Risk Aversion and Uncertainty in European Sovereign Bond markets[1]

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Abstract: Risk aversion and uncertainty are often both at play in market price determination, but it is empirically challenging to disentangle one from the other. In this paper we set up a theoretical model particularly suited for opaque over-the-counter markets that is shown to be empirically tractable. Based on high frequency data, we thus propose an evaluation of risk aversion and uncertainty inherent to the government bond markets in the euro area between 2007 and 2011. We particularly examine the impact of the European Central Bank Security Market Programme [SMP] implemented in May 2010 to ease the pressure on the European sovereign bond markets. We show how this programme has killed market uncertainty but raised risk aversion for all countries except Greece in a risk-pooling mechanism: this can therefore weaken the impact of market interventions over the long-term.

Key Words: Risk Aversion, Uncertainty, government bond market, Euro area.

Subject Classification: D40, D81, E58

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1. INTRODUCTION

European government bond markets in 2010 triggered interest of investors, and public authorities following the budget difficulties encountered by Greece and the threat to see this crisis spread all over Europe. The budgetary problems revealed by the end of 2009 by the Greek government launched a wave of hostility and mistrust of market participants. Unprecedented in the Euro area, the difficulties faced by Greece forced the European authorities to step in the market to restore market efficiency, liquidity and decrease upward tensions over bond rates. Indeed, the solvency of a state is based on the perception by market players of the quality of the traded bonds. In this sense, the most fragile countries are exposed to an increase in their probability of default given the increasingly degraded market conditions. As a result, other European countries started experiencing high credit rating discrimination, and therefore market tensions.

A central bank can intervene on the sovereign bond market to improve the transmission of the monetary policy signal along the yield curve. These interventions need a thorough understanding of the market microstructure in order to come to terms why market mechanisms may have adverse effects on price determination, or what behaviors may generate market freezes and jumps in prices. In addition, policy makers need to identify what the most appropriate actions to be taken to restore market functioning are, when the market tends not to reflect economic fundamentals anymore. The crisis has put some crucial questions to the fore, reflecting many a-priori we had before seeing this heterogeneity in sovereign bonds. Indeed, contrary to what it has been generally assumed so far, sovereign bonds seem not to be immune against liquidity scarcity. Defining the measures that can guarantee market efficiency is of primary importance for policy makers and especially how market makers use information and ensure market liquidity on these key markets.

As a consequence of the sovereign bond crisis, the Euro area countries collectively announced an intervention programme on Sunday the 9^{th} of May 2010 which consisted in two main measures. First, Euro area countries implemented the European Financial Stability Facility [EFSF]. This institution is composed of every Euro area countries and aims at providing financial support to countries in weak economic situations. The EFSF can raise funds by issuing bonds in the name of the Euro area. Second, the European Central Bank announced at the same time the beginning of market interventions on government bond segments to restore liquidity and to keep under control the efficient functioning of the market. This coupled announcements in a context of high tensions, appeared as a strong market commitment, and modified market participants behaviors. However, its specific impact on market efficiency, uncertainty, or risk aversion has not been explicitly analyzed in the literature.

To this prospect, we focus on the two key concepts that are risk aversion and uncertainty in the European government bond market. To adopt Knight (1921) view on risk and uncertainty, we define risk aversion as the fact that agents are reluctant to face choices with several probable outcomes. On the other hand, uncertainty (or ambiguity) would precise how numerous and probable are these different outcomes. In other words, the level of uncertainty is the degree of measurability of the possible outcomes when agents face risk. This explicit distinction between risk and uncertainty appears, in particular, in Gilboa and Schmeidler (1989) showing how decision makers may be influenced by some specific outcomes in a given set (e.g. the outcome with the lowest expected utility even though its probability is low). Going further, Klibanoff *et al.* (2005) propose a separation between ambiguity, embodied by agent's subjective beliefs about future outcomes and ambiguity aversion, which is reflected in her preferences.

In the microstructure analysis, Illeditsch (2009) or Easley and O'Hara (2010) provide key models concerning uncertainty and how this dimension may influence market dynamics. The fact that investors perceive uncertainty is that they consider not one but several possible distributions for the value of the traded asset. In this paper, we consider that this phenomenon may be especially exacerbated during crises when market makers face a lot of contradictory information, and may become reluctant to propose narrow bid-ask spreads in order to protect themselves from losses.

In this strand, we consider Biais (1993) model particularly suited for opaque and over the counter markets. This model framework particularly fits with the Euro area government bond secondary markets, for which scarce information on the best available prices is only displayed by screen process. We extend the model by introducing uncertainty on the shocks affecting the value of the asset, and show how this may directly influence the size of the bid-ask spreads given the risk aversion of the marginal market maker. We show the existence, under weak conditions, of a unique analytical solution for uncertainty and risk aversion, that is time-varying by exploiting volatilities of the bid and ask prices coupled with their durations. In particular, in the spirit of Domowitz and Wang (1994), we determine the bid-ask spread distribution in the context of our model and demonstrate that it can be well modeled by a non centered chi-square distribution whose moments depend on risk aversion, uncertainty and durations between quote revisions. This complements some other papers interested in bond market liquidity as Fleming (2003), Krishnamurthy (2002) or Goldreich et al. (2005) for the US or Dunne *et al.* (2007) for European countries.

This model extension is done for several objectives: (i) to propose a theoretical model that allows for empirical tractability of market makers' risk aversion and uncertainty on the government bond markets; (ii) to analyze the time series of the risk aversion and uncertainty measures provided by our model; (iii) to propose a reasonable family of distributions for the instantaneous quoted bid-ask spread as a function of risk aversion and uncertainty; (iv) and finally analyze the effect of the 9th of May 2010 announcement in the Euro Area government bond markets.

The paper is organized as follows. We first present our theoretical model with risk averse market makers facing uncertainty concerning the value of the asset. In a second section, we derive the properties of our model and we show, under weak conditions, how this model is empirically tractable. In particular, we derive the distribution properties of the instantaneous bid-ask spread. In section 3, we apply the model to France, Germany, Spain, Greece, Italy and Portugal government bond markets for three maturities (2,5 and 10 years) between 2007-2011 using high frequency data. We particularly focus on the effects of the Securities Market Programme. Section 4 concludes.

2. A MODEL WITH RISK AVERSION AND UNCERTAINTY

2.1. Model structure

2.1.1. Market maker population

Let consider a population of N market makers perfectly competing for quotes in a single-asset market. This market is an over-the-counter [OTC] market characterized by its opacity and the fact that transaction details are not reported. However, there is a screening process of quotes to the extent that agents may observe the historical data for the best bid (b_t^*) and ask (a_t^*) quotes, in real time. This situation prevails in the European government bond markets for example. Each market maker is indexed by $i = 1, ..., N$, and each new quote arrival on the public screen is indexed by $t = 1, ..., T$. Note that we develop the model on an irregularly spaced timeline since t is not the calendar time but the quotes arrivals. N is assumed to be reasonably large and determined exogenously, especially when one considers a market such as the sovereign bond market. Liquidity is provided by dealers who declare themselves ready to trade at their bid and ask prices when they appear on the screen.

Each market maker, indexed by i, has a utility function with constant absolute risk aversion A_i

as

$$U_i(x_t) = -e^{-A_i.x_{i,t}},\ \forall x_{i,t} \tag{1}$$

Note that for each agent, A_i is constant over time (at least is considered to be constant from a high frequency point of view), and $x_{i,t}$ denotes the time-varying wealth of agent i. Each market maker is endowed with cash $C_{i,t}$ that is net off a fixed cost related to market participation. Each agent has a current position in the traded asset $I_{i,t}$ such that the agent is long if $I_{i,t} > 0$ and short if $I_{i,t} < 0$. The final wealth of agent i at date t is

$$W_{i,t} = C_{i,t} + v_t I_{i,t} \tag{2}$$

with v_t being the value of the asset revealed by the market.

2.1.2. The asset

In this model, we consider a single asset market. This asset has a value v_t revealed by the market but is imperfectly known by agents. In addition, at each point in time, there is not necessarily a new quote arrival so that information blurs over time. In other words, the larger the time between two quote arrivals on the screen, the more uncertain each agent is about v_t, and higher potential for private information to exist.

Let consider z_t as the shock between the $t-1$ and t quote arrivals for the value v_t revealed by the market as

$$z_t = v_t - v_{t-1} \tag{3}$$

with z_t following a Gaussian distribution with zero mean and variance $\sigma^2_{z_t}$. We assume $\sigma^2_{z_t}$ to be directly proportional to the observed durations between two quote issuances similar in the empirical literature with an Autoregressive conditional duration model (ACD) as in Engle and Russel (1998). Even if z_t is never observed, agents build some expectations for $\sigma^2_{z_t}$ based on the expectations of the variances ($\tilde{\sigma}^2_{a^*_t}$ and $\tilde{\sigma}^2_{b^*_t}$) and covariance ($\tilde{\sigma}_{a^*_t b^*_t}$) for a^*_t and b^*_t the ask and bid prices that appear on screens:

$$\sigma^2_{z_t} = \pi_t^2 \tilde{\sigma}^2_{a^*_t} + (1-\pi_t)^2 \tilde{\sigma}^2_{b^*_t} + \pi_t(1-\pi_t)\tilde{\sigma}_{a^*_t b^*_t}$$

The parameter π_t is assumed to be the weight attributed to each price variation (at the bid and at the ask prices) and is defined as

$$\pi_t \sim N(\frac{1}{2}, \sigma^2_{\pi_t}) \tag{4}$$

The intuition behind this formula is the following: from only bid and ask prices appearing on the screen, it is difficult for market makers to accurately infer the fundamental value of the asset and therefore its variance. As a result, they anticipate what will be the variance of the shock based on what they know at time $t-1$, i.e. the past variances of bid and ask prices. Often, in the market microstructure literature, asset fundamental value is considered to coincide with the midquote offered. In our model, this hypothesis is not needed and we can therefore take into account divergences from the midquote for the fundamental value, given the uncertainty that we introduce at this stage of the model.

Indeed, $\sigma^2_{\pi_t}$ reflects, in this setup, the uncertainty surrounding the value of the asset v_t revealed by the market at transaction t. $\sigma^2_{\pi_t}$ gives the length of the market state spectrum, given the values that π_t can reach: in other words the set of distributions that agents may expect for the shock on the value of the asset v_t. For example, if we assume the absence of uncertainty so that $\sigma^2_{\pi_t} = 0$, finally $\pi_t = \frac{1}{2}$ and we obtain that $\sigma^2_{z_t} = var(\Delta M^*_{t-1})$ the volatility of the midquote variations: in this case the variations of the midquote reveals the variations of the asset value. However, the larger $\sigma^2_{\pi_t}$, the more uncertainty we have in the market.

Moreover, the role of market price volatility is crucial in our model since it directly influences the size of the spread and the ability of buyer and sellers to meet. The market characteristics considered in our paper, are close to the ones used in Duffie, Gârleanu and Pedersen (2005) to show how risk aversion and volatility have a price impact in OTC markets, even if we do not consider here any model of bargaining process.

2.1.3. Information set and quote sequences

As already mentioned, the only piece of information agents have about the traded asset is the one given by a centralized screen. We denote Ω_t the information set a time t that is common knowledge to any market maker as

$$\Omega_t = \{(a^*_u, b^*_u, \tau^*_u)\}_{u=1\ldots t} \tag{5}$$

which comprises the past observations for declared best ask and bid prices, and the past delays in time unit between two quote arrivals τ^*. In this setup, the time between two quote revisions is crucial. For example, consider an agent at time $t-1$ who determines her optimal levels of bid and ask prices for transaction t, she knows that the longer she waits, the higher the uncertainty about v_t is. As a consequence to be the one appearing on the screen at t, she needs to have the best expected bid-ask spread and needs to be the fastest. This appears in the next section when, given this setup, we derive the optimal level for the bid and ask prices.

2.2. Reservation quotes and quote revisions

2.2.1. Market expectations

Before deriving their optimal quotes, all agents build some market expectations given the available information at date $t-1$, Ω_{t-1}. The first quantity of interest is the instantaneous volatility of the ask price

$$E(\sigma^2_{a^*_t} \,|\Omega_{t-1}) = \sigma^2_{a^*_{t-1}} \tag{6}$$

where both expectations for the future instantaneous volatility and the past observed instantaneous volatility under the hypothesis of rational expectations are considered. In the empirical section, we explain in more details the estimation process of the instantaneous variances and covariance between the ask and the bid prices. Similarly, the expected instantaneous variance of the bid price and the expected instantaneous covariance of the bid and ask prices are given by

$$E(\sigma^2_{b^*_t} \,|\Omega_{t-1}) = \sigma^2_{b^*_{t-1}}. \tag{7}$$

and

$$E(\sigma_{a^*_t b^*_t} \,|\Omega_{t-1}) = \sigma_{a^*_{t-1} b^*_{t-1}}. \tag{8}$$

All these quantities are elements of the instantaneous variance covariance matrix for the ask and bid prices as

$$E(\Sigma_t \,|\Omega_{t-1}) = \Sigma_{t-1} = \begin{bmatrix} \sigma^2_{a^*_{t-1}} & \sigma_{a^*_{t-1} b^*_{t-1}} \\ \sigma_{a^*_{t-1} b^*_{t-1}} & \sigma^2_{b^*_{t-1}} \end{bmatrix} \tag{9}$$

Given this evaluation of market instantaneous variances and covariances, agents infer an implicit measure of the variations of v_t as a function of π_t such that

$$E(\sigma^2_{z_t} \,|\Omega_{t-1}, \pi_t) = \tilde{\sigma}^2_{z_t} = \left[\pi_t^2 \sigma^2_{a^*_{t-1}} + (1-\pi_t)^2 \sigma^2_{b^*_{t-1}} + \pi_t(1-\pi_t)\sigma_{a^*_{t-1} b^*_{t-1}}\right] \tilde{\tau}_t \tag{10}$$

Variances are increasing in durations as it is usually considered in the empirical literature of ACD models. In our theoretical framework, this is related to the price for immediacy as in Chacko *et al.* (2008) such that impatient market makers have to propose faster and narrower spreads than competitors to appear on the screen and this is a cost related to their impatience.

2.2.2. Reservation quotes

Reservation quotes are the bid and ask prices that make the market maker indifferent. Given the market expectations previously derived, each agent computes her reservation quotes such that she is indifferent between potentially buying the underlying asset at the bid reservation quote, selling at the ask reservation quote and doing nothing. These reservation quotes are not the optimal ones in a sense that they do not maximize the surplus in wealth that they could expect from trading with the public. However, in the case of perfect competition as it is in our case, the expected surplus is null. In other words, the aim of posting some quotes on the market is not perceived as an effective search for surplus, but more for reputation purposes. Indeed, appearing on the screen for market makers is important to signal to other market participants their presence on the trading of the asset even if this signal should not expose the market maker to excessive risk (inventory risk in our case) once her quotes are displayed.

At time $t-1$ each dealer is endowed with cash $C_{i,t-1}$ and has a current position in the traded asset $I_{i,t-1}$. In the event that she does not quote prices, her final wealth when a new quote is posted in t by a competitor is supposed to be $W_{i,t}(0)$ as:

$$W_{i,t}(0) = C_{i,t-1} + I_{i,t-1} v_t \tag{11}$$

Alternatively, facing a market buy order for a quantity Q_t at the ask price gives a final wealth of $W_{i,t}(a_{i,t})$ as:

$$W_{i,t}(a_{i,t}) = C_{i,t-1} + I_{i,t-1} v_t + (a_{i,t} - v_t) Q_t \tag{12}$$

On the contrary, if the dealer i buys a quantity Q_t at price $b_{i,t}$, her final wealth is $W_{i,t}(b_{i,t})$:

$$W_{i,t}(b_{i,t}) = C_{i,t-1} + I_{i,t-1}v_t + (v_t - b_{i,t})Q_t \tag{13}$$

The ask reservation quote and the bid reservation quote for each agent i at date t are denoted $\bar{a}_{i,t}$ and $\bar{b}_{i,t}$ respectively such that

$$E[U(W_{i,t}(0))\,|\Omega_{t-1}, \pi_t\,] = E[U(W_{i,t}(\bar{a}_{i,t}))\,|\Omega_{t-1}, \pi_t\,] = E[U(W_{i,t}(\bar{b}_{i,t}))\,|\Omega_{t-1}, \pi_t\,]. \tag{14}$$

Given our setup, and close to the Biais (1993) model, we obtain that

$$\bar{a}_{i,t} = \frac{A_i}{2}(Q_t - 2I_{i,t-1})E(\sigma^2_{z_t}\,|\Omega_{t-1}, \pi_t\,) + v_{t-1} \tag{15}$$

$$\bar{b}_{i,t} = -\frac{A_i}{2}(Q_t + 2I_{i,t-1})E(\sigma^2_{z_t}\,|\Omega_{t-1}, \pi_t\,) + v_{t-1} \tag{16}$$

And the reservation bid-ask spread is equal to:

$$\bar{S}_{i,t} = \bar{a}_{i,t} - \bar{b}_{i,t} = A_i Q_t E(\sigma^2_{z_t}\,|\Omega_{t-1}, \pi_t\,) \tag{17}$$

Even if we do not have any information concerning the level of v_{t-1}, the fundamental value of the asset, the spread reflects both the risk aversion coefficient and the expected instability of the asset value. An increase in the risk aversion component A_i tends to shift up the reservation ask price and exerts downward pressure on the bid price, which widens the reservation spread. Moreover, the higher $E(\sigma^2_{z_t}\,|\Omega_{t-1}, \pi_t\,)$, the larger the reservation spread. Such a phenomenon is quite intuitive since the dealer is willing to protect herself from significant variations of the asset's value by quoting a large spread.

Assuming perfect competition in the market for posting quotes, the smallest and fastest spread appears on the screen so that

$$S^*_t\,|\Omega_{t-1}, \pi_t\; = \min(\bar{S}_{i,t}, i = 1...N) \tag{18}$$

such that

$$S_t^* \,|\Omega_{t-1}, \pi_t \, = A^* Q_t^* \tilde{\tau}_t \left[\pi_t^2 \sigma^2_{a^*_{t-1}} + (1-\pi_t)^2 \sigma^2_{b^*_{t-1}} + 2\pi_t(1-\pi_t)\sigma_{a^*_{t-1} b^*_{t-1}} \right] |\Omega_{t-1}, \pi_t \; . \tag{19}$$

Assuming that $\pi_t \sim N(\frac{1}{2}, \sigma^2_{\pi_t})$, we can rewrite the expected spread only conditional on Ω_t as

$$E(S_t^* \,|\Omega_{t-1}\,) = A^* Q_t^* \tilde{\tau}_t \left[\sigma^2_{\pi_t} (\sigma^2_{a^*_{t-1}} + \sigma^2_{b^*_{t-1}} - 2\sigma_{a^*_{t-1} b^*_{t-1}}) + Var(\Delta M^*_{t-1} \,|\Omega_{t-1}\,) \right] \tag{20}$$

where $Var(\Delta M^*_{t-1})$ is the variance of the previous midquote. Equation (20) directly illustrates the impact of risk aversion and uncertainty on the expectation of the optimal spread for next quote update and thus on market liquidity. If higher risk aversion leads to a larger bid-ask spread, stronger uncertainty (large $\sigma^2_{\pi_t}$) also widens the expected best spread appearing on the screen at quotation t. The linear impact of $\tilde{\tau}_t$, the duration, on expected quoted spreads is related, as we said before, to some price for immediacy as in Chacko *et al.* (2008) but also on the fact that the spread is increasing given that there is no quote revision: as long as agents do not have enough valuable information to revise their quotes, they stay out of the market, bid-ask spreads increase and liquidity becomes scarce.

Note that in our model, if agents are not explicitly uncertainty averse, as it is in Easley and O'Hara (2010) for instance with the maximization of the minimum of the market maker utility, they are however reluctant to quote narrow spreads in case of high uncertainty. In the decision taking theory, there is not a clear consensus regarding this issue of disentangling uncertainty aversion from uncertainty itself as by definition these two components are intrinsically linked. Indeed, a change in the distribution set considered for the fundamental value of an asset has an impact on the degree of uncertainty and *de facto* on the decision taking process. In our setup, agents face uncertainty as given, and this appears in the spread because they are both risk and uncertainty averse. As a consequence, if A^*, the risk aversion of the marginal market maker is nought the spread cancels even if there is uncertainty in the market which is quite a reasonable result. However, we can have some situations of zero uncertainty, which does not mean that the spread is zero but just that the midquote of the spread is truly revealing the variations of the asset value.

At first glance, this spread equation shows that several combinations of risk aversion and uncertainty may lead to the same level of spread even if the market scenarii can be quite different. For example, considering a low level of risk aversion, with high uncertainty may lead to the same spread as high risk aversion with low uncertainty. However, these two situations are not comparable at all.

In the next section, we show how this model is empirically tractable and discuss the weak conditions for a unique solution $(A^*, \sigma_\pi^2)_t$ characterizing both risk aversion and uncertainty, but disentangling the two different concepts for the best spreads appearing, in real-time, on the quote-screen.

3. EMPIRICAL TRACTABILITY OF THE THEORETICAL MODEL

In this section, we first show that we can obtain under weak conditions an unique analytical solution for risk aversion and uncertainty at any point in time and then provide some empirical specifications for the requested quantities to derive $(A^*, \sigma_\pi^2)_t$.

3.1. Uncertainty and risk aversion mapping

As seen before in equation (20), the expected spread conditional on Ω_{t-1} can be written

$$E(S_t^* \,|\Omega_{t-1}) = A^* Q_t^* \tilde{\tau}_t \left[\sigma_{\pi_t}^2 (\sigma_{a_{t-1}^*}^2 + \sigma_{b_{t-1}^*}^2 - 2\sigma_{a_{t-1}^* b_{t-1}^*}) + \frac{1}{4}(\sigma_{a_{t-1}^*}^2 + \sigma_{b_{t-1}^*}^2 + 2\sigma_{a_{t-1}^* b_{t-1}^*})\right]. \quad (21)$$

or

$$E(S_t^* \,|\Omega_{t-1}) = A^* Q_t^* \tilde{\tau}_t \left[\sigma_{\pi_t}^2 Var(\Delta(a_{t-1}^* - b_{t-1}^*) \,|\Omega_{t-1}) + Var(\frac{\Delta(a_{t-1}^* + b_{t-1}^*)}{2} \,|\Omega_{t-1})\right]. \quad (22)$$

For the moment, we exclude the quantity effect. By this assumption, we assume that for a given bond the implicit quantity for which bid and ask prices are quoted by market makers is standardized to a pre-specified level. The volatility of the expected optimal spread is

$$Var(S_t^* \,|\Omega_{t-1}) = A^{*2} \tilde{\tau}_t^2 \left[2 \left[Var(\Delta(a_{t-1}^* - b_{t-1}^*) \,|\Omega_{t-1})\right]^2 \sigma_{\pi_t}^4 + (\sigma_{a_{t-1}^*}^2 - \sigma_{b_{t-1}^*}^2)^2 \sigma_{\pi_t}^2\right]. \quad (23)$$

Proposition 1. *Given equations (21) and (23), there exists a unique one-to-one mapping* (A^*, σ_π^2) *at any quote update* t *if*

$$Var(\Delta(a_{t-1}^* - b_{t-1}^*) \,|\Omega_{t-1}) = 0 \quad (24)$$

or if

$$Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1}\,) \neq 0 \;\; and \;\; \frac{Var(S^*_t\,|\Omega_{t-1}\,)}{[E(S^*_t\,|\Omega_{t-1}\,)]^2} < 2$$

so that the model is empirically tractable for any spread distribution whose coefficient of variation is lower than $\sqrt{2}$. *Proof of proposition 1 is reported in Appendix A.*

From the proposition stated above, we can derive another proposition.

PROPOSITION 2. *Assuming that* $\pi_t \sim N(\frac{1}{2}, \sigma^2_{\pi_t})$ *and* $Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1}\,) = 0$, *then the distribution of the spread* S_t *conditional on* Ω_{t-1} *is a Gaussian distribution with mean* μ_{S_t} *and variance* $\sigma^2_{S_t}$ *equal to*

$$\mu_{S_t} = A^*\tilde{\tau}_t\sigma^2_{a^*_{t-1}b^*_{t-1}} \tag{25}$$

$$\sigma^2_{S_t} = A^{*2}\tilde{\tau}^2_t(\sigma^2_{a^*_{t-1}} - \sigma^2_{b^*_{t-1}})^2\sigma^2_{\pi_t} \tag{26}$$

If $Var(\Delta(a^*_{t-1} - b^*_{t-1})) \neq 0$, *its distribution is a non centered and non reduced* χ^2 *distribution with one degree of freedom and with a non centrality parameter* $\lambda_t = \frac{1}{4}\left(\frac{\sigma^2_{a^*_{t-1}} - \sigma^2_{b^*_{t-1}}}{Var(\Delta(a^*_{t-1} - b^*_{t-1})|\Omega_{t-1}\,)}\right)^2$ *such that the mean* μ_{S_t} *and variance* $\sigma^2_{S_t}$ *are*

$$\mu_{S_t} = A^*\tilde{\tau}_t\left[Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1}\,)(\sigma^2_{\pi_t} + \lambda_t) + \sigma^2_{b^*_{t-1}} - (\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})^2\right] \tag{27}$$

and

$$\sigma^2_{S_t} = 2A^{*2}\tilde{\tau}^2_t Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1}\,)^2\left[\sigma^4_{\pi_t} + 2\lambda_t\sigma^2_{\pi_t}\right]. \tag{28}$$

It follows that the probability density function $f_{S_t}(s)$ *is,* $\forall s \in \mathbb{R}_+$,

$$f_{S_t}(s) \;=\; \frac{1}{A^*\tilde{\tau}_t Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1}\,)\sigma^2_{\pi_t}} \times \tag{29}$$

$$h^1_{\chi^2_{nc}}\left(\frac{Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1}\,)s + A^*\tilde{\tau}_t(\sigma^2_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{a^*_{t-1}}\sigma^2_{b^*_{t-1}})}{A^*\tilde{\tau}_t Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1}\,)^2\sigma^2_{\pi_t}}\right) \tag{30}$$

where $h^1_{\chi^2_{nc}}$ *is the probability density function of a non centered chi-square distribution with one degree of freedom and a non centrality parameter* λ_t. *Proof of proposition 2 is reported in Appendix B.*

In our theoretical framework, we model the distribution of the instantaneous bid-ask spread at each new quote arrival. However, we should keep in mind that our model takes into account both bid-ask spread in itself and durations. These two liquidity components are intrinsically linked and

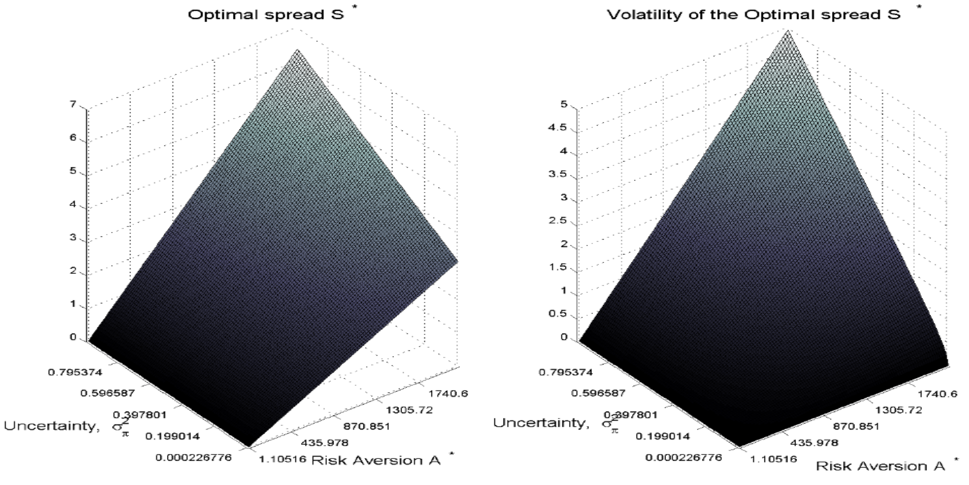

FIG. 1 First and second moments of the optimal spread for several levels of uncertainty and risk aversion.

both have an impact on the price discovery process: as explained before, a larger expected duration will increase market makers' expectations in terms of the volatility of the asset value revealed by the market, leading dealers to widen their spreads. Therefore, this time dimension has a sizeable effect on the expected optimal quoted bid-ask spread.

From the previous results, we can notice that both the expectation and the variance of the optimal bid-ask spread increase with risk aversion and uncertainty. They are also convex functions in the uncertainty. All these theoretical findings show us as expected that these two components have a tendency to widen the expected bid-ask spread but also the spectrum of possible outcomes associated. For illustration purposes, the following figure shows the effect of uncertainty and risk aversion on the level and volatility of the optimal spread.

As shown in Figure 1, the uncertainty starts having an effect on the spread, once the risk aversion increases. The effect of uncertainty is even stronger, when risk aversion increases to high levels. In other words, uncertainty does not influence market liquidity when agents are not highly risk averse. However, in period of crisis with risk averse market makers, the uncertainty creates an amplification phenomenon by deteriorating further market liquidity conditions, and rise spread volatility.

To determine the risk aversion, the uncertainty and then the distribution of the expected optimal spreads at any point in time we apply a two-step procedure. First we estimate $\tilde{\Sigma}_t$ given our specification of equation (31). Then by using the two first moments of the spread distribution and solving the quadratic expression derived in Appendix A at each point in time, we obtain $(A^*, \sigma^2_\pi)_t$.

3.2. Variance, covariances and durations

We first consider a set of model specifications to extract market instantaneous volatilities $\sigma^2_{a^*_t}$, $\sigma^2_{b^*_t}$, and $\sigma_{a^*_t b^*_t}$. Our model presents two variables of interest, the ask and the bid prices observed at each quote issuance. In the present paper, a non parametric approach is adopted to facilitate the empirical tractability of the model over time.

We fairly assume that market makers anticipate what future variances will be, based on the observations of the previous price variations. The non parametric specification retained here is rather simple and is based on the empirical variances and covariances. Therefore, the bid and ask prices vector, $(\Delta a_t, \Delta b_t)$, has a variance-covariance matrix according to the following formula :

$$\Sigma_t = \begin{pmatrix} \sigma^2_{a^*_t} & \sigma_{a^*_t b^*_t} \\ \sigma_{a^*_t b^*_t} & \sigma^2_{b^*_t} \end{pmatrix} \tag{31}$$

with

$$\sigma^2_{a^*_t} = \frac{1}{N-1} \sum_{j=1}^{J} \frac{(\Delta a_{t-j})^2}{\tau_{t-j}} \tag{32}$$

$$\sigma^2_{b^*_t} = \frac{1}{N-1} \sum_{j=1}^{J} \frac{\left(\Delta b_{t-j}\right)^2}{\tau_{t-j}} \tag{33}$$

$$\sigma_{a^*_t b^*_t} = \frac{1}{N-1} \sum_{j=1}^{J} \frac{(\Delta a_{t-j})(\Delta b_{t-j})}{\tau_{t-j}} \tag{34}$$

where N is the number of lag price variations considered by the market maker to infer volatility. This gives our instantaneous $t-$variance covariance matrix. We also need an expected duration $\tilde{\tau}_t$ to extract (A^*, σ^2_π) at each quote arrival. Durations are assumed to follow an $AR(p)$ process such that we have some clusters in durations, with the alternation of periods of high frequency revisions with low frequency revisions.

4. EMPIRICAL APPLICATION

4.1. Dataset and Stylized facts on the euro government bond market

To derive the historical paths of our measures of risk aversion and uncertainty, we use high frequency data provided by Thomson Reuters Tick History. We mainly focus our analysis on 6 European government bonds (France, Germany, Greece, Italy, Portugal, Spain) for three maturities (2, 5 and 10 years). The data cover the period from January 4^{th}, 2007 to May 10^{th}, 2011. The Thomson Reuters Tick History database contains date-and-time stamped bid-ask quotes. At each new quote issuance, the best ask (bid) price, which appears on the screen corresponds to the lowest (highest) price offered by market-makers. Therefore, having only access to best prices leads us to call upon the previous model based on a competitive market.

Before trying to extract relevant information from this dataset, we conduct some prior data processing in order to remove non-valid quotations. Observations whose either ask or bid price is equal to zero, bid-ask spread is negative or explosive due to reporting errors are deleted. Moreover, we reduce our sample by only taking quotes which are issued between 8:00 and 18:00 GMT. Indeed, even if government bonds are traded on over-the-counter markets and therefore transactions of such assets may take place all day long, numbers of observed quote issuances are only significant when European markets are open. We can reasonably assume that quotes issued for European bonds on the American or Asian markets, which are relatively scarce, would add no relevant value and could only disturb the running of our analysis. Finally, we remove days which count less than two hundred observations as they could lead to misleading findings.

Table 1 reports some descriptive statistics from the raw data of the different government bonds studied. Major differences appear between mean and median statistics, which indicates the presence of extreme values (confirmed by large standard deviations) for both bid-ask spreads and durations. This confirms that sovereign bond markets are marked by episodes of high volatilities that can result from uncertainty and/or risk aversion. We can notice some heterogeneity across countries and maturities for the quoted bid-ask spreads. In particular, countries recently under stress in the context of the sovereign bond crisis have unsurprisingly recorded larger than usual bid-ask spreads. This feature is clear looking at mean bid-ask spreads but also in the distributions presented in the appendix C. The different humps observed may indicate the presence of time-varying distributions. Durations also show significant differences. Quotes for the French and German bonds are globally revised less frequently than the ones of other countries (see Figures 2 and 3 in the Appendix C).

Table 1: Descriptive statistics of observed bid-ask spreads and durations (time between two consecutive quote issuances)

Notes: The table reports summary statistics for the different European government bonds studied. From high frequency data, mean, median and standard deviation are computed for the whole period considered (from the start of the sample to May 2011).

	Spread			**Duration (in seconds)**		
	Mean	Median	Standard Deviation	Mean	Median	Standard Deviation
DE2YTRR	0,042	0,04	0,084	33,74	20,91	67,11
DE5YTRR	0,052	0,04	0,069	17,68	13,85	36,06
DE10YTRR	0,051	0,04	0,096	15,76	13,09	32,61
FR2YTRR	0,05	0,03	0,06	52,60	41,27	86,49
FR5YTRR	0,059	0,03	0,077	51,32	40,36	74,84
FR10YTRR	0,094	0,05	0,148	35,18	33,04	45,82
ES2YTRR	0,104	0,04	0,146	14,04	9,06	32,80
ES5YTRR	0,116	0,04	0,172	12,23	8,07	26,96
ES10YTRR	0,135	0,05	0,207	11,21	8,06	25,12
IT2YTRR	0,079	0,04	0,01	16,64	9,21	41,74
IT5YTRR	0,089	0,05	0,115	14,79	9,73	29,16
IT10YTRR	0,093	0,05	0,119	13,85	9,93	25,78
PT2YTRR	0,173	0,05	0,324	11,03	7,06	29,14
PT5YTRR	0,225	0,06	0,437	11,35	7,75	27,13
PT10YTRR	0,299	0,05	0,576	13,73	9,94	25,79
GR2YTRR	0,22	0,04	0,369	17,60	10,84	38,09
GR5YTRR	0,416	0,05	0,839	19,09	10,08	48,96
GR10YTRR	0,499	0,06	0,936	17,03	11,67	36,88

One first striking feature is that we detect an impact of maturities on spreads and durations as in Elton and Green (1998). We particularly find that spreads increase with maturities whereas durations decrease with maturities. Such a phenomenon seems to be rather counterintuitive. Indeed, both bid-ask spreads and durations between two quote issuances can be seen as liquidity indicators. They should both rise when market conditions deteriorate: when an asset is assumed less liquid, risk averse and uncertain market makers widen the spread offered by way of hedging and transactions (and therefore quote revisions) occur less often. Nevertheless, an other explanation can be brought to the fore: presence of smaller durations does not mean that the asset in question is traded more frequently. This can reflect a high uncertainty and/or risk aversion environment in which market makers revise their quotes more often in order to protect themselves from potential large price variations or informed traders.

We display in Appendix C the historical evolutions of the daily median bid-ask spreads (Figure 4). Even if we observe some heterogeneity across countries and maturities regarding quoted spreads, the period before May 2010 was globally marked by a surge in this liquidity indicator and then a significant drop after the intervention of the ECB. This stylized fact is obvious for peripheral countries. We must however point out that spreads of France and Germany have broadened at the outbreak of the crisis and then shrunk maybe due to some *flight-to-quality* effects. Looking at the bid-ask spread volatilities (Figure 5), the past evolutions exhibit two groups of countries. For France and Germany, spread volatilities remain relatively steady over the whole period with some episodes of higher but contained standard deviations in the market. On the other hand, for the second group of countries, volatilities started sharply increasing with the emergence of the European sovereign bond crisis. Suddenly, after April-May 2010, spread volatilities seem to vanish, leading us to wonder whether this drop is due to lower risk aversion or uncertainty.

The historical paths of the daily median durations between two consecutive quote issuances are also provided in appendix C (Figure 6). Similar comments that the ones mentioned about the quoted bid-ask spreads can be made. In addition, for the very recent period and for all maturities, durations reached unprecedented low levels especially for Spain, Italy, Portugal, Greece and to a lesser extent for Germany. We may suppose that in a period of crisis the high level of uncertainty leads market maker to more frequently revise their quotes.

4.2. The European government bond crisis

The model previously presented and its empirical counterpart are applied to the analysis of the SMP of the ECB. To our knowledge, this is the first paper looking at the impact of the SMP in such a way. The graphical results are therefore displayed into two parts, before and after mid-January 2010, i.e. four months before the collective announcement of May the 9th, 2010 and when the first significant signs of the sovereign bond crisis can be perceived.

From equation 21, we directly see that risk aversion is key to the definition of the spread. Indeed, if the A coefficient is nought, then the spread is zero, whatever is the level of uncertainty in the market. On the other side, even if uncertainty is nought, it does not mean that the spread is zero but the market efficiently reveals the asset value at the midquote.

Figure 7 in the appendix D exhibits the evolutions of the daily median of risk aversion and uncertainty for the 2-year maturity. Firstly, we can notice that the variability of these measures are quite high but contained. Market makers' risk aversion towards Germany is the lowest one. For Italy, Greece and Portugal, some spikes in this measure appear during the summer-autumn 2009 until the beginning of 2010. On the other hand, for France, we observe high risk aversion only during summer-autumn 2009. In terms of uncertainty, France and Germany present the lowest levels, which means that the price discovery process is the most efficient for them. Follows Spain, Italy and finally Greece and Portugal are the more uncertain markets. Uncertainty regarding Spain, Italy and Portugal reaches high levels during the summer 2007 whereas the one towards France and Italy only spikes at the beginning of 2008.

We can notice that for Spain, Greece, Italy and Portugal, bid-ask spreads soared during October 2008 and the beginning of 2010. However, the increase during 2008 was smaller than the one observed during the sovereign bond crisis.

In general, for the 5-year and 10-year maturities, in Figures 8 and 9, we observe a significant decrease in risk aversion during October 2008 with the surge of the Lehman Brothers crisis. This remains true for the most of the countries until spring 2009. Most of the countries have taken advantage of some flight-to-quality and liquidity effects. From April 2009, we notice that in Portugal, Greece, Italy and Spain, risk aversion progressively increases without reaching the pre-crisis levels and to finally surge at the beginning of 2010 for Portugal and Greece.

Regarding uncertainty, we globally observe that the 5-year maturity bonds are the ones with the highest coefficients compared to the other maturities. We observe high levels of uncertainty during the 2007 crisis for the 2-year maturity whereas the same phenomenon is noticed for the 5 and 10-year maturities during the 2008 crisis. This indicates that agents have expected some long-lasting effects of the 2008 crisis compared to the 2007 turmoil.

Given the deteriorating situation on the Greek bond market and to annihilate any contagion phenomena to other countries, the Euro area Governing Council opted to implement the Security Market Programme [SMP]. This decision was suddenly taken in the weekend of May 9^{th} 2010 to be effective on Monday 10^{th}. This implied some striking changes in our indicators, reflecting a strong reversal in market makers' sentiment on every bond segment.

Following May 10^{th}, the communication at the Euro area level has killed global uncertainty for Greece, Portugal, Italy and Spain (Figures 10,11 and 12). Going further into the interpretation of these results, the following reasoning seems the most plausible. After the introduction of the SMP, durations, spread volatilities and, to a lesser extent, spread levels dramatically decreased. In our model, for some given risk aversion and uncertainty levels, duration is supposed to have the same impact, in terms of magnitude, on both expected spreads and spread volatilities. Therefore, if risk aversion and uncertainty had remained constant after May 2010, the observed drop in duration should have been coupled with equivalent falls in the expected spread and its volatility. Yet, the amplitudes of the falls are not similar: spread volatilities dropped much more than spreads themselves. This phenomenon reflects the fact that uncertainty has been mainly killed with the SMP. However, market makers, who became more risk averse, still kept on protecting themselves from informed traders by quoting relatively large bid-ask spreads whose size did not vary that much. On the contrary, for France and Germany, uncertainty remains at some similar levels or even rise at some point in time after the implementation of the programme. This is in line with the risk-pooling mechanism at the Euro area level, that is also revealed by the dynamics of the risk aversion measure. Indeed, risk aversion for Portugal, Italy, Spain and France (only for the 2 and 5-year maturities) increased after the announcement of the SMP whereas it decreased for Greece. Consequently, the collective support beyond this mechanism has lowered the threat of dramatic isolated situations in the Euro area.

In this direction, the Securities Market Programme is a success since it guaranteed that market uncertainty would not deteriorate liquidity conditions further. However, these market interventions cannot, stricto sensus, be a substitute for economic fundamental based programmes (as it is implemented at the same time to solve debt management issues) since risk aversion is a key parameter to control. One drawback of market interventions is to make credible and signal the threat of a default risk, so that public intervention may only have a limited impact if the decrease of market uncertainty is lower than the increase in risk aversion. At the extreme, this may even render market interventions useless over the long-term to restore sovereign bond market efficiency.

5. CONCLUSION

Market participants during crises usually face high risk and a lot of information, potentially contradictory, that may complicate the pricing of some assets. In this paper, we introduce, in a model for opaque OTC markets, the uncertainty dimension, that makes market makers reluctant to participate in the price discovery process. Even if participants are not explicitly ambiguity averse we show that the level of uncertainty, coupled with risk aversion, has a direct impact on the bid-ask spreads and thus can deteriorate market liquidity. In particular, the high levels of uncertainty and risk aversion during crises tend to give higher probability to very large bid-ask spreads. Thus, thanks to our model, we are able to extract at each quote arrival instantaneous bid-ask spread whose shape directly depends on the degrees of uncertainty and risk aversion that markets encounter. Beyond the theoretical analysis of uncertainty, the proposed model presents an empirical tractability that allows for the analysis of the euro area bond market crisis. In particular, we empirically analyzed risk aversion and uncertainty on a pool of six countries (France, Germany, Greece, Spain, Italy and Portugal) for three maturities (2, 5 and 10 years) and had a closer look on the impact of announcement of the Securities Market Programme in May 2010.

Our main conclusions are as follows. First, the historical evolutions of our risk aversion and uncertainty measures reveal periods of severe tensions that occurred in the sovereign bond market between 2007 and the beginning of 2011. As expected, they mainly coincide with the subprime crisis, the Lehman Brother's bankruptcy episode and the sovereign debt episode. Our model allows us to have a better understanding of what happened in these markets in terms of liquidity. For instance, during the summer 2007, market makers quoting short-term maturity European government bonds were both risk averse and uncertain regarding the outcome of the financial turmoil. On the other hand, medium and long-term maturities seem to have not been affected the same way by the subprime crisis: first a decrease in risk aversion tends to reflect the *flight-to-liquidity/quality* phenomena, but with a very high level of uncertainty surrounding market dynamics; from 2009 onwards, the first effects of a weak market sentiment against fiscal issues in sovereign bond markets started to be reflected in prices and liquidity. Finally, in 2010-2011, the crisis is clearly revealed by the risk aversion of the market makers, and the uncertainty coefficient until the implementation of the SMP by the ECB that killed market uncertainty and broadly improved liquidity conditions. However, one main adverse effect of these interventions is to increase the risk aversion across countries, even for those that are not targeted by credit rating discrimination among euro area countries.

In terms of monetary policy implications, this paper provides some insights to better understand the impact of the several measures that the ECB has adopted so far to ease off on the pressures in these markets. The Securities Market Programme has succeeded in restoring market efficiency by killing market uncertainty. The implicit resolution to not allow the default of any country in the Euro area and the ability to directly intervene to provide funding appeared as a really strong and collective commitment. However, the announcement has the adverse consequences to rise risk aversion over all Euro area members due to the risk-pooling mechanism beyond this commitment, except for Greece which has taken advantage of this decision. In this direction, public authorities need to be vigilant by using market intervention since the market liquidity gain by killing uncertainty, may be overcome by higher risk aversion, given the signal of a credible threat of default risk.

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6. APPENDIX

Appendix A.

Proof of Proposition 1. From the assumption that $\pi_t \sim N(\frac{1}{2}, \sigma^2_{\pi_t})$ and the formula (17) and (18), we get:

$$\begin{aligned} E(S^*_t \,|\Omega_{t-1}) &= E(A^*\tilde{\tau}_t\left[\pi_t^2\sigma^2_{a^*_{t-1}} + (1-\pi_t)^2\sigma^2_{b^*_{t-1}} + 2\pi_t(1-\pi_t)\sigma_{a^*_{t-1}b^*_{t-1}}\right]|\Omega_{t-1}). \qquad (35)\\ &= A^*\tilde{\tau}_t(E[\pi_t^2 Var(\Delta(a^*_{t-1}-b^*_{t-1})\,|\Omega_{t-1})\,|\Omega_{t-1}] + \\ &\quad 2E[\pi_t(\sigma^2_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})\,|\Omega_{t-1}] + \sigma^2_{b^*_{t-1}}) \\ &= A^*\tilde{\tau}_t[Var(\Delta(a^*_{t-1}-b^*_{t-1})\,|\Omega_{t-1})E(\pi_t^2\,|\Omega_{t-1}) + \\ &\quad 2(\sigma^2_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})E(\pi_t\,|\Omega_{t-1}) + \sigma^2_{b^*_{t-1}}] \\ &= A^*\tilde{\tau}_t[\sigma^2_{\pi_t}Var(\Delta(a^*_{t-1}-b^*_{t-1})\,|\Omega_{t-1}) + \frac{1}{4}Var(\Delta(a^*_{t-1}+b^*_{t-1})\,|\Omega_{t-1})] \end{aligned}$$

and

$$\begin{aligned} Var(S^*_t\,|\Omega_{t-1}) &= A^{*2}\tilde{\tau}_t^2(Var(\Delta(a^*_{t-1}-b^*_{t-1})\,|\Omega_{t-1})^2 Var(\pi_t^2\,|\Omega_{t-1}) + \qquad (36)\\ &\quad 4(\sigma^2_{b^*_{t-1}} - \tilde{\sigma}_{a^*_{t-1}b^*_{t-1}})^2 Var(\pi_t\,|\Omega_{t-1}) + \\ &\quad 4(\sigma^2_{b^*_{t-1}} - \sigma_{a^*_{t-1}b^*_{t-1}})Var(\Delta(a^*_{t-1}-b^*_{t-1})\,|\Omega_{t-1})Cov(\pi_t,\pi_t^2\,|\Omega_{t-1})) \\ &= A^{*2}\tilde{\tau}_t^2(2Var(\Delta(a^*_{t-1}-b^*_{t-1})\,|\Omega_{t-1})^2\sigma^4_{\pi_t} + (\sigma^2_{a^*_{t-1}} - \sigma^2_{b^*_{t-1}})^2\sigma^2_{\pi_t}) \end{aligned}$$

as $\pi_t \sim N(\mu, \sigma^2_{\pi_t})$ (with $\mu = \frac{1}{2}$) then $Var(\pi_t^2\,|\Omega_{t-1}) = E((\pi_t^2 - E(\pi_t^2))^2) = E(\pi_t^4) - E(\pi_t^2)^2 = E(\pi_t^4) - (\sigma^2_{\pi_t} + \mu^2)^2$ as $E(\pi_t^2) = \sigma^2_{\pi_t} + E(\pi_t)^2$

As the kurtosis coefficient of a Gaussian distribution equals 3, $E((\pi_t - E(\pi_t))^4) = 3\sigma^4_{\pi_t}$.

Moreover, the skewness coefficient is equal to 0, $E((\pi_t - E(\pi_t))^3) = 0$.

Then, after some mathematical computation, as $E(\pi_t^4) = 3\sigma^4_{\pi_t} - 6\sigma^2_{\pi_t}\mu^2 - 3\mu^4 + 4\mu E(\pi_t^3)$ and $E(\pi_t^3) = \mu^3 + 3\mu\sigma^2_{\pi_t}$, we obtain $Var(\pi_t^2\,|\Omega_{t-1}) = 2\sigma^4_{\pi_t} + 4\sigma^2_{\pi_t}\mu^2$ and $Cov(\pi_t, \pi_t^2\,|\Omega_{t-1}) = \sigma^2_{\pi_t}$.

If we have $Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1}) = 0$ (then $Var(\Delta(a^*_{t-1} + b^*_{t-1})\,|\Omega_{t-1}) \neq 0$ and $\sigma^2_{a^*_{t-1}} - \sigma^2_{b^*_{t-1}} \neq 0$ otherwise neither the ask nor the bid price move between $t-2$ and $t-1$, which is impossible) then from 35 and 36, $A^* = \frac{4E(S^*_t|\Omega_{t-1})}{\tilde{\tau}_t Var(\Delta(a^*_{t-1}+b^*_{t-1})|\Omega_{t-1})}$ and $\sigma^2_{\pi_t} = \frac{Var(S^*_t|\Omega_{t-1})}{A^{*2}\tilde{\tau}_t^2(\sigma^2_{a^*_{t-1}} - \sigma^2_{b^*_{t-1}})^2}$

From the system of two equations 35 and 36 of the first and second moment of the expected quoted spread at t conditional on the information set Ω_{t-1}, we can derive the value of the risk aversion coefficient A^* and the variance of the coefficient π_t (measure of uncertainty). By extracting an expression of A^* from Equation 35 as follows

$$A^* = \frac{E(S_t^* \,|\Omega_{t-1})}{\tilde{\tau}_t[\sigma^2_{\pi_t} Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1}) + \frac{1}{4} Var(\Delta(a^*_{t-1} + b^*_{t-1})\,|\Omega_{t-1})]},$$

we can obtain a quadratic equation for $\sigma^2_{\pi_t}$. Indeed from Equation 36 we have

$$\begin{aligned} Var(S_t^* \,|\Omega_{t-1}) &= \left(\frac{E(S_t^* \,|\Omega_{t-1})}{\tilde{\tau}_t[\sigma^2_{\pi_t} Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1}) + \frac{1}{4} Var(\Delta(a^*_{t-1} + b^*_{t-1})\,|\Omega_{t-1})]}\right)^2 \times \\ & \quad \tilde{\tau}_t^2 (2Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})^2 \sigma^4_{\pi_t} + (\sigma^2_{a^*_{t-1}} - \sigma^2_{b^*_{t-1}})^2 \sigma^2_{\pi_t}) \end{aligned} \tag{37}$$

so that the quadratic equation for $\sigma^2_{\pi_t}$ is the following:

$$\begin{aligned} & (Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})^2 Var(S_t^* \,|\Omega_{t-1}) - 2Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})^2 E(S_t^* \,|\Omega_{t-1})^2)\sigma^4_{\pi_t} + \\ & (\frac{1}{2} Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1}) Var(\Delta(a^*_{t-1} + b^*_{t-1})\,|\Omega_{t-1}) Var(S_t^* \,|\Omega_{t-1}) \\ & -(\sigma^2_{a^*_{t-1}} - \sigma^2_{b^*_{t-1}})^2 E(S_t^* \,|\Omega_{t-1})^2)\sigma^2_{\pi_t} + \frac{1}{16} Var(\Delta(a^*_{t-1} + b^*_{t-1})\,|\Omega_{t-1})^2 Var(S_t^* \,|\Omega_{t-1}) \\ = \;& 0. \end{aligned} \tag{38}$$

We denote

$$\alpha = Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})^2 Var(S_t^* \,|\Omega_{t-1}) - 2Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})^2 E(S_t^* \,|\Omega_{t-1})^2$$

$$\beta = \frac{1}{2} Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1}) Var(\Delta(a^*_{t-1} + b^*_{t-1})\,|\Omega_{t-1}) Var(S_t^* \,|\Omega_{t-1}) - (\sigma^2_{a^*_{t-1}} - \sigma^2_{b^*_{t-1}})^2 E(S_t^* \,|\Omega_{t-1})^2$$

$$\gamma = \frac{1}{16} Var(\Delta(a^*_{t-1} + b^*_{t-1})\,|\Omega_{t-1})^2 Var(S_t^* \,|\Omega_{t-1})$$

The discriminant Δ of this quadratic equation is equal to

$$\begin{aligned} \Delta &= \beta^2 - 4\alpha\gamma \\ &= [\frac{1}{2} Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1}) Var(\Delta(a^*_{t-1} + b^*_{t-1})\,|\Omega_{t-1}) Var(S_t^* \,|\Omega_{t-1}) \\ & \quad -(\sigma^2_{a^*_{t-1}} - \sigma^2_{b^*_{t-1}})^2 E(S_t^* \,|\Omega_{t-1})^2]^2 - \frac{1}{4} Var(\Delta(a^*_{t-1} + b^*_{t-1})\,|\Omega_{t-1})^2 Var(S_t^* \,|\Omega_{t-1}) \times \\ & \quad (Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})^2 Var(S_t^* \,|\Omega_{t-1}) - 2Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})^2 E(S_t^* \,|\Omega_{t-1})^2) \end{aligned}$$

Assume that the condition $\frac{var(S_t^* |\Omega_{t-1})}{[E(S_t^* |\Omega_{t-1})]^2} < 2$ is verified, then the second term of the discriminant is strictly positive and there exist two real roots of the quadratic equation. Besides, as the two roots are respectively equal to $x_1 = \frac{-\beta-\sqrt{\Delta}}{2\alpha}$ and $x_2 = \frac{-\beta+\sqrt{\Delta}}{2\alpha}$, we can compute

$$\begin{aligned} x_1 x_2 &= \frac{\beta^2 - \Delta}{4\alpha^2} \\ &= \frac{\gamma}{\alpha} \\ &= \frac{\frac{1}{16} Var(\Delta(a^*_{t-1} + b^*_{t-1}) \,|\Omega_{t-1})^2 Var(S^*_t \,|\Omega_{t-1})}{Var(\Delta(a^*_{t-1} - b^*_{t-1}) \,|\Omega_{t-1})^2 Var(S^*_t \,|\Omega_{t-1}) - 2Var(\Delta(a^*_{t-1} - b^*_{t-1}) \,|\Omega_{t-1})^2 E(S^*_t \,|\Omega_{t-1})^2} \end{aligned}$$

The product of the two roots is negative if $\frac{var(S^*_t|\Omega_{t-1})}{[E(S^*_t|\Omega_{t-1})]^2} < 2$ as assumed. Under such assumption, there exist one unique positive real root and one unique negative real root. It follows that we have a unique positive solution for the uncertainty $\sigma^2_{\pi_t}$, and necessarily a unique solution for A^* the risk aversion coefficient. Q.E.D. ∎

Appendix B.

Proof of Proposition 2. From equation 19 and assuming $Var(\Delta(a^*_{t-1} - b^*_{t-1})) \neq 0$ (the other case is straightforward and quite simple), we can rewrite the optimal bid-ask spread S^*_t(whose the random part comes from π_t and which is conditional on Ω_{t-1}: we do not precise it at each step of the proof):

$$\begin{aligned} S^*_t &= A^*\tilde{\tau}_t \left[\pi_t^2 \sigma^2_{a^*_{t-1}} + (1-\pi_t)^2 \sigma^2_{b^*_{t-1}} + 2\pi_t(1-\pi_t)\sigma_{a^*_{t-1}b^*_{t-1}}\right]. \\ &= A^*\tilde{\tau}_t \left[Var(\Delta(a^*_{t-1} - b^*_{t-1}) \,|\Omega_{t-1})\pi_t^2 + 2\pi_t(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}}) + \sigma^2_{b^*_{t-1}}\right]. \\ &= A^*\tilde{\tau}_t \left[Var(\Delta(a^*_{t-1} - b^*_{t-1}) \,|\Omega_{t-1}) \left[\pi_t^2 + 2\pi_t \frac{(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})}{Var(\Delta(a^*_{t-1} - b^*_{t-1}) \,|\Omega_{t-1})}\right] + \sigma^2_{b^*_{t-1}}\right]. \\ &= A^*\tilde{\tau}_t [Var(\Delta(a^*_{t-1} - b^*_{t-1}) \,|\Omega_{t-1})[\left(\pi_t + \frac{(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})}{Var(\Delta(a^*_{t-1} - b^*_{t-1}) \,|\Omega_{t-1})}\right)^2 \\ &\quad - \frac{(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})^2}{Var(\Delta(a^*_{t-1} - b^*_{t-1}) \,|\Omega_{t-1})^2}] + \sigma^2_{b^*_{t-1}}]. \\ &= A^*\tilde{\tau}_t [Var(\Delta(a^*_{t-1} - b^*_{t-1}) \,|\Omega_{t-1}) \left(\pi_t + \frac{(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})}{Var(\Delta(a^*_{t-1} - b^*_{t-1}) \,|\Omega_{t-1})}\right)^2 \\ &\quad + \sigma^2_{b^*_{t-1}} - \frac{(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})^2}{Var(\Delta(a^*_{t-1} - b^*_{t-1}) \,|\Omega_{t-1})}]. \end{aligned}$$

As $Y_t = \pi_t + \frac{(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})}{Var(\Delta(a^*_{t-1} - b^*_{t-1})|\Omega_{t-1})}$ follows a Gaussian distribution whose mean is $\frac{1}{2}\frac{\sigma^2_{a^*_{t-1}} - \sigma^2_{b^*_{t-1}}}{Var(\Delta(a^*_{t-1} - b^*_{t-1})|\Omega_{t-1})}$

and variance $\sigma^2_{\pi_t}$, $Z_t = \left(\pi_t + \frac{(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})}{Var(\Delta(a^*_{t-1}-b^*_{t-1})|\Omega_{t-1})}\right)^2$ follows a non centered and non reduced χ^2_1 distribution with one degree of freedom. According to the formula of a non centered χ^2 distribution and after some calculations, its mean and variance are

$$\mu_{Z_t} = \sigma^2_{\pi_t} + \lambda_t$$

$$\sigma_{Z_t} = 2\sigma^4_{\pi_t} + 4\lambda_t \sigma^2_{\pi_t}$$

where $\lambda_t = \frac{1}{4}\left(\frac{\sigma^2_{a^*_{t-1}} - \sigma^2_{b^*_{t-1}}}{Var(\Delta(a^*_{t-1}-b^*_{t-1})|\Omega_{t-1})}\right)^2$.

Therefore, S^*_t follows a non centered and non reduced χ^2 distribution with one degree of freedom and with mean and variance equal to

$$\mu_{S_t} = A^*\tilde{\tau}_t \left[Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})(\sigma^2_{\pi_t} + \lambda_t) + \sigma^2_{b^*_{t-1}} - \frac{(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})^2}{Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})} \right]$$

$$\sigma^2_{S_t} = 2A^{*2}\tilde{\tau}^2_t Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})^2 \left[\sigma^4_{\pi_t} + 2\lambda_t \sigma^2_{\pi_t}\right]. \tag{39}$$

In addition, for $\forall s \in \mathbb{R}_+$,

$$
\begin{aligned}
F_{S_t}(s) &= P(S_t \leq s) \\
&= P(\left(\pi_t + \frac{(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})}{Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})}\right)^2 - \frac{(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})^2}{Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})^2} + \\
&\quad \frac{\sigma^2_{b^*_{t-1}}}{Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})} \\
&\leq \frac{s}{A^*\tilde{\tau}_t Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})}) \\
&= P(\left(\pi_t + \frac{(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})}{Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})}\right)^2 \leq \\
&\quad \frac{s}{A^*\tilde{\tau}_t Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})} + \\
&\quad \frac{(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})^2}{Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})^2} - \frac{\sigma^2_{b^*_{t-1}}}{Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})}) \\
&= P(\frac{1}{\sigma^2_{\pi_t}}\left(\pi_t + \frac{(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})}{Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})}\right)^2 \leq \\
&\quad \frac{s}{A^*\tilde{\tau}_t \sigma^2_{\pi_t} Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})} + \frac{(\sigma_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{b^*_{t-1}})^2}{\sigma^2_{\pi_t} Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})^2} \\
&\quad - \frac{\sigma^2_{b^*_{t-1}}}{\sigma^2_{\pi_t} Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})})
\end{aligned}
$$

As $\frac{Z_t}{\sigma^2_{\pi_t}}$ follows a non centered χ^2_1 distribution, $\forall s \in \mathbb{R}_+$ we have,

$$
\begin{aligned}
f_{S_t}(s) &= \frac{dF_{S_t}(s)}{ds} \\
&= \frac{1}{A^*\tilde{\tau}_t Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})\sigma^2_{\pi_t}} \times \\
&\quad h^1_{\chi^2_{nc}}(\frac{Var(\Delta(a^*_{t-1} - b^*_{t-1}))\,|\Omega_{t-1}\; s + A^*\tilde{\tau}_t(\sigma^2_{a^*_{t-1}b^*_{t-1}} - \sigma^2_{a^*_{t-1}}\sigma^2_{b^*_{t-1}})}{A^*\tilde{\tau}_t Var(\Delta(a^*_{t-1} - b^*_{t-1})\,|\Omega_{t-1})^2\sigma^2_{\pi_t}})
\end{aligned}
$$

where $h^1_{\chi^2_{nc}}$ is the probability density function of a non centered chi-square distribution with one degree of freedom. Q.E.D. ∎

Appendix C. Stylized facts

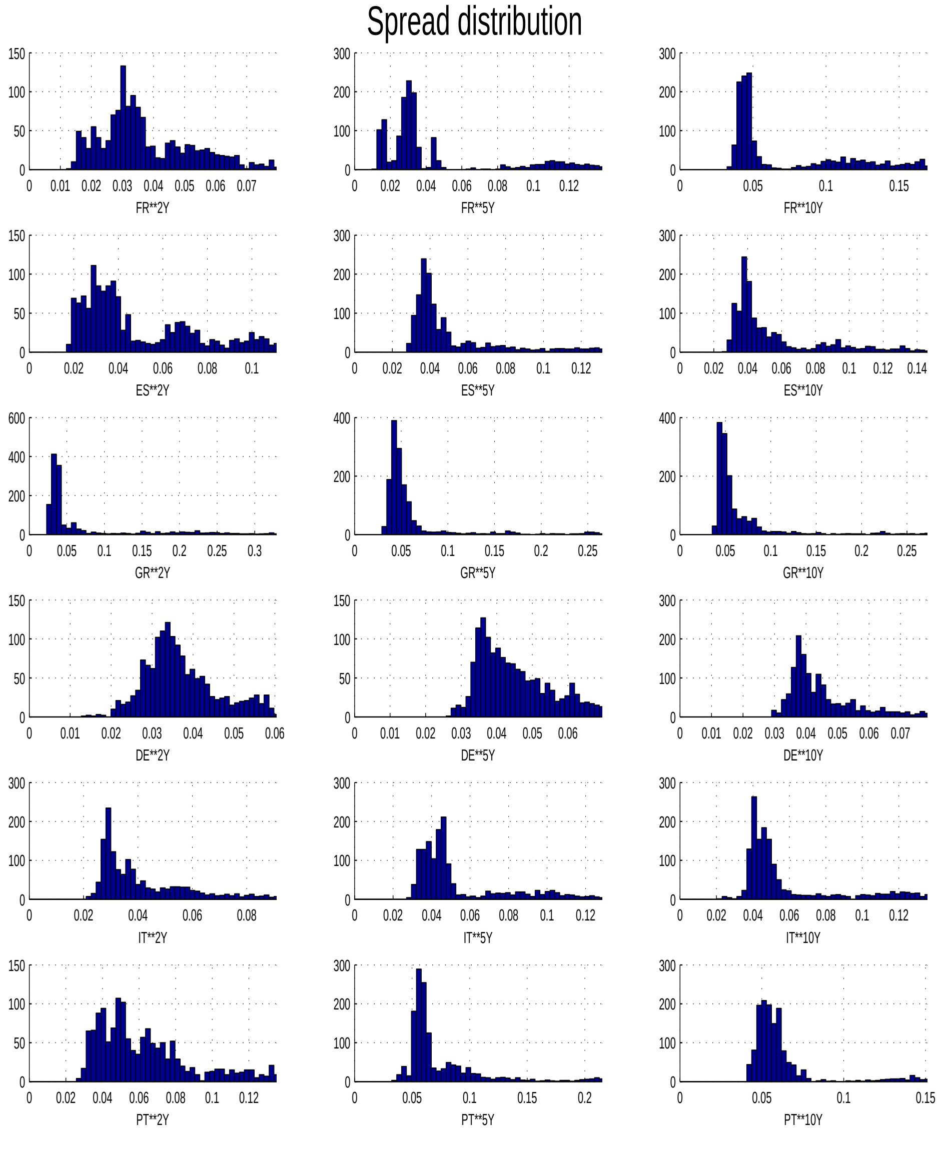

FIG. 2 Histograms of the daily median bid-ask spreads.

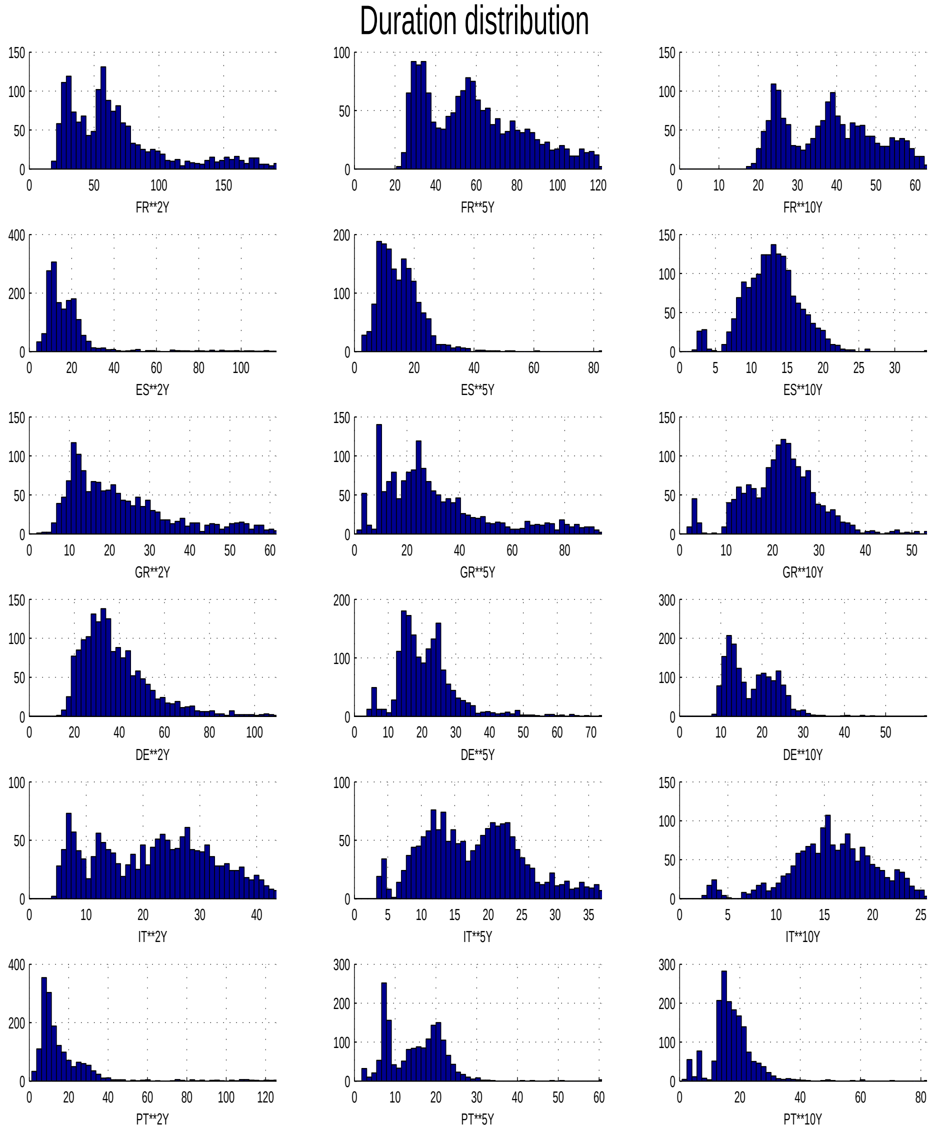

FIG. 3 Histograms of the daily median durations.

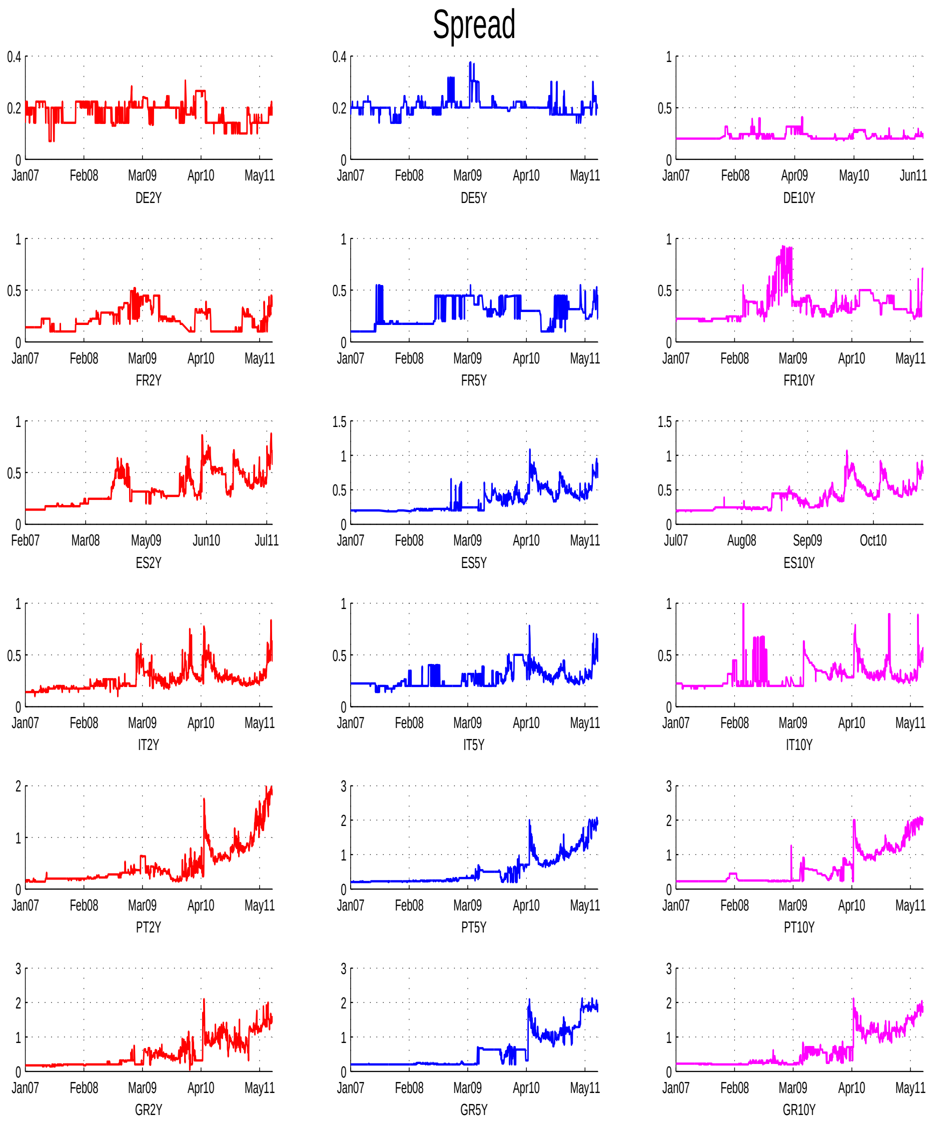

FIG. 4 Evolutions of the daily median bid-ask spread, 2007-2011.

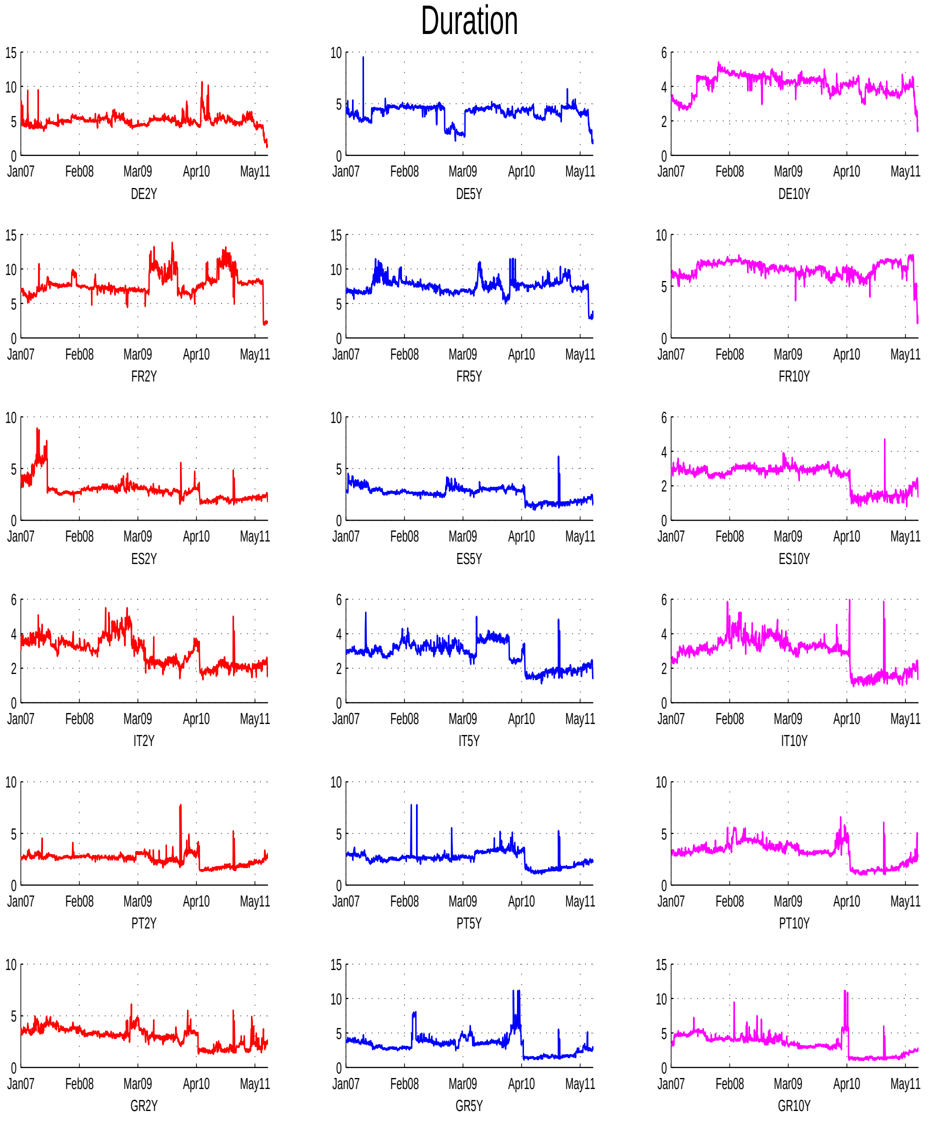

FIG. 5 Evolutions of the daily median durations between two quote arrivals, 2007-2011.

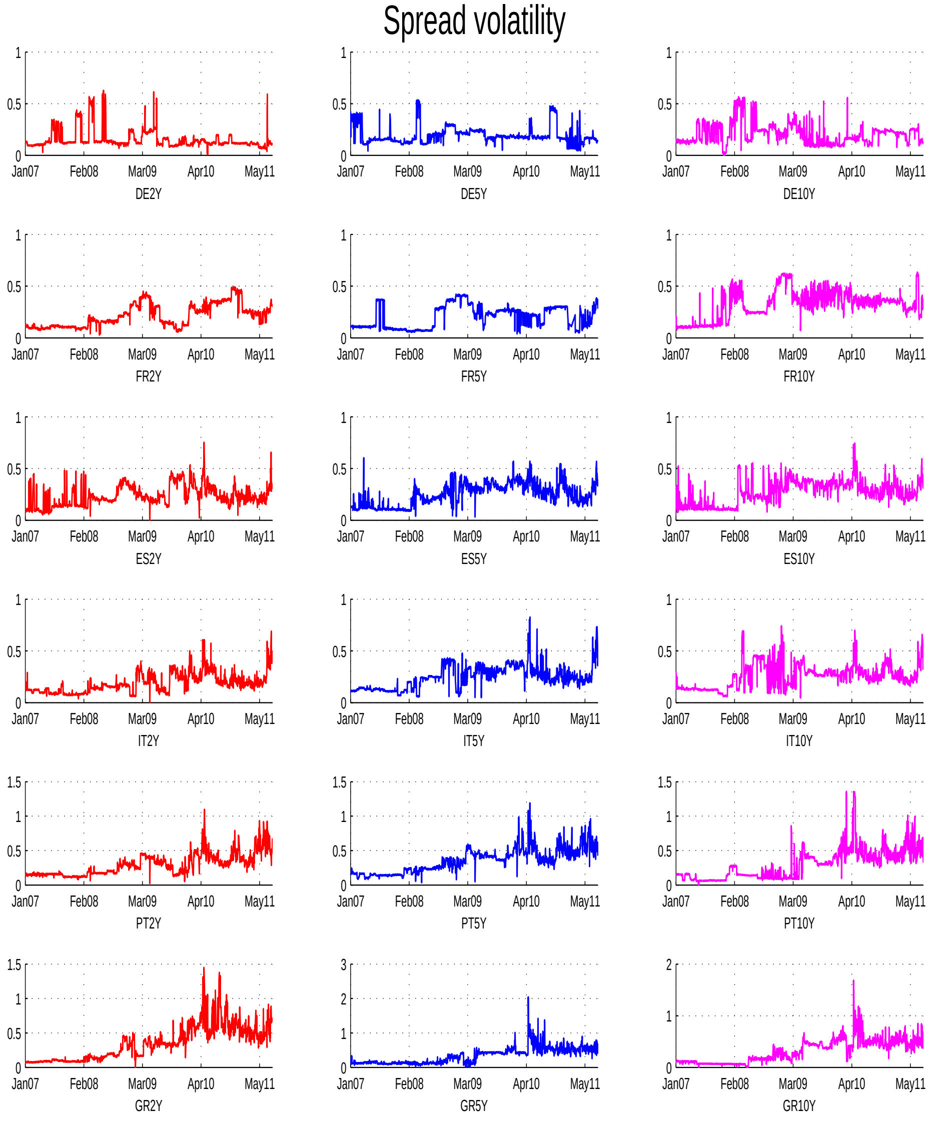

FIG. 6 Evolutions of the daily median bid-ask spread volatilities, 2007-2011.

Appendix D. Uncertainty and risk aversion

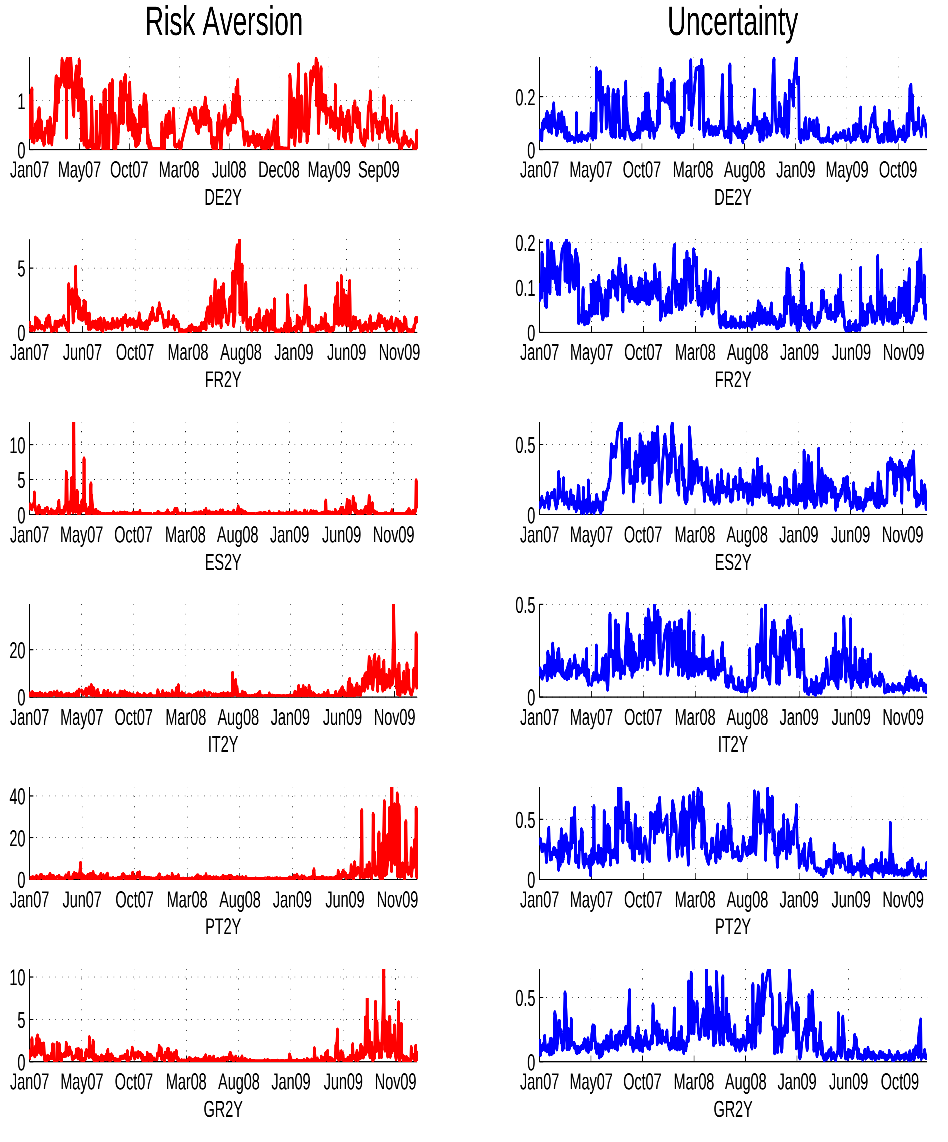

FIG. 7 Risk aversion and uncertainty for 2-year-maturity rates, 2007-January 2010.

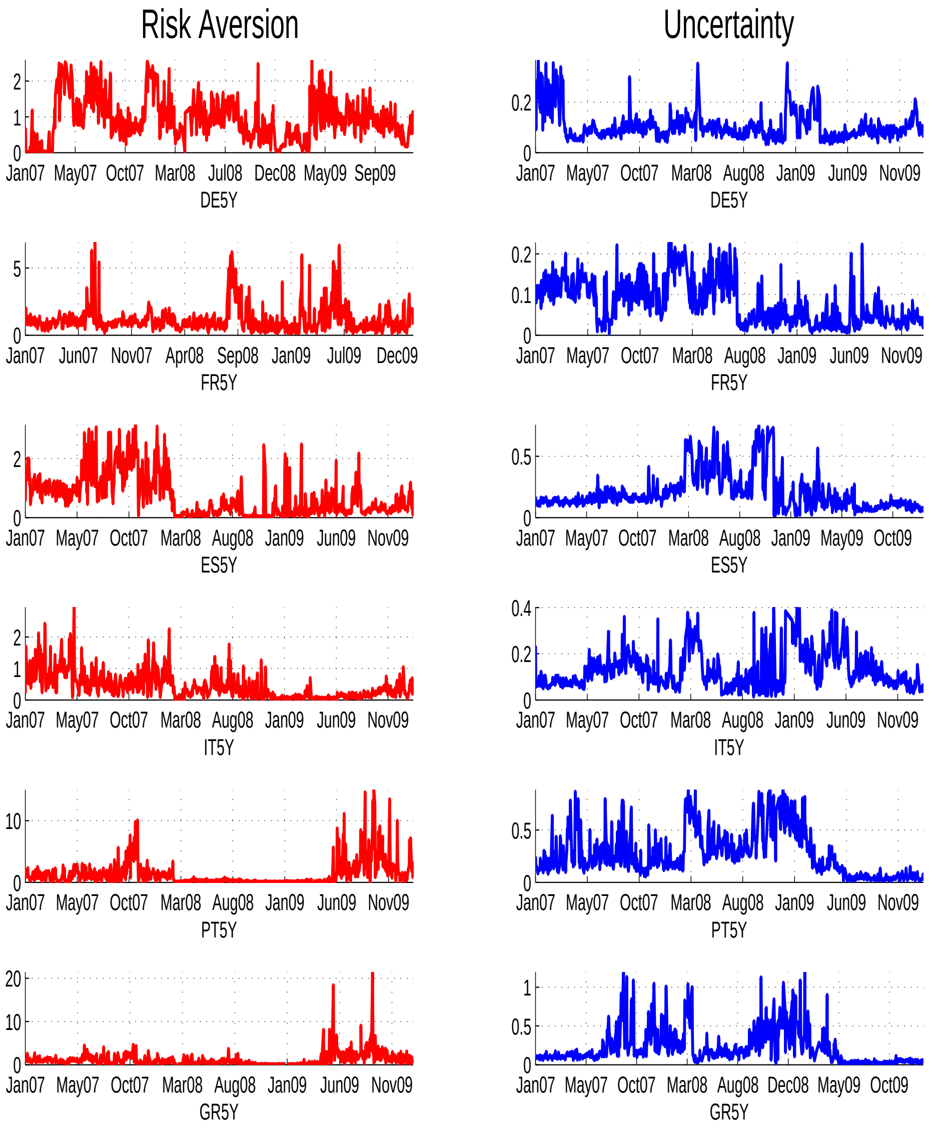

FIG. 8 Risk aversion and uncertainty for 5-year-maturity rates, 2007-January 2010.

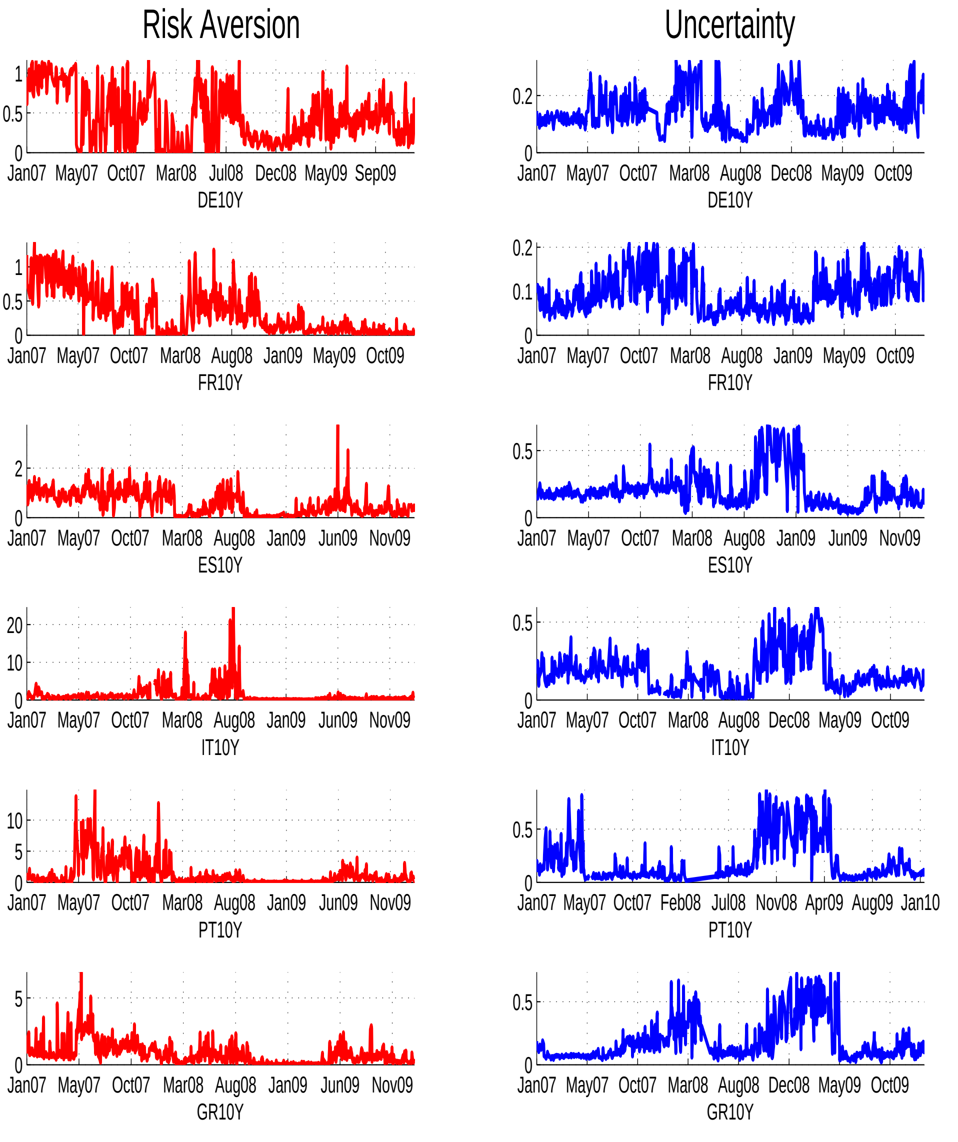

FIG. 9 Risk aversion and uncertainty for 10-year-maturity rates, 2007-January 2010.

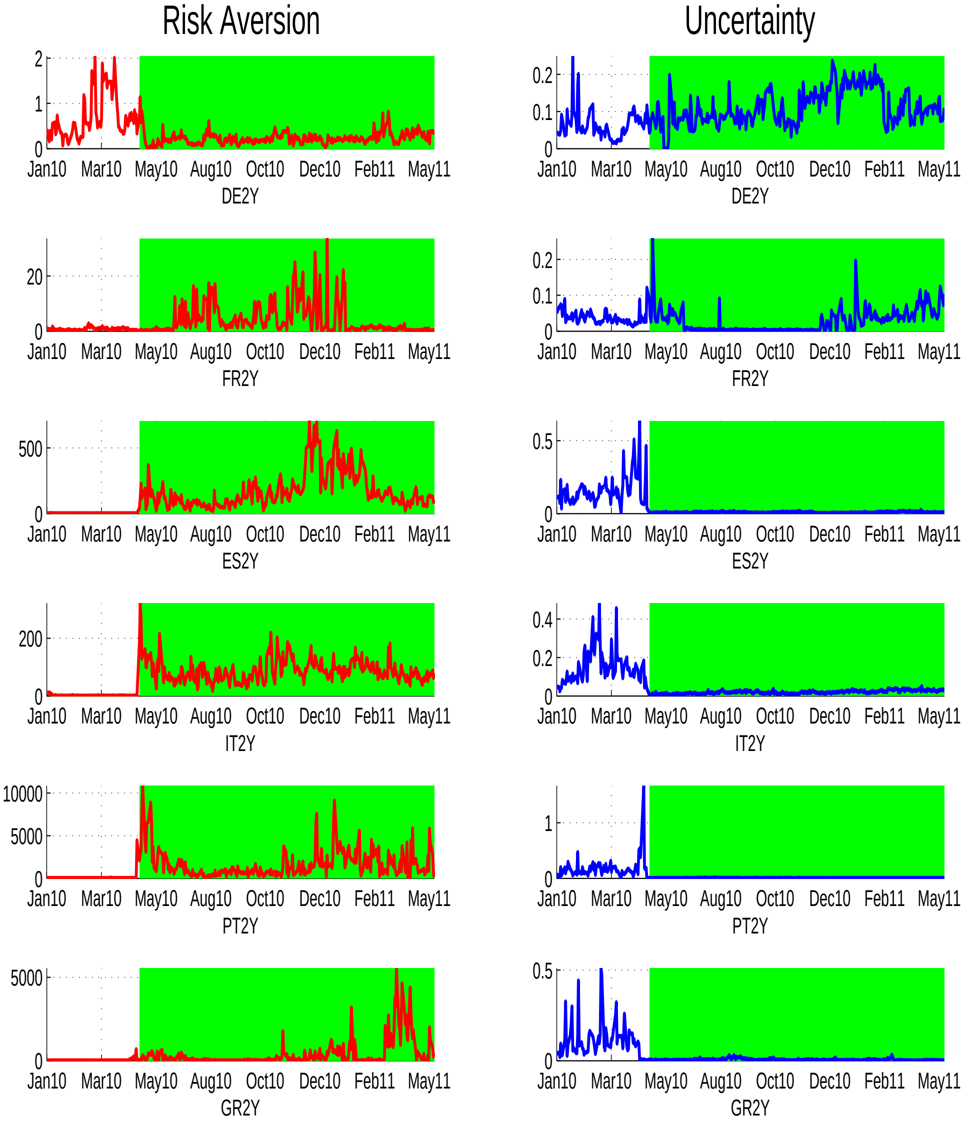

FIG. 10 Risk aversion and uncertainty for 2-year-maturity rates, 2010-2011.

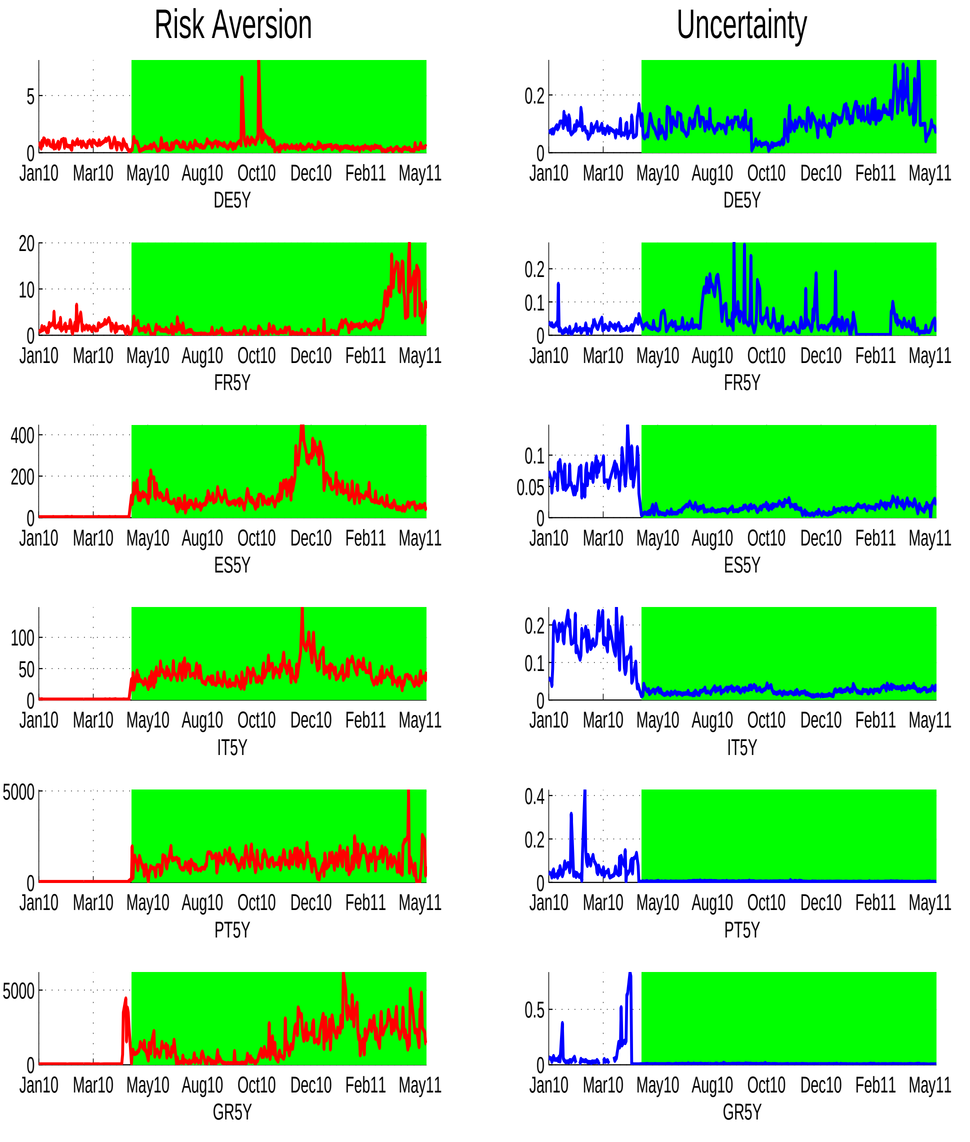

FIG. 11 Risk aversion and uncertainty for 5-year-maturity rates, 2010-2011.

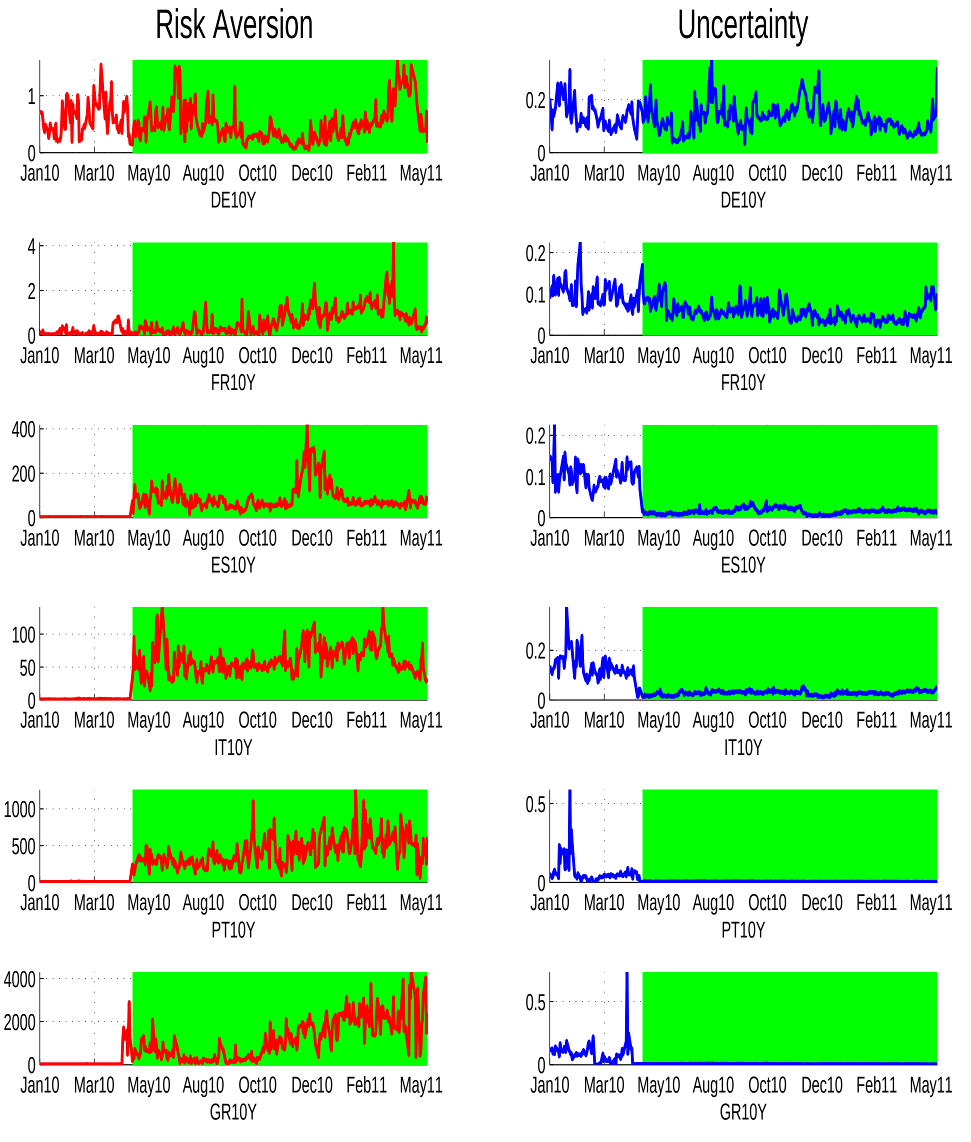

FIG. 12 Risk aversion and uncertainty for 10-year-maturity rates, 2010-2011.