2 \tag{29}$$

which is the desired transition function.

The transition functions of the cross-sectional moments (19), (28), and (29) are an integral component of the model. They impact the economy's dynamics, private expectations and policy efficacy.

4 Interaction of diverse beliefs with policy

Before proceeding we explain why private expectations result in non-monotonicity and singularity of the economy's response to changes in policy parameters. Though, Kurz et al. (2013) address the issue of aggregation under diverse rational beliefs, they did not focus on the policy implications and the nature and source of non-monotonicity and singularity. This is a relevant consequence of diversity of expectations. In order to explain it, we momentarily go back to the linear approximation of the model: we consider the linear component of the equilibrium map and show how it changes with policy parameters.

We start by computing the equilibrium parameters of the technology shock ζ. To do that we first linearize the Euler equations (4) and (6) (see Kurz et al. (2013) for details and notation) and insert into them (20a)-(20f) and the linearized monetary rule (7). Next we use the perception model (17) to compute expectations of all state variables. Finally, since these are linear difference equations we match coefficients (A_y^ζ, A_π^ζ) of $\hat{\zeta}_t$. The equations that determine these parameters are

$$\kappa(\eta+\sigma)A_y^\zeta - [1-\beta\lambda_\zeta]A_\pi^\zeta = \kappa(1+\eta) \tag{30}$$

$$\left[1-\lambda_\zeta + \frac{\xi_y}{\sigma(1+\bar{r})}\right]A_y^\zeta + \left(\frac{\dfrac{\xi_\pi}{(1+\bar{r})} - \lambda_\zeta}{\sigma}\right)A_\pi^\zeta = \left(\frac{\xi_y}{\sigma(1+\bar{r})}\right)\frac{1+\eta}{\eta+\sigma} \tag{31}$$

where $\kappa = (1-\omega)/\omega(1-(\beta\omega))$. Now define

$$T(\hat{\zeta}) = \begin{bmatrix} \left(1-\lambda_\zeta + \dfrac{\xi_y}{\sigma(1+\bar{r})}\right) & \dfrac{\dfrac{\xi_\pi}{(1+\bar{r})} - \lambda_\zeta}{\sigma} \\ \kappa(\eta+\sigma) & -(1-\beta\lambda_\zeta) \end{bmatrix} \tag{32}$$

and the solution of the above equation is

$$\begin{pmatrix} A_y^\zeta \\ A_\pi^\zeta \end{pmatrix} = T(\hat{\zeta})^{-1}\begin{pmatrix} \left(\dfrac{\xi_y}{\sigma(1+\bar{r})}\right)\dfrac{1+\eta}{\eta+\sigma} \\ \kappa(1+\eta) \end{pmatrix} \tag{33}$$

This solution exists and is unique since

$$\mid T(\hat{\zeta}) \mid = -\left[\left(1-\lambda_\zeta+\frac{\xi_y}{\sigma(1+\bar{r})}\right)(1-\beta\lambda_\zeta) + \frac{\dfrac{\xi_\pi}{(1+\bar{r})} - \lambda_\zeta}{\sigma}\kappa(\eta+\sigma)\right] < 0$$

for all policies. But then, equilibrium parameters (A_y^ζ, A_π^ζ) of $\hat{\zeta}_t$ depend only on the direct and indirect effects the exogenous shock $\hat{\zeta}_t$ has in the economy and do not depend upon the model's expectations.

The above points to a conclusion we labeled *decomposition*. Equilibrium parameters of each exogenous shock are independent of expectations hence are the same in models with diverse beliefs and under RE. But this result does not hold with respect to the effect of diverse belief on equilibrium. For simplicity we carry out computations for $(\zeta_t \neq 0, \nu_t = 0)$ but it will be clear how to modify computations for more shocks.

To determine $(A_y^Z, A_\pi^Z, A_y^g, A_\pi^g)$ we follow the same procedure used for the technology shock and deduce a system of equations written in matrix form with $\Xi_1 = \frac{\xi_y}{\sigma} - (A_y^b + \tau_b)\left[1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}\right]$, $\Xi_2 = -(1 - \lambda_Z - \lambda_Z^\zeta + (A_y^b + \tau_b))\left[1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}\right]$ and $\Xi_3 = \beta\omega(\lambda_Z + \lambda_Z^\zeta) + \beta(1-\omega)(\lambda_Z^\zeta + \lambda_Z^g) - 1$ and

$$T(Z,g) = \begin{pmatrix} \left(1 - \lambda_Z + \frac{\xi_y}{\sigma(1+\bar{r})}\right) & \frac{\frac{\xi_\pi}{1+\bar{r}} - \lambda_Z}{\sigma} & \Xi_1 & \frac{\frac{\xi_\pi}{1+\bar{r}} - \lambda_Z}{\sigma} \\ \kappa(\eta+\sigma) & -(1-\beta\lambda_Z) & \kappa(\eta+\sigma) & \beta(1-\omega)\lambda_Z \\ (\lambda_Z + \lambda_Z^g) & \frac{(\lambda_Z + \lambda_Z^g)}{\sigma} & \Xi_2 & \frac{(\lambda_Z + \lambda_Z^g)}{\sigma} \\ 0 & \frac{(\lambda_Z + \lambda_Z^g)}{\sigma} & 0 & \Xi_3 \end{pmatrix}$$

$$T(Z,g)\begin{pmatrix} A_y^Z \\ A_\pi^Z \\ A_y^g \\ A_\pi^g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -A_y^\zeta - \frac{A_\pi^\zeta}{\sigma} \\ -\beta A_\pi^\zeta \end{pmatrix} \tag{34}$$

Both (33) and (34) depend upon policy but there are two crucial differences between (33) and (34). The first explains why decomposition fails to hold here: the effect of belief works *through* the effects of exogenous shocks. That is, since beliefs are about exogenous shocks their effects on equilibrium depend upon how exogenous shocks affect the economy. That is to assess the effect of beliefs one must first solve for $\left(A_y^\zeta, A_\pi^\zeta\right)$ which represent the effects of the exogenous shocks. These are then used on the right hand side of (34) to determine the effect of beliefs.

Second, the interaction of policy and beliefs is a crucial component of our theory and it was never explicitly addressed before. Whereas in (33) the matrix is non-singular and (A_y^ζ, A_π^ζ) are monotonic in each of the policy instruments (ξ_y, ξ_π), these two properties do not hold for (34): the matrix may be singular and the parameters are not monotonic with respect to (ξ_y, ξ_π). Indeed, there exists a curve of (ξ_y, ξ_π) along which the matrix is singular and as we approach this singularity curve in the policy space, volatility becomes unbounded. Also, equilibrium parameters $(A_y^Z, A_\pi^Z, A_y^g, A_\pi^g)$ are not monotonic with respect to policy instruments (ξ_y, ξ_π). These are departures from the common conclusions about policy trade-off under RE. To explain it further we first examine the interaction of policy with expectations and then specify when such interaction is reduced.

Private beliefs affect equilibrium via two channels. First are the effects of g_t^j and Z_t on individual decisions and inspection of (20a)-(20c) show they are measured by $(A_y^Z, A_\pi^Z, A_y^g, A_\pi^g)$. Second, the effect on aggregates is more complex since they are functions of Z_t only and (20d)-(20f) show they are measured by $(A_y^Z + A_y^g, A_\pi^Z + A_\pi^g)$. Also note that (A_y^g, A_π^g) and (A_y^Z, A_π^Z) play different roles in the agent's decision functions compared to their role in aggregates. In decision functions (A_y^g, A_π^g) measure the effect of g_t^j on j's current consumption and pricing decisions while (A_y^Z, A_π^Z) measure an agent's known externality effect of market belief on the aggregates: output, inflation, interest rate and wage rate. That is, a more optimistic agent faces two conflicting incentives. Expected higher future wage causes income effect to increase today's consumption but a substitution effect to work less today and more tomorrow when the wage is higher. Similarly, a more optimistic monopolistic competitor expects lower future prices due to higher productivity (that exceeds the rise of wages due to labor supply elasticity). Risk of being unable to lower future prices motivates him to lower prices today. But expected higher future demand imply higher future prices and for the same sticky price effect an opposite motive exists to raise today's prices. These two private motives are based on an agent's forecasts of aggregates that depend on $(A_y^Z + A_y^g, A_\pi^Z + A_\pi^g)$ which are impacted - and this is the important point here - by the policy in place.

With opposite private motives the dominant effect depends upon two factors. One is market interest rate since all choices are between today and the future. The second is the set of agent's forecasts of future aggregates such as wage rate and income. Both these factors depend upon the policy, hence different policies will result in different $(A_y^Z, A_\pi^Z, A_y^g, A_\pi^g)$ and what our theory shows is that as policy shifts between inflation stabilization and output stabilization, the matrix in (34) becomes, at some intermediate point, singular and $(A_y^Z, A_\pi^Z, A_y^g, A_\pi^g)$ switch sign reflecting the change in the dominant effect discussed above. This establishes a deep interaction between private expectations and efficacy of policy. Expectations may be supportive of the policy but a conflict may also exist between policy and private expectations and this may result in undesirable volatility outcomes of the aggregates.

Note that since signs of (A_y^g, A_π^g) are typically opposite to the signs of (A_y^Z, A_π^Z) and since aggregates are functions of $(A_y^Z + A_y^g, A_\pi^Z + A_\pi^g)$, the market carries out some cancelation of conflicting private motives. This does not prevent the sum of the parameters from changing sign.

What is the component of private expectations that accounts for the fact that the matrix in (34) can, in general, be singular? To answer this question we propose the following definition:

Definition 1 *An economy has no private beliefs about market belief (i) if beliefs about market belief Z_{t+1} are not diverse hence no agent uses his g_t^j to forecast Z_{t+1}, and (ii) if there is no learning feed-back from current data. These two require $\lambda_Z^\zeta = 0, \lambda_Z^\nu = 0, \lambda_Z^g = 0$ and hence $\mathbb{E}_t^j [Z_{t+1}] = \lambda_Z Z_t$.*

The following is the answer to the above question which we state without proof:

Proposition 1 *If an economy has no private belief about market belief, the matrix in* (34) *is non singular for all feasible policies $\xi_y \geq 0, \xi_\pi \geq 1$ and therefore it does not change sign in the feasible policy space.*

What does the singularity mean? Since ours is not a single agent model and aggregate variables play a role, an equilibrium of the linearized economy is not an approximation of a saddle point solution of a single agent maximization. Hence, Blanchard-Kahn instability is not the issue. Mathematically speaking, the problem is one of multiplicity: the equilibrium of the linear approximation does not have a unique solution. When policy parameters are close to the critical singularity curve, equilibrium parameters become inaccurate, causing growing errors in the Euler equations. Hence, computed equilibria close to the critical singularity curve exhibit high volatility but also have large and rising errors in the Euler equations. It all implies irregularity in the equilibrium of the original, non linear, economy and the linearization exposes this fact. Such irregularity may take one of several familiar forms, but which of these cannot be ascertained since we cannot solve the non-linear system. The most likely cause is the familiar curvature of the excess demand function, leading to a discontinuous jump in the equilibrium map of the original economy at a threshold. The curvature and threshold effect is caused by interaction of market belief with policy parameters and is exhibited by changes in policy parameters causing equilibrium to transit from a region where intertemporal substitutions dominates to a region where intertemporal income effect dominates, *with a jump at the threshold*.

5 Monetary policy with diverse beliefs

The impact of diverse expectations on the efficacy of monetary policy varies with the policy in place. As illustrated in Figures 1 and 2, agents hold diverse forecasts of real variables and disagree when forecasting inflation. Therefore, as we illustrate later, also the level of (dis)agreement affects the efficacy of monetary policy. In what follows, we first give a summary of the parameter choice and then present the simulation results of the model. In particular, we focus on the aggregate effects of changes in the mean market belief and in its cross-sectional standard deviation. We do not aim at a precise calibration of the model, but we provide examples that highlight some of the qualitative equilibrium features of the interaction between diverse private expectations and monetary policy.

The quarterly model parameters are set according to standard values in the literature (e.g. Galí (2008), Walsh (2010)): $\beta = 0.99$, $\sigma = 0.9$, $\eta = 1.0$, $\tilde{\tau}_b = 10^{-3}$, $\omega = 2/3$, $\theta = 6$, $\lambda_\zeta = 0.90$. Unlike the standard Real Business Cycle (in short, RBC) assumption about the technology shock $\sigma_\zeta = 0.0072$ (as measured by the Solow residual) we set it to $\sigma_\zeta = 0.0045$ on the ground that the larger measure contains other endogenous factors. We calibrate the cost push to be consistent with Smets and Wouters (2007), as well as most other literature on cost push shocks, with $\lambda_\nu = 0.9$, $\sigma_\nu = 0.0045$. Parameters of the monetary policy rule are also in accord with standard range used in the literature ($\xi_y \geq 0$, $\xi_\pi > 1$). Finally, some of the belief parameters, which are discussed and motivated in Appendix A, common to all four scenarios are set as follows: $\lambda_Z = 0.8$, $\lambda^g_\zeta = 1$, $\lambda^g_\nu = -0.1$, $\sigma_g = 0.0026$ and $\rho = 0.9$. All results reported are statistics of model simulations over 10,000 periods.

5.1 The economic environments under study

We start our evaluation by discussing results of the standard RE version of our model under which (11) is believed to be the truth. Table 1 shows[9] there is no *Divine Coincidence* since, in accord

		Output Volatility ($\sigma_{\hat{y}}$)										Inflation Volatility (σ_π)							
		ξ_y										ξ_y							
		0.00	**0.25**	**0.50**	**0.75**	**1.00**	**1.25**	**1.50**	**2.00**			**0.00**	**0.25**	**0.50**	**0.75**	**1.00**	**1.25**	**1.50**	**2.00**
	1.20	1.122	1.109	1.098	1.089	1.082	1.076	1.071	1.063		**1.20**	1.314	1.515	1.756	2.003	2.241	2.466	2.677	3.057
	1.25	1.137	1.125	1.115	1.106	1.099	1.092	1.087	1.078		**1.25**	1.137	1.322	1.543	1.772	1.994	2.206	2.406	2.770
	1.30	1.150	1.138	1.128	1.120	1.112	1.106	1.100	1.090		**1.30**	1.002	1.173	1.377	1.589	1.797	1.996	2.185	2.533
	1.35	1.159	1.148	1.139	1.131	1.123	1.117	1.111	1.101		**1.35**	0.896	1.053	1.243	1.440	1.635	1.823	2.002	2.334
	1.40	1.167	1.157	1.148	1.140	1.133	1.126	1.120	1.110		**1.40**	0.810	0.956	1.132	1.316	1.499	1.677	1.847	2.164
	1.45	1.173	1.164	1.155	1.148	1.141	1.134	1.129	1.119		**1.45**	0.739	0.875	1.040	1.213	1.385	1.553	1.714	2.017
ξ_π	**1.50**	1.179	1.170	1.162	1.154	1.147	1.141	1.136	1.126	ξ_π	**1.50**	0.679	0.807	0.961	1.124	1.287	1.446	1.600	1.889
	1.55	1.183	1.175	1.167	1.160	1.153	1.147	1.142	1.132		**1.55**	0.629	0.749	0.894	1.047	1.201	1.353	1.499	1.776
	1.60	1.187	1.179	1.172	1.165	1.159	1.153	1.148	1.138		**1.60**	0.585	0.699	0.835	0.981	1.127	1.271	1.411	1.676
	1.65	1.190	1.183	1.176	1.169	1.163	1.158	1.152	1.143		**1.65**	0.547	0.654	0.784	0.922	1.061	1.198	1.332	1.587
	1.70	1.193	1.186	1.179	1.173	1.167	1.162	1.157	1.148		**1.70**	0.514	0.616	0.739	0.870	1.002	1.134	1.262	1.507
	2.00	1.206	1.200	1.195	1.190	1.185	1.181	1.176	1.168		**2.00**	0.376	0.454	0.548	0.649	0.753	0.856	0.958	1.156

Table 1: Output and inflation volatility trade-off and monotonicity under RE.

with Blanchard and Galí (2007), the presence of cost-push shocks brings about a trade-off between output and inflation volatility. In the table this trade-off is point-wise which means that at each (ξ_y, ξ_π) any small change in policy parameters that increases $\sigma_{\hat{y}}$ must decrease σ_π. In the case of RE the market responses to policy are monotonic in both (ξ_y, ξ_π): $\sigma_{\hat{y}}$ *decreases* in ξ_y and *increases* in ξ_π while σ_π *decreases* in ξ_π and *increases* in ξ_y and this pattern of monotonicity implies point-wise trade-off.

A weaker concept of trade-off at any given initial policy (ξ_y^0, ξ_π^0) with outcome $(\sigma_{\hat{y}}^0, \sigma_\pi^0)$ does not require a *small change in policy*. Instead, it requires that *there should exist* some other policy (ξ_y^1, ξ_π^1) with outcome $(\sigma_{\hat{y}}^1, \sigma_\pi^1)$ such that if $\sigma_{\hat{y}}^1 > \sigma_{\hat{y}}^0$ then $\sigma_\pi^1 < \sigma_\pi^0$ and if $\sigma_{\hat{y}}^1 < \sigma_{\hat{y}}^0$ then $\sigma_\pi^1 > \sigma_\pi^0$. The policy (ξ_y^1, ξ_π^1) may be very different from (ξ_y^0, ξ_π^0) hence trade-off may require drastic change of policy or imply trade-off among regions of the policy space. Testing for such weaker concept can be done by projecting feasible policy outcomes $(\sigma_{\hat{y}}, \sigma_\pi)$ as illustrated in Figure 3 where we project all outcomes of a denser grid of policy points (ξ_y, ξ_π) which includes those of Table 1. A weaker trade-off then exists at a point $(\sigma_{\hat{y}}, \sigma_\pi)$ if (i) the set of outcomes is locally convex and (ii) $(\sigma_{\hat{y}}, \sigma_\pi)$ has neighbors on a negatively sloped segment. It is clear from Figure 3 that such trade-off exists at all points in the space since the stronger point-wise trade-off implies the weaker concept.

Assuming the bank's objective is a monotonic function of $(\sigma_{\hat{y}}, \sigma_\pi)$, efficient outcomes on the boundary of Figure 3 are generated by the efficient policies highlighted in Table 1. In Table 1 and

[9] The shaded areas in Table 1 and later in Tables 2-3 represent efficient policies, which are formally mentioned in the text and throughout the Section.

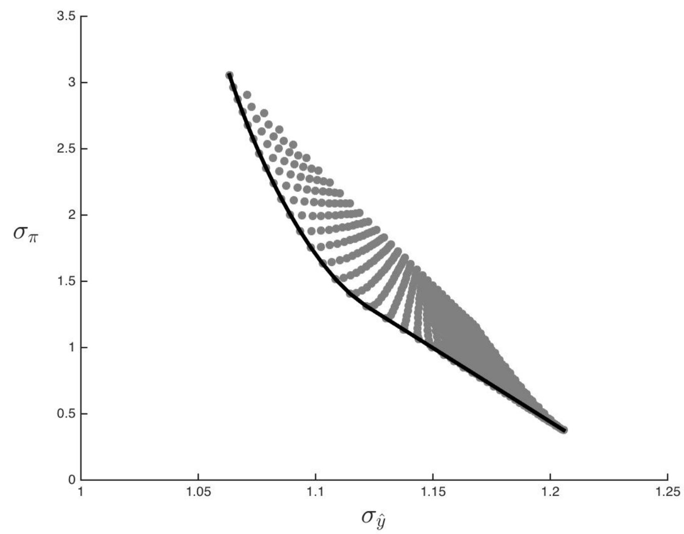


Figure 3: Trade-off between output and inflation stabilization under RE.

all following tables we report results for the intervals $[0 \leq \xi_y \leq 2.0]$ and $[1.2 \leq \xi_\pi \leq 2.0]$. $\xi_\pi > 1$ guarantees stability while the intervals for which we report our results were chosen as politically relevant in accordance to common view among central bankers but our results are unchanged for larger intervals. It is thus surprising that *efficient policies are boundary policies*: either inflation stabilization ($\xi_y = 0, 1.2 \leq \xi_\pi \leq 2.0$) or output stabilization ($0 \leq \xi_y \leq 2.0, \xi_\pi = 1.2$). Hence, mixed, dual-mandate, policies like ($\xi_y = 0.5, \xi_\pi = 1.4$), presumed to be followed by the Fed, are inefficient and this surprising conclusion will be further discussed later. The result is presented here first in an RE context but it remains true under diverse beliefs. Belief diversity changes the efficient frontier but does not alter the conclusion that efficient policies are boundary policies. However, the *boundary* in that case consists of the politically feasible boundary as well as an endogenous boundary which is explained in Section 5.3 on trade-off and efficiency. Such an endogenous boundary is formed by belief diversity. Why surprising? Since it is common to view the existence of trade-off as ensuring that both policy instruments are used. This view is false: existence of trade-off is only necessary but not sufficient.

Turning to belief diversity, it clearly entails diverse agents' consumption and investment decisions as well as diverse desires to work. Change in expectations generate income and inter-temporal substitution effects in these decisions that alter both the levels and timings of these activities. These vary with belief intensity and with policy parameters, giving rise to interaction between policy and private expectations which is central to this paper: it will be repeatedly noted later. On the other hand, although central banks are concerned with excessive borrowing and with asset prices, these play no role in the dynamics of the representative RE agent in Table 1 since such an agent does not borrow.

Equilibrium aggregates depend upon market belief hence expected aggregates depend in a central way upon private expectations of future market beliefs. The key expression of this heterogeneity is agent j's belief about date $t+1$ mean beliefs of others, (i.e. Z_{t+1}), and modeled by the parameter λ_Z^g which measures how $\mathbb{E}_t^j [Z_{t+1}]$ in (17) changes with g_t^j. Now suppose that $g_t^j > 0$: agent j is optimistic about the current state of the economy and its effect on future state variables. Given observed market belief Z_t, the question at hand is: what does j expect to believe tomorrow relative to Z_t? To quantify this issue let us define the *conditional optimism of j relative to the mean market belief* by the difference $Z_t - g_t^j$. Note the term *conditional* since all comparisons are made given known information at t. We know from (17) that relative optimism has a persistence rate $\lambda_Z < 1$ so that, apart from other factors, it tends to decline. We also know from (17) that

$$\mathbb{E}_t^j \left[Z_{t+1} - g_{t+1}^j \right] = \lambda_Z \left[Z_t - g_t^j \right] + \lambda_Z^g g_t^j$$

and this shows that the sign of λ_Z^g alters the speed at which conditional relative optimism declines. This implies two different and opposing belief patterns of the agent.

Case a: $\lambda_Z^g > 0$. Such optimistic j expects others to be relatively *more* optimistic than him at $t+1$ while a pessimistic j expects others to be relatively *less* optimistic (more pessimistic) than him. In short, he has a reserved view, expecting others to react more sharply to shocks than he does.

Case b: $\lambda_Z^g < 0$. This pattern is the opposite of **Case a**. It implies that shifting belief of an agent lead him to expect others to shift less than himself: when optimistic he expects the market to be *less* relatively optimistic and when pessimistic he expects the market to be *more* relatively optimistic. He has an active view, expecting others to react less sharply to shocks than he does.

Case a reflects agents with less active view of the market and **Case b** a more active role. Which of these is more relevant? Kent and Piccillo (2017) use Bayesian estimation, aggregate macro data and survey data for the US economy to estimate the parameters of a NKM with Rational Beliefs and conclude the parameter λ_Z^g must be positive. The same conclusion is reached by identifying the belief parameter λ_Z^g using forecast data on GDP growth rate collected by the *Survey of Professional Forecasters*, which we have partially illustrated in Figure 1. We therefore regard **Case a** to be the one relevant to the US economy.

However, in order to give also a picture of the efficacy of monetary under other belief scenarios, which may be relevant to other economies, we consider below four representative beliefs scenarios indexed by the beliefs parameter λ_Z^g and learning feed-back parameters λ_Z^χ. We choose positive and negative values of λ_Z^g and interact these with high and low values of the feed-back parameters λ_Z^ζ and λ_Z^ν. This results in the following four economic environments:

Case a-1: $\lambda_Z^g = 0.05$, $\lambda_Z^\zeta =$0.1, $\lambda_Z^\nu =$-0.1;
Case a-2: $\lambda_Z^g = 0.05$, $\lambda_Z^\zeta =$0.25, $\lambda_Z^\nu =$-0.25;
Case b-1: $\lambda_Z^g =$-0.05, $\lambda_Z^\zeta =$0.1, $\lambda_Z^\nu =$-0.1;
Case b-2: $\lambda_Z^g =$-0.05, $\lambda_Z^\zeta =$0.25, $\lambda_Z^\nu =$-0.25.

5.2 Macro volatility under different market beliefs scenarios

Tables 2-3 respectively illustrate the simulated standard deviations of output and inflation under the four alternative economic environments listed above. The tables reveal a more complex structure of interaction between monetary policy and diverse beliefs when compared to the RE case in Table 1. We proved in Section 4 the policy space contains a curve of singularity[10] which divides the policy parameter space into sub-regions and this clearly shows up under **Case a-2** and **Case b-2** in Tables 2-3 where points of singularity are marked by "–". Approaching this curve results in rising volatility and at the singularity points $\sigma_{\hat{y}}$ or σ_{π} become unbounded. Thus, the outcome of any monetary policy action is no longer monotonic and straightforward to predict! Moreover, a careful examination of Tables 2-3 reveals that apart from the singularity, some form of non-monotonicity arises in all four cases. Hence, the first general conclusion to draw from Tables 2-3 is that under diverse beliefs non-monotonicity in the market's response to credible changes in policy parameters is widespread over the policy parameter space. The main cause for non-monotonicity are the competing income effects and inter-temporal substitution effects of beliefs, as explained in detail in Section 4. Let

Output Volatility ($\sigma_{\hat{y}}$)

Case a-1

	ξ_y							
ξ_π	**0.00**	**0.25**	**0.50**	**0.75**	**1.00**	**1.25**	**1.50**	**2.00**
1.20	1.292	1.265	1.243	1.224	1.208	1.194	1.182	1.162
1.25	1.278	1.257	1.239	1.223	1.209	1.197	1.186	1.168
1.30	1.268	1.251	1.235	1.222	1.210	1.199	1.189	1.172
1.35	1.261	1.246	1.233	1.221	1.210	1.200	1.191	1.176
1.40	1.255	1.242	1.230	1.220	1.210	1.201	1.193	1.179
1.45	1.251	1.239	1.229	1.219	1.210	1.202	1.195	1.181
1.50	1.248	1.237	1.228	1.219	1.211	1.203	1.196	1.184
1.55	1.245	1.236	1.227	1.219	1.211	1.204	1.198	1.186
1.60	1.243	1.234	1.226	1.219	1.211	1.205	1.199	1.188
1.65	1.241	1.233	1.226	1.219	1.212	1.206	1.200	1.189
1.70	1.240	1.232	1.225	1.219	1.212	1.207	1.201	1.191
2.00	1.235	1.230	1.224	1.220	1.215	1.211	1.206	1.198

Case a-2

	ξ_y							
ξ_π	**0.00**	**0.25**	**0.50**	**0.75**	**1.00**	**1.25**	**1.50**	**2.00**
1.20	1.148	1.219	1.540	–	11.289	4.464	6.199	–
1.25	1.130	1.135	1.176	1.344	2.214	–	3.282	3.030
1.30	1.134	1.126	1.126	1.146	1.229	1.601	–	1.855
1.35	1.142	1.131	1.124	1.122	1.131	1.170	1.332	4.014
1.40	1.150	1.139	1.131	1.124	1.121	1.124	1.142	1.633
1.45	1.157	1.147	1.139	1.131	1.126	1.122	1.121	1.161
1.50	1.162	1.154	1.146	1.139	1.132	1.127	1.123	1.124
1.55	1.167	1.159	1.152	1.145	1.139	1.134	1.129	1.123
1.60	1.172	1.164	1.157	1.151	1.145	1.140	1.135	1.128
1.65	1.175	1.168	1.162	1.156	1.151	1.146	1.141	1.133
1.70	1.178	1.172	1.166	1.160	1.155	1.150	1.146	1.139
2.00	1.190	1.186	1.182	1.177	1.174	1.170	1.167	1.160

Case b-2

	ξ_y							
ξ_π	**0.00**	**0.25**	**0.50**	**0.75**	**1.00**	**1.25**	**1.50**	**2.00**
1.20	1.431	1.386	1.348	1.317	1.290	1.267	1.247	1.215
1.25	1.410	1.373	1.341	1.314	1.290	1.269	1.251	1.222
1.30	1.391	1.359	1.332	1.308	1.288	1.269	1.253	1.226
1.35	1.373	1.347	1.323	1.303	1.284	1.268	1.253	1.228
1.40	1.358	1.336	1.315	1.297	1.281	1.266	1.253	1.230
1.45	1.346	1.326	1.308	1.292	1.277	1.264	1.252	1.231
1.50	1.335	1.317	1.301	1.287	1.274	1.262	1.251	1.231
1.55	1.325	1.310	1.296	1.283	1.271	1.260	1.250	1.232
1.60	1.317	1.303	1.290	1.279	1.268	1.258	1.249	1.232
1.65	1.310	1.297	1.286	1.275	1.265	1.256	1.247	1.232
1.70	1.303	1.292	1.282	1.272	1.263	1.254	1.246	1.232
2.00	1.279	1.272	1.265	1.258	1.252	1.246	1.241	1.230

Case b-2

	ξ_y							
ξ_π	**0.00**	**0.25**	**0.50**	**0.75**	**1.00**	**1.25**	**1.50**	**2.00**
1.20	–	–	–	2.290	2.041	1.878	1.766	1.636
1.25	1.675	–	–	–	1.450	1.359	1.275	1.198
1.30	1.225	1.211	1.201	1.197	1.229	–	1.257	1.166
1.35	1.226	1.214	1.203	1.194	1.186	1.180	1.174	1.166
1.40	1.226	1.215	1.206	1.198	1.191	1.185	1.180	1.171
1.45	1.225	1.216	1.208	1.201	1.194	1.189	1.184	1.175
1.50	1.224	1.216	1.209	1.202	1.196	1.191	1.187	1.179
1.55	1.223	1.216	1.209	1.203	1.198	1.193	1.189	1.182
1.60	1.223	1.216	1.210	1.205	1.200	1.195	1.191	1.184
1.65	1.223	1.216	1.211	1.205	1.201	1.197	1.193	1.186
1.70	1.222	1.216	1.211	1.206	1.202	1.198	1.194	1.188
2.00	1.221	1.217	1.213	1.209	1.206	1.203	1.200	1.194

Table 2: Output volatility in the four economic environments under study.

[10] Under the linear approximation in Section 4 there exist a single curve of singularity which is a linear function in ξ_π and ξ_y. When considering a second order approximation, the enlarged state space (25) contains cross-sectional moments with transition equations whose coefficients are implicit functions of the policy parameters ξ_π and ξ_y as shown in (28) and (29). Therefore the pattern of singularity becomes non-linear and can exhibit more than a single curve.

Inflation Volatility (σ_π)

Case a-1

ξ_π \ ξ_y	0.00	0.25	0.50	0.75	1.00	1.25	1.50	2.00
1.20	2.433	2.270	2.225	2.269	2.374	2.517	2.680	3.028
1.25	2.264	2.128	2.084	2.112	2.193	2.309	2.446	2.749
1.30	2.098	1.988	1.950	1.973	2.041	2.140	2.260	2.529
1.35	1.947	1.856	1.827	1.848	1.909	1.997	2.104	2.349
1.40	1.811	1.737	1.714	1.736	1.791	1.872	1.970	2.196
1.45	1.690	1.629	1.613	1.635	1.687	1.762	1.853	2.063
1.50	1.583	1.532	1.521	1.543	1.593	1.664	1.750	1.947
1.55	1.487	1.445	1.437	1.461	1.509	1.576	1.657	1.844
1.60	1.402	1.366	1.362	1.386	1.433	1.497	1.574	1.751
1.65	1.325	1.295	1.294	1.318	1.364	1.425	1.498	1.668
1.70	1.256	1.230	1.232	1.256	1.301	1.360	1.430	1.592
2.00	0.955	0.945	0.953	0.978	1.016	1.065	1.122	1.252

Case a-2

ξ_π \ ξ_y	0.00	0.25	0.50	0.75	1.00	1.25	1.50	2.00
1.20	5.943	6.368	7.740	–	13.834	16.340	16.382	–
1.25	4.654	4.811	5.171	6.233	13.459	–	12.820	15.484
1.30	3.845	3.899	4.037	4.320	5.026	8.077	–	9.172
1.35	3.285	3.296	3.355	3.473	3.690	4.153	5.647	12.049
1.40	2.872	2.863	2.889	2.950	3.053	3.220	3.532	8.526
1.45	2.555	2.536	2.546	2.582	2.643	2.734	2.868	3.557
1.50	2.302	2.279	2.280	2.304	2.347	2.409	2.492	2.764
1.55	2.096	2.071	2.068	2.085	2.119	2.168	2.230	2.402
1.60	1.924	1.899	1.894	1.907	1.936	1.977	2.030	2.165
1.65	1.779	1.754	1.748	1.759	1.784	1.822	1.869	1.988
1.70	1.655	1.630	1.624	1.633	1.656	1.691	1.735	1.845
2.00	1.168	1.149	1.143	1.151	1.169	1.197	1.232	1.322

Case b-1

ξ_π \ ξ_y	0	0.25	0.5	0.75	1	1.25	1.5	2
1.20	2.540	2.348	2.286	2.319	2.417	2.552	2.707	3.034
1.25	2.482	2.293	2.207	2.206	2.265	2.365	2.489	2.771
1.30	2.390	2.215	2.125	2.106	2.141	2.217	2.318	2.562
1.35	2.285	2.127	2.040	2.013	2.035	2.093	2.178	2.392
1.40	2.177	2.036	1.955	1.925	1.939	1.986	2.058	2.248
1.45	2.072	1.947	1.873	1.843	1.852	1.891	1.954	2.124
1.50	1.972	1.861	1.794	1.766	1.772	1.805	1.861	2.016
1.55	1.879	1.780	1.719	1.693	1.698	1.728	1.778	1.919
1.60	1.793	1.704	1.649	1.625	1.629	1.656	1.702	1.833
1.65	1.712	1.633	1.583	1.562	1.565	1.590	1.633	1.755
1.70	1.638	1.566	1.521	1.502	1.506	1.530	1.569	1.684
2.00	1.292	1.250	1.226	1.217	1.223	1.243	1.273	1.360

Case b-2

ξ_π \ ξ_y	0	0.25	0.5	0.75	1	1.25	1.5	2
1.20	–	–	–	20.421	14.209	11.957	11.221	11.387
1.25	13.470	–	–	–	10.041	8.589	7.660	7.287
1.30	7.378	7.277	7.264	7.446	8.358	–	7.137	5.605
1.35	5.634	5.463	5.310	5.174	5.052	4.941	4.840	4.652
1.40	4.774	4.631	4.507	4.399	4.304	4.221	4.148	4.027
1.45	4.190	4.069	3.964	3.874	3.795	3.728	3.670	3.574
1.50	3.747	3.641	3.550	3.472	3.405	3.349	3.301	3.227
1.55	3.394	3.299	3.219	3.151	3.093	3.046	3.007	2.950
1.60	3.103	3.019	2.947	2.887	2.837	2.796	2.764	2.722
1.65	2.860	2.783	2.719	2.665	2.622	2.587	2.561	2.530
1.70	2.653	2.583	2.524	2.477	2.438	2.409	2.387	2.365
2.00	1.853	1.809	1.773	1.746	1.726	1.714	1.709	1.717

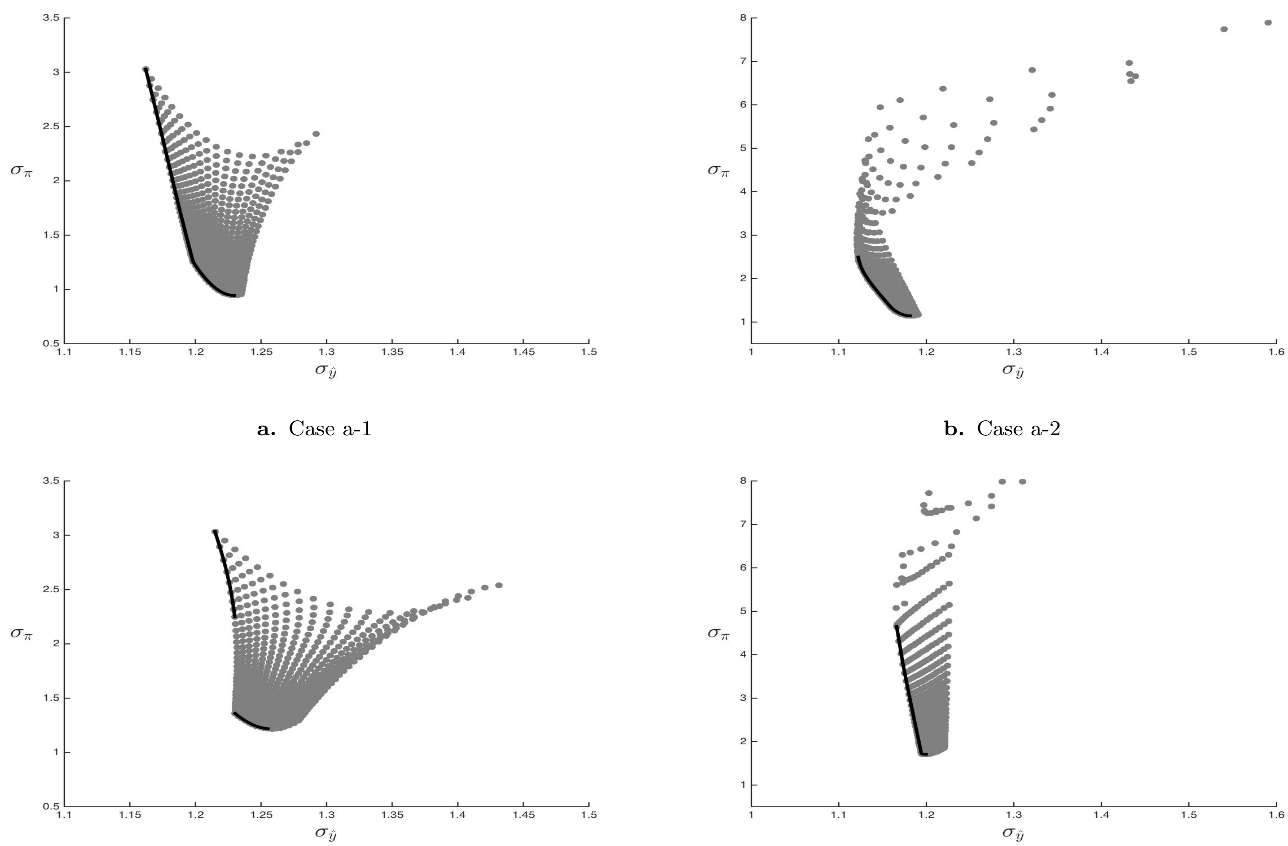
Table 3: Inflation volatility in the four economic environments under study.

us examine **Case b-2**: for $1.2 \leq \xi_\pi \leq 1.3$ and a low ξ_y any relatively small change towards a more aggressive output stabilization policy utilizing larger values of ξ_y is self defeating since it causes an increased output volatility rather than a decrease. However, for $\xi_\pi = 1.25$, if we let $\xi_y \to \infty$ we achieve lower output volatility $\sigma_{\hat{y}} \to 1.09$ but at the cost of higher volatility of inflation and of individual consumption: $\sigma_\pi \to 11.93$ and $\sigma_{\hat{c}^j} \to 6.29$. On the other hand, a more aggressive inflation stabilizing policy reduces the volatility of inflation: as $\xi_\pi \to \infty$ inflation volatility $\sigma_\pi \to 0$. In spite of this, such a policy is subject to limitations. Its effect on output volatility is bounded below: as $\xi_\pi \to \infty$, output volatility $\sigma_{\hat{y}} \to 1.22$ which is bounded away from zero and higher than volatility achievable with less aggressive policies (for instance, see the outcome in Table 2 when $1.3 \leq \xi_\pi \leq 1.5$ and $0.25 \leq \xi_\pi \leq 1$). Similar patterns occur for the other three **Cases**. Changes of policy rule may induce unexpected interactions between private expectations and the aggregate economy. As advocated by Blanchard (2014), commenting on the current financial crisis, there exist "*dark corners* - situations in which the economy could badly malfunction." Our results offer the view that we can get closer to or into those *dark corners* just because of unexpected market results unleashed by the effect of private expectations on policy outcomes.

We comment that individual consumption exhibits volatility patterns that differ from those of aggregate income: standard deviations $\sigma_{\hat{c}^j}$ are uniformly higher across the policy parameter space. This results from the fact that an individual's consumption responds to his own belief as well as to economy wide measures of belief diversity. In such an economy trading volume in financial markets is large and we used a penalty as a proxy for credit regulations to ensure equilibrium exists (see Kurz and Motolese (2001) and Kurz et al. (2005a)).

Our results show that in an economy with diverse beliefs monetary policy must find a balance between stabilization of inflation, output and individual consumption. While this choice is absent from a single agent economy, it is understandable under diverse beliefs since now the central bank acts in a more complex world, where its policies are amplified or muffled by expectations of agents (see also Motolese (2003)). There exists a deep interaction between market expectations and efficacy of policy and Tables 2-3 show substantially higher aggregate volatility than the volatility obtained under RE in Table 1.

5.3 Private expectations and trade-off between inflation and output volatility



a. Case a-1

b. Case a-2

c. Case b-1

d. Case b-2

Figure 4: Inflation-output volatility trade-off in the four economic environments under study[a]

[a] To better illustrate the trade-off in **Case a-2** and **Case b-2** points in the north-east corner with volatility of inflation higher than 8% have been left out. Also notice the scaling of axis in **Cases a-1** and **b-1** is smaller than in **Cases a-2** and **b-2** which are more volatile.

The non-monotonicity of response under diverse beliefs implies that point-wise trade-off is not present. There are many pairs (ξ_y, ξ_π) with *neighboring* policies which do not offer trade-off. For example $\xi_y = 0.50, \xi_\pi = 1.50$ in **Case a-1**. If the central bank chooses $\xi_y = 0, \xi_\pi = 1.50$ aiming to accept *higher* output volatility for *lower* inflation volatility it would find that this change *increases* both volatilities since such point-wise trade-off is not feasible! The problem is even more dramatic in **Cases a-2** and **b-2** as illustrated earlier. In short, when private expectations matter, a central bank cannot assume that a more aggressive policy in some direction will have predictable outcome based on an assumed local trade-off. Changes of policy parameters or comparisons across economies with different policy parameters must be based on some understanding of the structure of private expectations and the interaction between policy and private expectations.

Leaving aside the general point-wise trade-off, we explore weaker forms of trade-off that may be available under diverse beliefs. To that end we project in Figure 4 the set of outcomes that materialize in the four **Cases** under study and it is clear that they vary widely. These outcome sets do not have the convex structures as under RE since now a trade-off may imply that a neighboring policy outcome is not in the policy space neighborhood and a trade-off often requires a choice between two different regions of the parameter space.

To characterize efficient policies we need to discuss our *boundary* concepts of policy parameters. To that end note the *ridge* of high volatilities associated with singularity in **Cases a-2** and **b-2** of Tables 2-3. This *ridge* causes non-monotonicity of outcomes: output volatility is low before reaching the *ridge* but then it rises sharply, creating an endogenous boundary at the bottom of which efficient outcomes can be formed. We thus distinguish between *hard* boundaries formed by political feasibility and *soft* boundaries formed by ridges of singularity. We say *ridges* rather than *a ridge* (as in Tables 2-3) since out in the larger policy space beyond the acceptable hard boundary used in Tables 2-3 we find other singularities which generate additional non-monotonicity in outcomes and, constituting *soft* boundaries, additional efficient policies. It is obvious that expansion of the policy space renders inefficient some points in Tables 2-3 which are efficient in the smaller space.

With this broader concept of *boundary*, our general conclusion is that *all efficient policies are boundary policies but with a crucial difference*. The effect of diverse beliefs, as distinct from RE, is *to move efficient policies to the aggressive boundaries defined by* $\xi_\pi = 2.0$, *by* $\xi_y = 2.0$ *or by the singularity* since policy must now address the higher volatility and complexity of outcomes due to private expectations. In simple terms, now policy aims to *subdue* private expectations and these boundary policies achieve this goal. However, note again that although widespread trade-off exists here, efficient policies are typically not mixed. The question is: why?

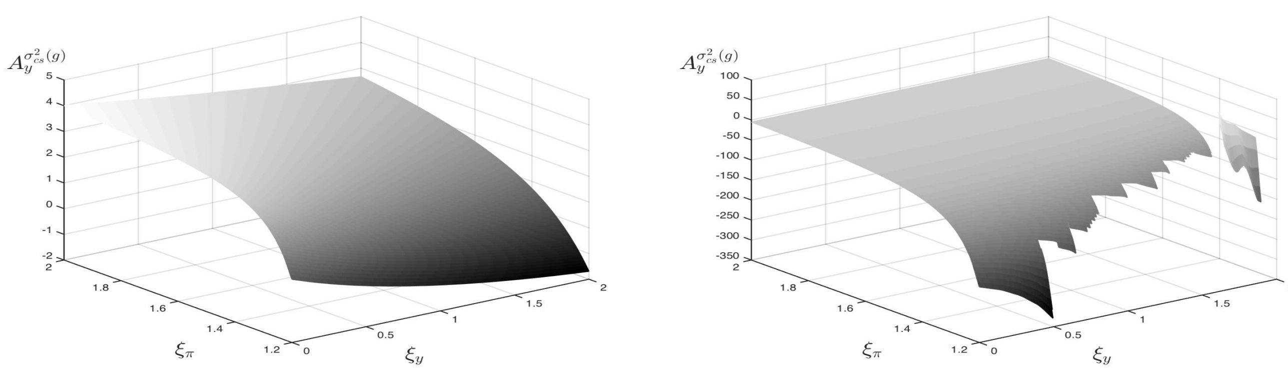
The evidence supports the view that a central bank does not alter interest rates in response to short term fluctuations in inflation or unemployment if such fluctuations are centered around the bank's goal. On the other hand, a central bank acts when it perceives major cyclical changes in inflation or unemployment which means it reacts only to assessed major changes in the *direction* of these variables. This implies a bank's low concern with long term *average* volatilities $(\sigma_{\hat{y}}, \sigma_\pi)$ but high concern with those parts of volatility that reflect sharp changes in the state of the economy. Under this interpretation the monetary rule we have used is an appropriate description of the policy in use but it does not imply that policy efficiency is measured by $(\sigma_{\hat{y}}, \sigma_\pi)$. In addition, the rule actually employed cannot be supported by the standard argument showing it minimizes a cost function which is monotonic in the long term average volatility $(\sigma_{\hat{y}}, \sigma_\pi)$.

5.4 The direct equilibrium effect of diversity of beliefs

As pointed out earlier, empirical evidence reveals belief scenarios under **Case a** to be those relevant to the US economy. Therefore, we now turn our attention to those belief environments and discuss the effects of the distribution of beliefs on aggregate output and inflation. As explained in Section 2.6, both the mean market belief Z_t and the cross-sectional distribution of individual beliefs are state variables. How do these belief state variables interact with monetary policy?

5.4.1 The case of the cross-sectional distribution of individual beliefs $\sigma^2_{cs}(g)_t$

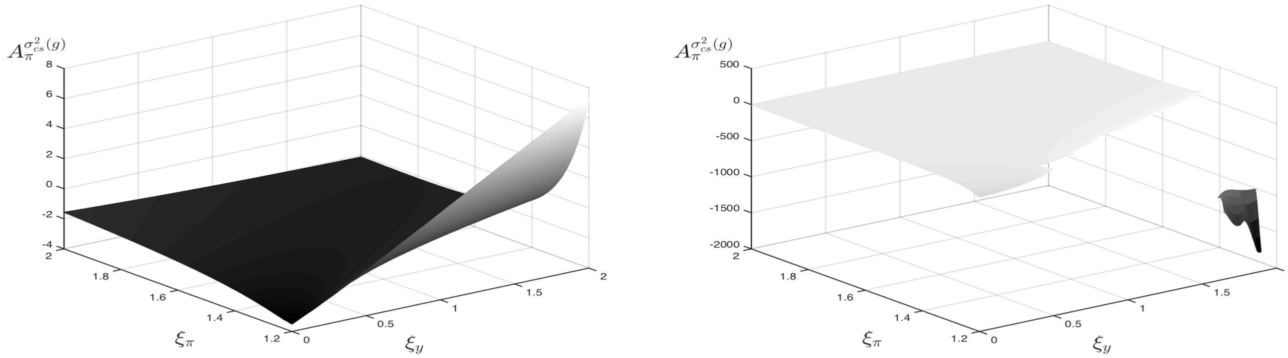
Let us begin by analyzing the impact of the cross-sectional distribution of individual beliefs, which is one of the novelties introduced by this paper. We report in Figures 5 and 6 the computed equilibrium elasticities of aggregate output and inflation with respect to $\sigma^2_{cs}(g)_t$. Any positive shock to $\sigma^2_{cs}(g)_t$ increases disagreement across agents. What is then the aggregate impact of increased disagreement? Figures 5 and 6 reveal the impact varies across the policy parameter space.



a. Case a-1 **b.** Case a-2

Figure 5: Equilibrium coefficients of aggregate output with respect to $\sigma^2_{cs}(g)_t$.

We need to think of the variable $\sigma^2_{cs}(g)_t$ as a market externality which can be either positive or negative. The effect of increased belief dispersion on real activity is positive and results in higher aggregate output almost everywhere under **Case a-1**. The contrary occurs under **Case a-2** where

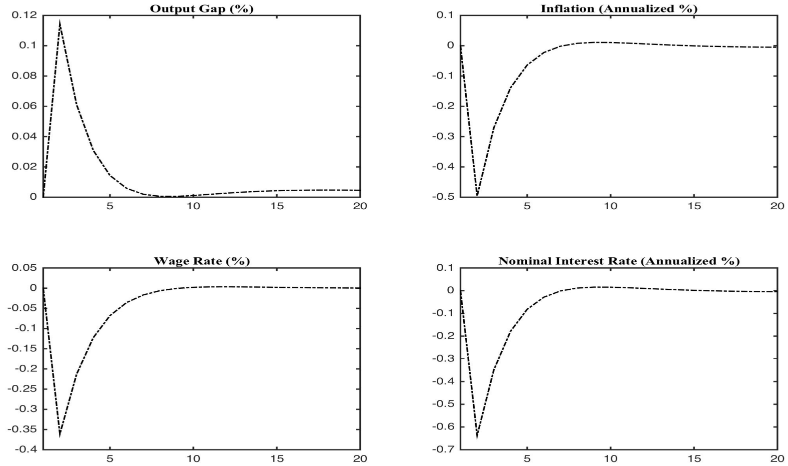


a. Case a-1 **b.** Case a-2

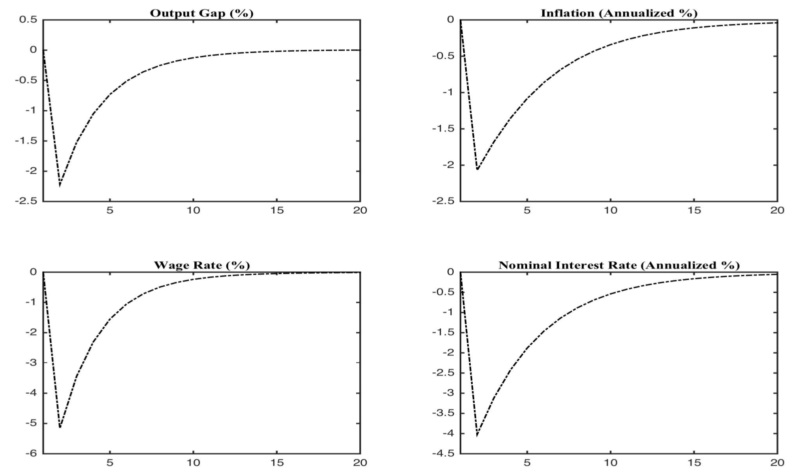
Figure 6: Equilibrium coefficients of inflation with respect to $\sigma^2_{cs}(g)_t$.

increased belief dispersion is uniformly associated with a decline in real activity. The positive effect of increased diversity is compatible with the results in Kurz and Motolese (2001, 2011) where it is shown an increase in diversity of market opinions decreases the risk premium. In this case markets are more stable since more dispersed beliefs cancel each other out, resulting in reduced volatility.

We focus in Figure 7 on assessing the macro-economic effect of increased diversity of expectations, holding the mean fixed. This question has recently been discussed in relation to the effect of changes in aggregate risk on which we comment below. Figure 7 shows that in all cases increased disagreement lowers *(i)* the wage rate *(ii)* the inflation rate and *(iii)* the nominal rate. The difference between **Case a-1** and **Case a-2** is the effect on output: in **Case a-1** output is increased and in **Case a-2** it is decreased. To understand these results observe that a symmetric increase in diversity means that some become more optimistic and some more pessimistic *about the future state*. Concavity of utility implies that under full symmetry the pessimists' intensity is greater than the optimists' intensity hence the aggregate effect is similar to the expectation of lower future incomes and wages. This increases present day labor supply, lowers wages and marginal cost, and hence lowers inflation and nominal rate. The harder question is the effect on output, determined by the equilibrium effect on total employment. Dominant pessimism not only implies increased labor supply, but also brings an opposing force by which employment is lowered due to lower wages and demand associated with lower prices and inflation. **Case a-1** and **Case a-2** differ in the intensity in which expectations respond to current information: in **Case a-1** $\lambda^{\zeta}_{Z} = 0.1$ while in **Case a-2** $\lambda^{\zeta}_{Z} = 0.25$ and this difference alters the dominance of income vs. the inter-temporal substitution effects of expectations. In **Case a-1** the negative income effect of a more pessimistic outlook enhances the effect of increased current supply of labor (reduction of leisure) which results in *increased* output. In **Case a-2** the inter-temporal substitution effect dominates to *weaken* the current rise in labor supply in favor of lower future labor supply, thus strengthening the effects of present lower prices and deflation which cause a *decrease* in output.



a. Case a-1



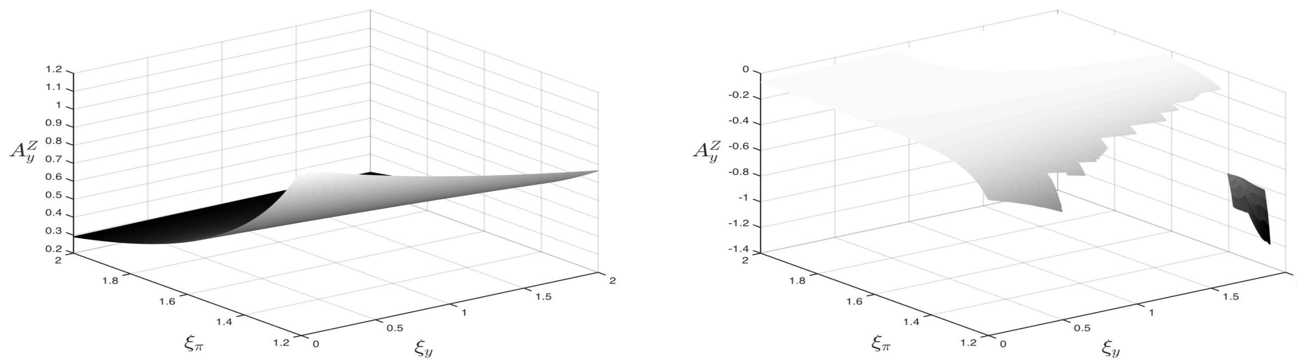
b. Case a-2

Figure 7: Effects of a 0.05% quarterly increase to dispersion of private expectations $\sigma^2_{cs}(g)_t$ at $(\xi_\pi = 1.4, \xi_y = 0.5)$.

The above discussion shows that increased disagreement is not an expression of aggregate risk. Increased aggregate risk could cause increased disagreement but the causality does not go the other way around. We thus conclude that from observing higher belief dispersion in the market we cannot deduce we are in a state of higher macroeconomic uncertainty as strongly advocated by Bloom (2009). We agree with Jurado et al. (2015) and conclude that increases in cross-sectional dispersion are not necessarily associated with increases in macroeconomic uncertainty and *may vary for many reasons that are unrelated to broad-based macroeconomic uncertainty.*

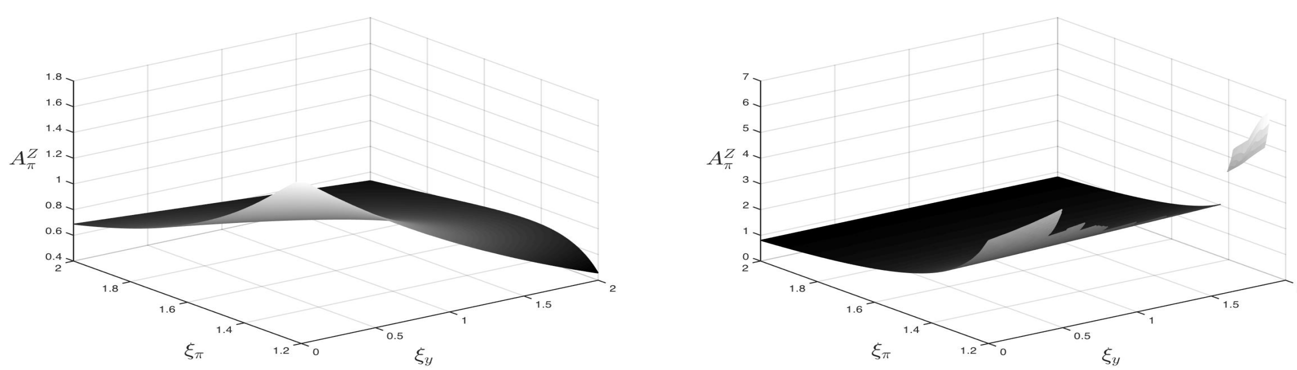
5.4.2 The case of the mean market belief Z_t

We now turn our attention to the direct effect of the observed mean of the distribution of beliefs, i.e. Z_t. We consider first the effects of optimism about the future state on aggregate output and inflation and report in Figures 8-9 the equilibrium elasticities of aggregate output and inflation with respect to mean market belief variable Z_t. The impact of optimism (a positive shift to Z_t) is not uniform across **Case a-1** and **Case a-2**. Shifts to Z_t change the inter-temporal allocation of consumption and labor between date t and future dates. While a positive shift to Z_t boosts inflation in both **Case a-1** and **Case a-2** (though with different intensity), its effects on aggregate output varies with different beliefs configurations (f.i. under **Case a-2** optimism lowers aggregate output).



a. Case a-1 **b.** Case a-2

Figure 8: Equilibrium coefficients of aggregate output with respect to mean market belief Z_t.

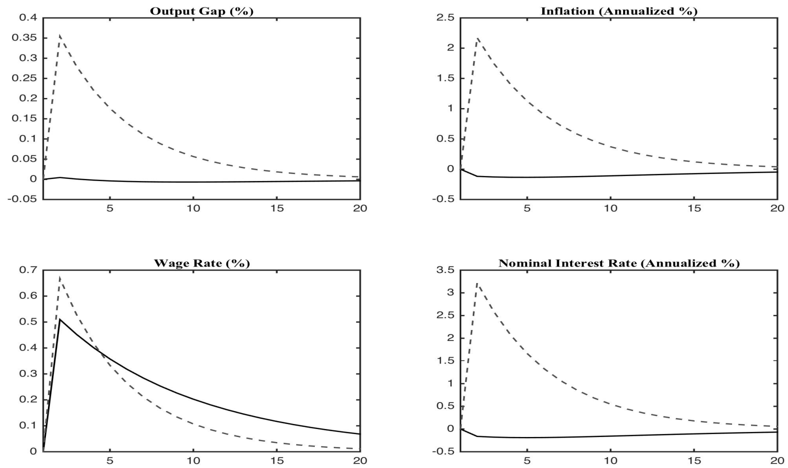


a. Case a-1 **b.** Case a-2

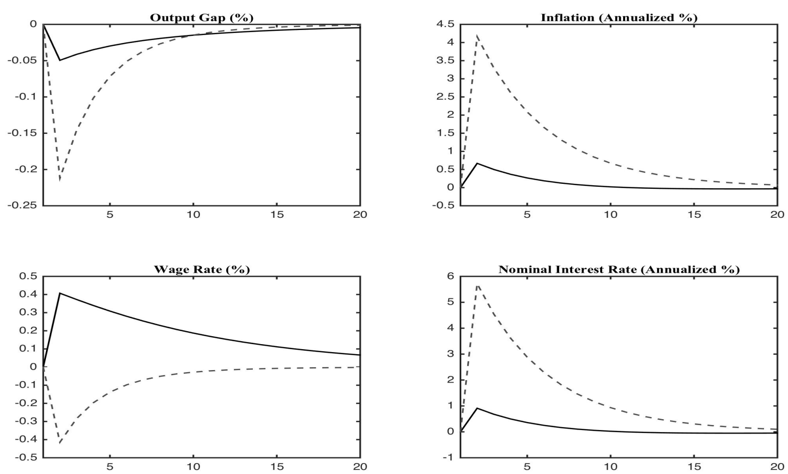
Figure 9: Equilibrium coefficients of inflation with respect to mean market belief Z_t.

In **Case a-2** the inter-temporal substitution effect prevails over the income effect from expected higher future income and lowers present consumption, pushing aggregate output lower. In all other

Cases the income effect prevails. However, the intensity of the impact of any shift to Z_t depends upon the monetary policy configuration. In sum, the sign and the intensity of the response of aggregate output to market optimism varies with the beliefs environment and policy.



a. Case a-1



b. Case a-2

Figure 10: Effects of a 0.5% shock to productivity $\hat{\zeta}_t$ (solid line) and mean market belief Z_t (dashed line) at $(\xi_\pi = 1.4, \xi_y = 0.5)$.

To further clarify the aggregate effects of shifts to market belief Z_t we report in Figure 10 impulse responses of output gap, inflation, wage rate and nominal interest rate to a 0.5% shock to mean market belief Z_t and to productivity $\hat{\zeta}_t$ under policy $(\xi_\pi = 1.4, \xi_y = 0.5)$. The difference between the two panels of Figure 10 is due to two factors: *(i)* the dominant income effect in Figure 10.a vs. the dominant substitution effect in Figure 10.b and *(ii)* the interaction of these effects with

$\hat{\zeta}_t$, whose impact is augmented or shrunk via the positive revision of mean market belief Z_t caused by the increased forecast error in $\hat{\zeta}_t$. Any positive unanticipated shock to $\hat{\zeta}_t$ results in a higher level of productivity than forecasted. Agents revise upward their beliefs according to (13) and by aggregation this results in higher optimism in the mean market belief Z_t according to (15). In the economy of **Case a-2** in Figure 10.b agents give higher weight (i.e. λ_Z^{ζ}=0.25) to the forecast error than under the economy of Figure 10.a. Therefore, an unexpected rise in productivity causes agents to expect higher income in the future making the substitution effect dominate. This accounts for the sign reversal of the impulse response of the output gap to a $\hat{\zeta}_t$ shock. Moreover, the responses under **Case a-2** in Figure 10.b, when output and inflation respond with opposite signs, demonstrate there exist circumstances when the economy exhibits a *stagflation response* to a shock. In such circumstances policy is partly muted since the effects of higher inflation and lower output on the nominal interest rates are partly canceled.

6 Conclusions and policy implications

Our NKM incorporates diverse beliefs and is solved with a quadratic approximation. Hence, aggregation of quadratic decision functions naturally turns the belief distribution into a new set of state variable expressed by the mean and variance. The model has two exogenous shocks and under RE smooth trade-off between inflation and output volatility exists. Under diverse beliefs this does not hold: both policy efficacy and the nature of trade-off change drastically. The key results and policy implications, which results from the diversity of beliefs, are as follows:

(i) The policy space contains curves of singularity. Each is a collection of policy parameters at which volatilities are unbounded and which divide the space into sub-regions. Some trade-off between output and inflation volatility exists within each region and some across regions. Hence, some trade-off is made with jumps in policy.

(ii) Each singularity causes volatility outcomes to be complex and non-monotonic. Hence a policy-maker cannot assume that a more aggressive policy parameter will change outcomes in a smooth and monotonic manner as predicted by a single agent RE model.

(iii) Despite monotone response and trade-off under RE, all efficient policies relative to monotone utility of output and inflation volatilities are *lower boundary policies* that use only one instrument: either $\xi_y = 0$ or $\xi_\pi = 1.2$ (the lowest parameter value). This result remains true under diverse beliefs but with a key difference: all efficient policies are on the *upper*, *aggressive*, *boundary* defined by $\xi_\pi = 2.0$ or $\xi_y = 2.0$ or by the singularity. The reason is that policy faces increased outcome volatility and complexity due to private expectations and policy now aims to *subdue* the effects of private beliefs. Contrary to common view, *full trade-off does not guarantee the efficiency of mixed instruments*. Indeed, the policy ($\xi_y = 0.5, \xi_\pi = 1.4$), viewed as approximating the current Fed policy, is not efficient with respect to a utility that minimizes a monotonic cost function of $(\sigma^2_{\hat{y}}, \sigma^2_{\pi})$.

(iv) Belief diversity implies diverse individual consumption and diverse bond holdings. Central bank policy must therefore consider volatility of individual consumption, in addition to volatility of output and inflation. Policy with volatile nominal rate leads to high consumption volatility which is then a proxy for the volatility of financial markets.

(v) Higher market optimism Z_t generates an income and an inter-temporal substitution effect on current variables. In most cases income effects dominate, causing increased demand, higher wage and higher output. In cases when unexpected high productivity intensifies (via inference) private expectations about the future, inter-temporal substitution dominates and increased optimism results in lower present employment and *lower present output but higher inflation*. This *stagflation* effect is stronger the stickier prices are. The policy response is then muted since the effects of higher inflation and lower output on the interest rate partially offset. To be more effective a central bank may need to target beliefs directly or their impact in asset markets. Some evidence suggest that in practice central banks do respond to asset price movements even if temporarily and not systematically.

(vi) Belief dispersion is measured by cross-sectional variance of belief indices, holding the mean constant. We show that increased belief dispersion lowers inflation. It causes *higher* present output when income effects dominate and *lower* output when inter-temporal substitution effects dominate, leading to increased demand for current leisure. It demonstrates that increased disagreement is not an expression of aggregate risk. Higher aggregate risk could cause increased disagreement but the causality does not work the other way around. We thus disagree with recent attempts to identify disagreement with higher macroeconomic uncertainty.

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Appendix A RBE restrictions and the role of learning feed-back

The RB principle (see Kurz (1994)) is a model of rational agents who deviate from m but reproduce it with sufficiently long data. An RB model of the exogenous shocks exhibits the same volatility as the empirical model and it implies the following restrictions (for details, see Kurz et al. (2013)):

$$\begin{aligned}
\text{Var}[\lambda_\zeta^g g_t^j + \rho_{t+1}^{j\zeta}] &= \text{Var}[\rho_{t+1}^{\zeta}] \Rightarrow (\lambda_\zeta^g)^2 \,\text{Var}(g) + \hat{\sigma}_\zeta^2 = \sigma_\zeta^2 \\
\text{Var}[\lambda_\nu^g g_t^j + \rho_{t+1}^{jv}] &= \text{Var}[\rho_{t+1}^{\nu}] \Rightarrow (\lambda_\nu^g)^2 \,\text{Var}(g) + \hat{\sigma}_\nu^2 = \sigma_\nu^2 \\
\text{Var}[\lambda_Z^g g_t^j + \rho_{t+1}^{jZ}] &= \text{Var}[\rho_{t+1}^{Z}] \Rightarrow (\lambda_Z^g)^2 \,\text{Var}(g) + \hat{\sigma}_Z^2 = \sigma_Z^2
\end{aligned} \tag{A-1}$$

By normalization $\lambda_\zeta^g = 1$, therefore the rationality conditions (A-1) imply

$$\text{Var}(g) \le \sigma_\zeta^2,\ (\lambda_\nu^g)^2 \,\text{Var}(g) \le \sigma_\nu^2,\ (\lambda_Z^g)^2 \,\text{Var}(g) \le \sigma_Z^2,\ \tilde{\sigma}_\zeta \le \sigma_\zeta,\ \tilde{\sigma}_\nu \le \sigma_\nu,\ \tilde{\sigma}_Z \le \sigma_Z \tag{A-2}$$

In addition, the variance of ρ_{t+1}^Z is restricted by σ_g^2 and is specified as

$$\sigma_Z^2 \le \sigma_g^2 \text{ with } \sigma_Z = \rho\sigma_g \text{ and } \rho = \text{Corr}(\rho_{t+1}^{jg}, \rho_{t+1}^Z) > 0 \tag{A-3}$$

The unconditional variance of g_t^j is

$$\text{Var}(g) = \frac{1}{1-\lambda_Z^2}\left[(\lambda_Z^\zeta)^2\sigma_\zeta^2 + (\lambda_Z^\nu \lambda_\nu^g)^2\sigma_\nu^2 + \sigma_g^2\right] \tag{A-4}$$

The parameters $(\lambda_Z^\zeta, \lambda_Z^\nu)$ are important. They measure learning feed-back from current data with which agents deduce changes in estimated value of s_t in (10) from forecast errors in (17). This causes revisions of the belief index g_t^j which is j's subjective conditional expectations of s_t. The variance of g which ignores such feed-back is therefore

$$\text{Var}^{NF}(g) = \frac{\sigma_g^2}{1-\lambda_Z^2} \tag{A-5}$$

Interest in (A-5) arises from the need to reconcile learning feed-back with the RB principle. Learning feed-back increases the variance of g_t^j. In addition, comparing (10) with perception (17) shows learning feed-back causes g_t^j to introduce into (17) correlation with observed data which does not exist in (11). Hence, on the face of it, a learning feed-back violates the RB principle. But a rational agent who adjusts his belief about exogenous shocks in response to most recent data knows there is no learning within the actual data of exogenous shocks hence he must purge (A-1)-(A-5) from the effect of learning feed-back. This is the reason why we use (A-5) and not (A-4) as the basis for restricting individual beliefs. Sufficient conditions implied by this procedure and used in this paper are

$$\sigma_g \le \sigma_\zeta\sqrt{(1-\lambda_Z^2)}, |\lambda_\nu^g| \le \frac{\sigma_\nu}{\sigma_\zeta}, |\lambda_Z^g| \le \frac{\sigma_Z}{\sigma_\zeta} \le \rho\sqrt{(1-\lambda_Z^2)} \tag{A-6}$$

$$(\lambda_\zeta^g)^2 \,\text{Var}^{NF}(g) + \hat{\sigma}_\zeta^2 \ge \sigma_\zeta^2, (\lambda_\nu^g)^2 \,\text{Var}^{NF}(g) + \hat{\sigma}_\nu^2 \ge \sigma_\nu^2, (\lambda_Z^g)^2 \,\text{Var}^{NF}(g) + \hat{\sigma}_Z^2 \ge \sigma_Z^2 \tag{A-7}$$

To clarify the effect of learning feed-back return to (13) where future belief of j depends upon realized future data from which he will deduce g_{t+1}^j. Such dependence of belief upon current data amplifies the effect of beliefs. To see why suppose $g_t^j > 0$ thus j is optimistic today about larger s_t and hence he perceives a larger $\hat{\zeta}_{t+1}^j$ and a larger forecast error $(\hat{\zeta}_{t+1}^j - \lambda_\zeta \hat{\zeta}_t)$ at $t+1$. With learning feed-back he also knows that he expects to interpret this larger forecast error as a larger value of s_{t+1} as well, and the larger is the learning feed-back parameter λ_Z^ζ the larger is this secondary effect of $g_t^j > 0$ on the expected value of s_{t+1}. The mathematical result of this fact is seen by taking expectations of (13) using (12):

$$\mathbb{E}_t^j \left[g_{t+1}^j \right] = \lambda_Z g_t^j + \lambda_Z^\zeta g_t^j + \lambda_Z^\nu \lambda_\nu^g g_t^j = \left(\lambda_Z + \lambda_Z^\zeta + \lambda_Z^\nu \lambda_\nu^g \right) g_t^j$$

Due to learning feed-back from current data the parameter $(\lambda_Z + \lambda_Z^\zeta + \lambda_Z^\nu \lambda_\nu^g)$ of g_t^j in j's expected g_{t+1}^j may exceed 1. Being only within the agent's model it does not have any effect on the actual dynamic movements of either g_t^j or Z_t hence has no effect on market instability or violation of Blanchard-Kahn conditions. It is merely a result of interaction between learning feed-back and persistence of beliefs.

Since we assume only one unobserved state variable, belief parameters must be oriented in sign so as to have comparable meaning. For technology it is clear being *optimistic* means $\lambda_\zeta^g > 0$ hence we set $\lambda_\zeta^g = 1$. Higher values of cost push shock would result in an increase in inflation, which should be perceived as a bad state, therefore we set $\lambda_\nu^g < 0$. Put differently, a larger technology shock and a lower cost push shock would define a *good state* of the economy, therefore we set $\lambda_\zeta^g > 0$ and $\lambda_\nu^g < 0$. But what about the learning feed-back parameters λ_Z^ζ and λ_Z^ν? By using the Bayes Updating theorem in Kurz et al. (2013) we conclude that the above configuration that defines a *good state*, that is better technology and lower cost, also requires that we set $\lambda_Z^\zeta > 0$ and $\lambda_Z^\nu < 0$.

Appendix B More details about the second order approximation and solution approach

The final set of equations and solution of the equilibrium

The final system of equations that defines an equilibrium before any approximation, is now stated. The form which we use is somewhat different from the system described for the linear model.

$$\text{optimal bond holding} \quad \tilde{\tau}_b b_t^j + \left(C_t^j\right)^{-\sigma} \frac{1}{1+r_t} = \mathbb{E}_t^j \left[\beta (C_{t+1}^j)^{-\sigma} \frac{1}{\pi_{t+1}} \right] \tag{B-1a}$$

$$\text{optimal labor supply} \quad \left(C_t^j\right)^{-\sigma} W_t = \left(L_t^j\right)^\eta \tag{B-1b}$$

$$\text{budjet constraint} \quad C_t^j + \frac{b_t^j}{1+r_t} = W_t L_t^j + \frac{b_{t-1}^j}{\pi_t} + [Y_t - W_t L_t] \tag{B-1c}$$

$$\text{optimal pricing} \quad q_{jt} = \frac{\theta}{\theta - 1} \frac{S_{j,t}}{V_{j,t}} e^{\nu_t} \tag{B-1d}$$

$$S_{j,t} = \left(C_t^j\right)^{-\sigma} Y_t \varphi_t + \beta \omega \, \mathbb{E}_t^j \left[S_{j,t+1} \pi_{t+1}^\theta \right]$$

$$V_{j,t} = \left(C_t^j\right)^{-\sigma} Y_t + \beta \omega \, \mathbb{E}_t^j \left[V_{j,t+1} \pi_{t+1}^{\theta - 1} \right]$$

inflation identity $$1 = (1-\omega)\int_0^1 q_{jt}^{(1-\theta)}dj + \omega\left(\pi_t\right)^{-(1-\theta)} \tag{B-1e}$$

the monetary rule $$r_t = \bar{r} + \left(\frac{\pi_t}{\bar{\pi}}\right)^{\xi_\pi}\left(\frac{Y_t}{Y_t^F}\right)^{\xi_y} - 1 \tag{B-1f}$$

market clearing $$\int_0^1 C_t^j dj = C_t = Y_t, \int_0^1 N_t^j dj = N_t = L_t = \int_0^1 L_t^j dj, \tag{B-1g}$$
$$\int_0^1 B_t^j dj = 0.$$

To clarify developments below note the difficulties of solving for an equilibrium with heterogenous agents. Standard optimization techniques can be used to solve agent j's problem *if the agent knows the equilibrium map of the aggregates and of the prices*. Distinct from an optimization, an equilibrium requires two additional conditions: aggregation of individual decision functions that define equilibrium aggregates and second, an imposition of market clearing conditions that define prices. Here it amounts to the wage rate and the inflation rate. These are the two central problems we address next.

Our general equilibrium solution consists of two parts: the first is a solution of all individual decision functions employing a second order approximation of the optimum conditions given hypothetical parameter values of equilibrium prices and aggregate variables. One can consider this a *conjectured* equilibrium. Second, we aggregate the solved individual decision functions, using the market clearing conditions as restrictions on the aggregation. We then deduce from the aggregates implied new parameter values of the maps of prices and aggregate variables. With these new parameters we repeat the first part and seek convergence of the sequence of conjectured equilibria to equilibrium for which no further iterations improve the estimates. The algorithm to compute an equilibrium can be described as follows:

Step (i) Solve the linear model in (20a)-(20f) and store the coefficients;

Step (ii) deduce from step (i) coefficients of cross-sectional variances (19), (28), and (29);

Step (iii) set initial values for coefficients of the second order terms of aggregate variables and other cross-sectional moments all needed for our postulated law of motion of the iterative procedure;

Step (iv) solve (B-1a)-(B-1g) for agent j decision functions using perturbation method;

Step (v) aggregate over all j decision functions to update the coefficients of the law of motion of aggregate variables and other cross-sectional moments of step (iii);

Step (vi) iterate until convergence.

In order to solve for agent decision functions in step (iv) we need to specify the law of motion required by step (iii). We aggregate over the second order expansion of the labor supply equation (B-1b) and obtain:

$$-\sigma\hat{y}_t + \hat{w}_t + \frac{\sigma(1+\sigma)}{2}\int_0^1 (\hat{c}_t^j)^2 dj - \sigma\hat{y}_t\hat{w}_t = \eta\hat{\ell}_t + \frac{\eta(\eta-1)}{2}\int_0^1 (\hat{\ell}_t^j)^2 dj, \tag{B-2}$$

where the integrals, by decomposition and by Assumption 4, are computed by aggregating the squared linear approximation of the individual choice function. Aggregate income and aggregate optimal price are respectively approximated by the following second order polynomials:

$$\hat{y}_t = P^y(Z_t, \hat{\zeta}_t, \nu_t, \sigma_{cs}^2(g)_t, \sigma_{cs}^2(b)_t, Cov(b,g)_t, Z_t^2, \hat{\zeta}_t^2, \nu_t^2, Z_t\hat{\zeta}_t, Z_t\nu_t, \hat{\zeta}_t\nu_t; A^y); \tag{B-3}$$
$$\hat{q}_t = P^q(Z_t, \hat{\zeta}_t, \nu_t, \sigma_{cs}^2(g)_t, \sigma_{cs}^2(b)_t, Cov(b,g)_t, Z_t^2, \hat{\zeta}_t^2, \nu_t^2, Z_t\hat{\zeta}_t, Z_t\nu_t, \hat{\zeta}_t\nu_t; A^q). \tag{B-4}$$

Aggregating over the second order expansion of the labor demand equation implied by firm i production function in (1) we get:

$$\hat{y}_t - \frac{1}{2\theta}(\hat{y}_t)^2 = \hat{\zeta}_t - \frac{1}{2\theta}\hat{\zeta}_t^2 + \hat{\ell}_t - \frac{1}{2\theta}\int_0^1 (\hat{n}_{it})^2 di + \frac{\theta-1}{\theta}\hat{\zeta}_t\hat{\ell}_t. \tag{B-5}$$

Our last task is to define the relationship between aggregated optimal price $\hat{q}_t$ in (B-4) and inflation π_t. We aggregate prices across firms i, as in Kurz et al. (2013) (see equation (9)), by

$$\int_{S_t} (q_{it})^{1-\theta} di + \omega\left(\frac{1}{\pi_t}\right)^{1-\theta} = 1, \tag{B-6}$$

where S_t is the set of firms in $[0,1]$ which can change prices and has measure $(1-\omega)$. The second order approximation of (B-6) leads to:

$$\int_{S_t} [\hat{q}_{it} - \frac{\theta}{2}(\hat{q}_{it})^2] di = \omega[\pi_t + \frac{\theta-2}{2}(\pi_t)^2]. \tag{B-7}$$

The assumption of Calvo pricing creates an artificial heterogeneity in the optimal pricing of the firms, which is in addition to the heterogeneity arising from expectations. However, Assumption 1 ensures that the integral on the left-hand side of (B-7) is independent of sets S_t. For all random sets S_t in $[0,1]$,

$$\int_{S_t}\left[\hat{q}_{it} - \frac{\theta}{2}(\hat{q}_{it})^2\right] di = (1-\omega)\int_0^1 \left[\hat{q}_{it} - \frac{\theta}{2}(\hat{q}_{it})^2\right] di.$$

therefore, we can rewrite (B-7) as

$$\hat{q}_t - \frac{\theta}{2}\int_0^1 (\hat{q}_{it})^2 di = \frac{\omega}{1-\omega}\left[\pi_t + \frac{\theta-2}{2}(\pi_t)^2\right]. \tag{B-7$'$}$$

Hence, equations (B-2)-(B-5) and (B-7′) specify the laws of motion required by step (iii).

To solve the system of equations (B-1a)-(B-1g) given (B-2)-(B-5) and (B-7′) we use the iterative procedure described above and compute solutions at each iteration by perturbation method. To initiate iteration process we pick initial values for the set of coefficients $A = (A^y, A^q)$ in (B-3) and (B-4).

Aggregate variables in agent j's world are taken as given. The algorithm iterates on the unknown coefficients $A = (A^y, A^q)$ of the polynomials (B-3) and (B-4) until equilibrium conditions are satisfied. The iterative procedure can set coefficients at each iteration k using fixed-point iteration $A_{k+1} = \delta A_k + (1-\delta)A^*$, where δ is a damping parameter and A^* are the coefficients obtained after aggregating the optimal solution of agent j given aggregate parameters A_k. An alternative procedure, which we follow, is that of using the embedded iterative procedure of a Gauss-Newton non-linear equations solver. Convergence is achieved when equilibrium conditions are satisfied with an error of less than 10^{-8}.

As extensively pointed out by Schmitt-Grohé and Uribe (2004) and Preston and Roca (2007) when using a second-order perturbation approximation the presence of uncertainty affects the constant terms of the decision rules. This requires a risk correction to the mean of individual and aggregate variables which is computed according to Schmitt-Grohé and Uribe (2004) and adapting to our case their Matlab routines. We also use their routines to compute all other coefficients of our second-order approximation. All simulations and impulse response analysis conducted in the text take into account such a correction in the mean.

Appendix C Errors in the Euler equations

We measure the quality of a candidate solution by computing the mean, the max and the standard deviation of the errors in the Euler Equations. If the economic significance of these errors is small, we accept the solution. We test the accuracy of the approximation solution obtained from the second order iteration procedure according to Judd (1998): errors in the Euler equations of agent's consumption and labor decisions are constructed as unit-free measurement by a division of the marginal utility of consumption. For the optimal pricing equation of the firms, we do not measure it in units of consumption, since price ratio already provides a comparison baseline of 1.

$$\varepsilon_\tau^C \equiv \left[\tilde{\tau}_b\left(b_\tau^j\right) + \left(C_\tau^j\right)^{-\sigma}\frac{1}{1+r_\tau} - \beta\,\mathbb{E}_\tau^j\left[\left(C_{\tau+1}^j\right)^{-\sigma}\frac{1}{1+\pi_{\tau+1}}\right]\right] \Big/ \left(C_\tau^j\right)^{-\sigma}; \tag{C-1}$$

$$\varepsilon_\tau^\ell \equiv \left[\left(C_\tau^j\right)^{-\sigma} w_\tau - \left(L_\tau^j\right)^\eta\right] \Big/ \left(C_\tau^j\right)^{-\sigma}; \tag{C-2}$$

$$\varepsilon_\tau^q \equiv q_{j\tau} - \frac{\theta}{\theta-1}\frac{S_{j,\tau}}{V_{j,\tau}}e^{\nu_t}. \tag{C-3}$$

For each point, we need to compute the expectations in (C-1) to measure the errors. We do this by using the monomial integration rule with $2N^2+1$ nodes according to Judd (1998) and the Matlab code provided by Judd et al. (2011). Here, we adapt it to the case of correlated random variables using the change of variables via Cholesky decomposition. Since, according to Assumption 3, (Υ, ε) are the primitive shocks following normal distributions that are used to construct ρ^{jg} and the stochastic component of $\sigma^2_{cs}(g)_t$, we use the perception model with random normal variables $(\rho^{j\zeta}, \rho^{j\nu}, \rho^{jZ}, \Upsilon, \varepsilon)$ to measure the errors in the Euler equations. Note that expectations are formed for each agent j. The variance-covariance matrix of the perception model is specified through the rationality constraints discussed in Appendix A and by the conditions explained in footnote 8.

In our simulations across the policy space and across **Cases** we find that equation (C-2) is uniformly satisfied with error $\varepsilon^{\ell}_{\tau} \approx 10^{-15}$; excluding points which are singular or close to singularity, the average error over the simulated path in equation (C-1) is $10^{-3} \leq \varepsilon^{C}_{\tau} \leq 10^{-4}$ and the average error over the simulated path in equation (C-3) is $10^{-4} \leq \varepsilon^{q}_{\tau} \leq 10^{-5}$.