Nash Implementation with Lottery Mechanisms*

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Abstract

Consider the problem of exact Nash Implementation of social choice correspondences. Define a mechanism in which the planner can randomize on alternatives out of equilibrium while pure alternatives are always chosen in equilibrium. We call such a game form a lottery mechanism. When preferences over alternatives are strict, we show that Maskin Monotonicity (Maskin, 1999) is both necessary and sufficient for a social choice correspondence satisfying unanimity to be Nash implementable. We discuss how to dispense with unanimity and relax the assumption of strict preferences by modifying the mechanism we consider. We then study some examples of monotonic social choice correspondences violating no-veto power. Finally, we apply our method to the issue of voluntary implementation (Jackson-Palfrey, 2001).

Keywords: Lottery mechanism, Nash implementation, no-veto power, Maskin monotonicity.

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1 Introduction

The goal of implementation theory is to design institutions that eliminate strategic manipulations, on part of the agents, in order to implement desirable *social choice correspondences* (henceforth SCC). *Maskin monotonicity* (Maskin, 1999) is a necessary condition for (exact) Nash implementation. When there are at least three players, it is also sufficient if coupled with the assumption of *no-veto power*[1]. In economic environments with a perfectly divisible good (e.g. money), where there is typically conflict of interest, *no-veto power* is vacuously satisfied since agents will never agree on a best allocation. For more abstract environments, the gap between necessity and sufficiency was closed by Moore-Repullo (1990), Sjöström (1991) or Danilov[2] (1992). The construction involves a complex necessary and sufficient condition that may be hard to interpret. We suggest to close the gap between necessity and sufficiency by looking at a different class of mechanisms than the canonical one used in Maskin's theorem. We call it the class of *lottery mechanisms*. While equilibrium messages should still deliver (pure) alternatives selected by the SCC, we allow the planner to use non-degenerate lotteries out of equilibrium. We construct an alternative version of the canonical mechanism in which agents can obtain a lottery on two alternatives in some region of the message space. A consequence is that, under some assumptions on the SCC and on the environment, every *Maskin monotonic* SCC is implementable by this mechanism. In the main theorem, we consider for simplicity SCCs satisfying *unanimity*, and restrict our attention to linear orderings over alternatives. Preferences should also be extended to preference over lotteries. However, the theorem does not require agents to have preferences satisfying the Von-Neumann-Morgenstern axioms. Next, we discuss how we can easily relax the assumptions of *unanimity* and of strict preferences. By complexifying the mechanism, *unanimity* can be dropped. The restriction to strict preferences can be relaxed by using an assumption that we call *top strict*

[1] No-veto power states that if at least $n-1$ agents agree on a best alternative, it should be selected by the SCC. This condition is sometimes restrictive. For example, the individually rational SCC both in problems of indivisible goods assignement and in voting settings, fail to satisfy it.

[2] Danilov (1992) derives an elegant necessary and sufficient condition, for Nash implementation in the case of linear orders on alternatives, called *essential monotonicity*. In such environment, his condition is equivalent to Moore-Repullo's (1990) or Sjostrom's (1991) condition.

difference. It says that if an allocation is ranked top by at least $(n-1)$ agents, then the set of top allocations should be a singleton for at least two agents. By further changing the mechanism, one can relax this assumption to requiring it for only one agent. Next, we present some examples of SCCs that violate *no-veto power*. Finally, we consider the case of voluntary implementation (Jackson-Palfrey, 2001) and show that no-veto power can also be dispensed with by considering a *lottery mechanism*.

2 The set-up

There is a fixed finite set of agents $N \equiv \{1, ..., n\}$, with $n \geq 3$, and a fixed finite set of alternatives A, with $|A| \equiv \ell$ and $\ell \geq 3$. Let $\mathcal{L} \equiv \Delta^{\ell-1}$ be the set of lotteries over A. In lottery $x = (x_a)_{a \in A} \in \mathcal{L}$, alternative a occurs with probability x_a. Abusing notation, we write a both for the pure alternative $a \in A$ and for the lottery $x \in \mathcal{L}$ with $x_a = 1$. The support of a lottery $x \in \mathcal{L}$ is the set of pure alternatives receiving a strictly positive probability in $x : \text{supp}x \equiv \{a \in A \mid x_a > 0\}$.

The set of admissible preference profiles over A and over $\mathcal{L}$ are, respectively, Θ and Γ. For any $\theta \in \Theta$, $R_i(\theta)$ stands for the (weak) preference of agent $i \in N$ over alternatives in A. We denote by $P_i(\theta)$ and $I_i(\theta)$ the associated strict and indifference relations, respectively. Similarly, define, for any $\gamma \in \Gamma$, $R_i(\gamma), P_i(\gamma)$ and $I_i(\gamma)$ in the same fashion. Given $\theta \in \Theta$, let $\Sigma(\theta) \subseteq \Gamma$ be the set of preferences over $\mathcal{L}$ that agree on the ranking over A. We make two assumptions on preferences.

Strictness: For each $\theta \in \Theta$, each $a, b \in A$ and each $i \in N$, if $aR_i(\theta)b$, then $aP_i(\theta)b$ or $a = b$.

Monotonicity in probabilities: Preferences over lotteries are *monotonic in probabilities*, that is, shifts in probability to strictly preferred alternatives yield strictly preferred lotteries. For each $i \in N$, and each $k \in \{1, \ldots, m\}$, let $p_{ik} : \Theta \to A$ be defined by

$$p_{ik}(\theta) = a \iff |\{b \in A \mid b \, R_i(\theta) \; a\}| = k.$$

That is, $p_{i1}(\theta)$ is the preferred alternative of agent i in state θ, $p_{i2}(\theta)$ her second preferred, etc. Then, if two lotteries $x \equiv (x_a)_{a \in A}$ and $y \equiv (y_a)_{a \in A}$ are such that for each $k^* \in \{1,...,m\}$, $\sum_{k\leq k^*} x_{p_{ik}(\theta)} \geq \sum_{k\leq k^*} y_{p_{ik}(\theta)}$, then $xR_i(\gamma)y$ for any $\gamma \in \Sigma(\theta)$, and whenever one inequality is strict, $xP_i(\gamma)y$ for all $\gamma \in \Sigma(\theta)$.

Denote by $LC_i(\theta, a)$ the lower contour set of agent $i \in N$ at profile $\theta \in \Theta$ and alternative $a \in A$, i.e.,

$$LC_i(\theta, a) \equiv \{b \in A : aR_i(\theta)b\}.$$

A *social choice correspondence* (SCC) is a correspondence $f : \Theta \twoheadrightarrow A$ that associates to each preference profile a non-empty subset of alternatives.

Unanimity: a SCC f is *unanimous* if for each pair $(a, \theta) \in A \times \Theta$,

$$[aR_i(\theta)b \text{ for each } i \in N, \text{ each } b \in A] \implies [a \in f(\theta)].$$

No-Veto power: Fix $i \in N$. a SCC f satisfies *no-veto power* if for each pair $(a, \theta) \in A \times \Theta$,

$$[aR_j(\theta)b \text{ for each } j \neq i, \text{ each } b \in A] \implies [a \in f(\theta)].$$

Maskin Monotonicity: a SCC f is *Maskin monotonic* (Maskin (1999)), if for each pair $(\theta, \phi) \in \Theta$ and each $a \in f(\theta)$,

$$[LC_i(\theta, a) \subseteq LC_i(\phi, a) \text{ for each } i \in N] \implies [a \in f(\phi)].$$

For each agent $i \in N$, each $\theta \in \Theta$, and each $a \in A$, define

$$C_i(\theta, a) \equiv \left\{ \begin{array}{c} c \in LC_i(\theta, a) : \text{for all } \phi \in \Theta, \text{ if } LC_i(\theta, a) \subseteq LC_i(\phi, c) \text{ and,} \\ \text{for each } j \neq i, \ LC_j(\phi, c) = A, \text{ then } c \in f(\phi). \end{array} \right\}$$

Strong Monotonicity: a SCC f is *strongly monotonic* (see Danilov[3], 1992, for a different but equivalent definition) if for each pair $(\theta, \phi) \in \Theta$ and $a \in f(\theta)$,

$$[C_i(\theta, a) \subseteq LC_i(\phi, a) \text{ for each } i \in N] \implies [a \in f(\phi)].$$

[3] When *strictness* is satisfied, *essential monotonicity* and *strong monotonicity* coincide.

A *mechanism (or game form)* is a pair $G = (M, g)$ where $M \equiv \underset{i \in N}{\times} M_i$, and M_i is the message space of agent $i \in N$, and $g : M \to A$ is an outcome function that associates an alternative to every profile of messages. A *game* for G is a pair (G, γ) for some $\gamma \in \Gamma$. We will restrict our attention to Nash equilibria of games (G, γ), denoted $NE(G, \gamma)$. A mechanism $G = (M, g)$ is *ordinal* if the set of Nash equilibria only depends on agents' preferences over pure alternatives, that is, for each $\theta \in \Theta$, each $m \in M$ and all $\gamma, \delta \in \Sigma(\theta)$, $NE(G, \gamma) = NE(G, \delta)$. We confine our attention to *ordinal* game-forms. Therefore, abusing notation, for any $\gamma \in \Sigma(\theta)$, let $NE(G, \theta)$ denote the Nash equilibria of game (G, γ). A mechanism G is a *lottery mechanism* if for each $\theta \in \Theta$, and each $m \in M$, $g(m) \in \mathcal{L}$. Hence, $g : M \to \mathcal{L}$. That is, any outcome of the mechanism is a lottery –whether this lottery is degenerate or non-degenerate. Indeed, since our focus is on exact implementation of deterministic SCC, Nash equilibrium outcome of the mechanism should be degenerate lotteries. Let $\mathcal{G}$ be the class of lottery mechanisms.

A SCC f is *Nash implementable* if there exists a mechanism $G = (M, g)$ such that the Nash equilibrium outcomes of each game coincides with outcomes chosen by f. That is, for each $\theta \in \Theta$, and each $\gamma \in \Sigma(\theta), f(\theta) = g(NE(G, \theta))$.

3 Enlarging the Class of Mechanisms

To understand why enlarging the class of admissible mechanisms could help dispensing with the *no-veto power* assumption, it is useful to recall Maskin's theorem and his canonical mechanism. The necessity part *Maskin monotonicity* of SCCs is omitted and we only prove sufficiency since it is the focus of our paper.

Theorem 1 *(Maskin, 1999) If $n \geq 3$, any Maskin monotonic SCC that satisfies no-veto power is Nash implementable.*

Proof: For each $i \in N$, $M_i \equiv A \times \Theta \times \mathbb{N}$. A typical message is $m_i \equiv (m_i^1, m_i^2, m_i^3) \equiv (a, \theta, n)$. The outcome functions is described as follows.

Rule 1: If $m_i = (a, \theta, \cdot)$ for all $i \in N$ and $a \in f(\theta)$, then $g(m) = a$.

Rule 2: If for some $i \in N$, $m_j = (a, \theta, \cdot)$ for each $j \neq i$ and $m_i = (b, \phi, \cdot) \neq m_j$, then the outcome is

$$g(m) = \begin{cases} b \text{ if } b \in LC_i(\theta, a) \\ a \text{ otherwise.} \end{cases}$$

Rule 3: In all other cases, the outcome is $m^1_{i^*}$ where $i^* \equiv \min \{i \in N : n_i \geq n_j \; \forall j \in N\}$.

The proof that this mechanism implements any monotonic SCC that satisfies *no-veto power* is well-known but we sketch it here for the sake of completeness.

Suppose that the true state is $\theta \in \Theta$. We have different cases to consider.

1) $g(m)$ is given by *Rule 3*

Note that any agent $j \neq i^*$ could obtain his top ranked outcome by announcing $n_j > n_{i^*}$. If $g(m)$ is a Nash equilibrium outcome of the mechanism, then $LC_i(\theta, g(m)) = A$ for each $i \in N$. By *no-veto power*, $g(m) \in f(\theta)$.

2) $g(m)$ is given by *Rule 2*

Note that any agent $j \neq i$ could deviate, trigger the integer game and obtain his top ranked outcome, say a, by announcing $n_j > n_k \; \forall k \neq j$. If $g(m)$ is a Nash equilibrium outcome, then $LC_j(\theta, g(m)) = A \; \forall j \neq i$. By *no-veto power*, $g(m) \in A$.

3) $g(m)$ is given by *Rule 1*

If θ is announced truthfully, then $g(m) \in f(\theta)$. So, suppose instead that $m_i = (a, \phi, \cdot)$ and $a \in f(\phi)$. Any agent i could deviate and obtain any alternative b such that $b \in LC_i(\phi, a)$. If $g(m)$ is a Nash equilibrium outcome, then for each $i \in N$, $aR_i(\phi)b \iff aR_i(\theta)b$. By *Maskin monotonicity*, $a \in f(\theta)$.

Now, suppose the true state is $\theta \in \Theta$ and fix $a \in f(\theta)$. We leave it to the reader to show that $m_i = (a, \theta, 0)$ is a Nash equilibrium of the mechanism with outcome a.

Q.E.D.

Note that in the proof, *no-veto power* is in fact used only to rule out undesirable equilibria in *Rule 2*. In *Rule 3*, only *unanimity* –obviously implied by *no-veto power*– would be needed. What happens if *no-veto power* is not satisfied[4]? Suppose that the true state is θ. Messages reported are, say,

[4] A similar discussion can be found in the excellent survey of Maskin-Sjoström (2003). We follow here their terminology.

$m_j = (a, \phi, \cdot)$ $\forall j \neq i$, $m_i = (c, \theta, \cdot)$ and $c \in LC_i(\phi, a)$. The outcome is $g(m) = c$ given by *Rule 2*. If *no-veto power* is not satisfied, it could be the case that for each $j \neq i$, $LC_j(\theta, c) = A$, $cP_i(\theta)a$ and $c \notin f(\theta)$. Note that, in such a case, no deviations from m are possible: any agent $j \neq i$ could trigger the integer game but c is top ranked for any such agent. Message m is a Nash equilibrium. Using the terminology of Maskin-Sjöström (2002), c is an *awkward* outcome for agent i in $LC_i(\theta, c)$. In the canonical mechanism, given message $m \in M$ with $m_i = (a, \theta, \cdot)$ for all $i \in N$, the attainable set from *Rule 1* is $LC_i(\theta, a)$ for each $i \in N$. When f violates *no-veto power*, the planner should restrict the attainable sets by removing *awkward* alternatives. This entails constructed (personalized) attainable sets,

$$C_i(\theta, a) \equiv \left\{ \begin{array}{c} c \in LC_i(\theta, a) : \text{for all } \phi \in \Theta, \text{ if } LC_i(\theta, a) \subseteq LC_i(\phi, c) \text{ and,} \\ \text{for each } j \neq i, \ LC_j(\phi, c) = A, \text{ then } c \in f(\phi) \end{array} \right\}.$$

In *Rule 2*, one should replace $LC_i(\theta, a)$ by $C_i(\theta, a)$ for each $i \in N$. In the environment we consider, a necessary and sufficient condition for Nash implementation is *strong monotonicity* (see Danilov, 1992, for a different version of this condition, or Maskin-Sjöström, 2003, for a more detailed discussion). Indeed, *strong monotonicity* implies *Maskin monotonicity* but the converse is not true. However, *Maskin monotonicity* and *no-veto power* imply *strong monotonicity*. By considering a larger class of mechanisms that we call *lottery mechanisms*, *strong monotonicity* is only necessary when for each pair $(a, \theta) \in A \times \Theta$ such that $m = ((a, \theta, \cdot), (a, \theta, \cdot), ..., (a, \theta, \cdot)) \in NE(G, \theta)$, we have that $C_i(\theta, a) = LC_i(\theta, a)$ for each $i \in N$. But this is simply *Maskin monotonicity,* which remains obviously necessary. By using a *lottery mechanism* –see Theorem 2 below– it is therefore not necessary to restrict the attainable sets. The planner does not need to construct personalized attainable sets by removing every *awkward* outcome. To illustrate our approach, we consider an important example from Maskin (1985, 1999) of a SCC that is *Maskin monotonic* but does not satisfy *no-veto power*. We use it here to show that this particular SCC is implementable in Nash equilibrium if the planner uses a *lottery mechanism*. Thus, checking whether the SCC satisfies *Maskin monotonicity* is enough to know if it is Nash implementable.

Example 1 (Maskin, 1985 and 1999)**:**

$N \equiv \{1, 2, 3\}$, $\Theta \equiv \{\theta, \phi\}$ and $A \equiv \{a, b, c\}$. The preferences are described in the table below:

θ				ϕ	
1	2	3	1	2	3
b	a	a	b	c	c
a	c	c	c	a	a
c	b	b	a	b	b

The SCC is described as follows. Given $\theta' \in \Theta$, if a majority prefers a to b then $a \in f(\theta')$; if a majority prefers b to a, then $b \in f(\theta')$; and $c \in f(\theta')$ if $LC_i(\theta, c) = A$ for each $i \in N$. Indeed, this SCC is *Maskin monotonic* but does not satisfy *no-veto power*. It is thus not Nash implementable by Maskin's mechanism. In this example, $f(\theta) = \{a\} = f(\phi)$. However, $LC_1(a, \theta) = LC_1(c, \phi) = \{a, c\}$ and $LC_i(\phi, c) = A$ for each $i \in N \setminus \{1\}$. Hence, alternative c is an *awkward* outcome in $LC_1(a, \theta)$. In Maskin's mechanism, if the true state is ϕ, then the profile of message $m \in M$, with $m_j = (a, \theta, \cdot)$ for $j \neq 1$ and $m_1 = (c, \phi, \cdot)$ is a Nash equilibrium. *Rule 2* prescribes the outcome $c \notin f(\phi)$. Agents $j \neq 1$ do not have a profitable deviation from m since $LC_j(\phi, c) = A$. Clearly, what needs to be modified is *Rule 2.* Instead of the outcome being c, suppose *Rule 2* gives a lottery $(1-\varepsilon)\,a+\varepsilon c$, with $\varepsilon \in (0,1)$. By *monotonicity in probabilities*, when the true state is ϕ, agent 2 and 3 would rather get c with probability one than a lottery on a and c. Therefore, when the true state is ϕ, the profile $m \in M$, with $m_j = (a, \theta, \cdot)$ for $j \neq 1$ and $m_1 = (c, \phi, \cdot)$ is no longer a Nash equilibrium. We have just informally shown that we can, on one-hand, dispense with *no-veto power,* and on the other hand, that we do not need to construct restricted attainable sets. In fact, with linear orderings, we only need to check whether a SCC is *Maskin monotonic* to know if it is Nash implementable. Having possibly non-degenerate lotteries in *Rule 2* eliminates the need for *no-veto power*. Moreover, in the necessary and sufficient condition for Nash implementation, only *Maskin monotonicity* will have bite[5]. We are now ready to state our result.

Theorem 2 *Any SCC f satisfying unanimity is implementable in Nash equilibrium by a (ordinal) lottery mechanism if and only if it is Maskin monotonic*[6]*.*

[5] This statement is not entirely true at this stage. It was clear from the proof of theorem 1 that unanimity was needed in *Rule 3* of Maskin's mechanism. However, in the discussion following theorem 2, we show how unanimity can also be dropped by modifying *Rule 3* of the *lottery mechanism* we contruct.

[6] Alternatively, we could also drop the restriction to ordinal game forms. By dropping

Proof: We first prove that if f is implementable by a (ordinal) lottery mechanism, then f is *Maskin monotonic*. If f is implementable in Nash equilibrium by a *lottery mechanism*, there exists a mechanism (M, g) that implements it. Consider $\theta \in \Theta$ and $a \in f(\theta)$. Since f is implemented, there is $m \in NE(G, \theta)$ such that $g(m) = a$. Suppose there exists $\phi \in \theta$ with $a \notin f(\phi)$. Again, implementation of f requires $m \notin NE(G, \phi)$. Hence, there is $i \in N$ and $m'_i \in M_i$ such that $g(m'_i, m_{-i})P_i(\gamma)a$ for some $\gamma \in \Sigma(\phi)$. By *monotonicity in probabilities*, there exists at least one $b \in \text{supp}g(m'_i, m_{-i})$ such that $bP_i(\phi)a$. Since m is a Nash equilibrium under θ, it is the case that $aR_i(\theta)b$. Therefore, f is *Maskin monotonic*.

We now prove the sufficiency part. We construct the following *lottery mechanism*. The message space[7] of each agent $i \in N$ is $M_i \equiv A \times \Theta \times A \times \mathbb{N}$. A typical message will be denoted $m_i \equiv (m_i^1, m_i^2, m_i^3, m_i^4) \equiv (x, \theta, x', n_i)$. Fix a number $\varepsilon \in (0, 1)$.

Rule 1: If $m_i = \left(\bar{x}, \bar{\theta}, \bar{a}, .\right)$ for all $i \in N$ and $\bar{x} \in f(\bar{\theta})$, then $g(m) = \bar{x}$.

Rule 2: If $m_j = \left(\bar{x}, \bar{\theta}, \bar{a}, \cdot\right)$ for each $j \neq i$ and $m_i = (c, \phi, b, \cdot) \neq m_j$, then the outcome is

$$g(m) = \begin{cases} (1-\epsilon)\bar{x} + \epsilon b \text{ if } b \in LC_i(\bar{\theta}, \bar{x}) \\ \bar{x} \text{ otherwise} \end{cases}$$

Rule 3: In all other cases, $g\left(m\right) = m_{i^*}^1$ where $i^* \equiv \min\left\{i \in N : n^i \geq n^j \ \forall j \in N\right\}$.

We show that this game form Nash implements any *unanimous* SCC f that is *Maskin monotonic*. First, suppose that the true profile is θ and that $x \in f(\theta)$. The message profile $m \in M$ with $m_i = (x, \theta, x, 0)$ for each $i \in N$ is a Nash Equilibrium of G. By unilaterally deviating, an agent $i \in N$ can only trigger *Rule 2* and obtain either x with probability 1 or a lottery on x and another alternative $b \in LC_i(\theta, x)$. In that case, the deviation decreases the probability of $x \in f(\theta)$ and increase the probability of a worse alternative $b \in A$. By *monotonicity in probabilities*, this deviation is not

this assumption, one can look at a richer set of SCCs that uses cardinal information; that is SCCs that may vary with changes in preferences over lotteries while ordinal preferences remain the same, $f : \Gamma \twoheadrightarrow \mathcal{L}$. In that case, any cardinal SCC satisfying unanimity is implementable in Nash equilibrium by a lottery mechanism if and only if it is Maskin monotonic (in the simplex).

[7] In fact, the message spaces can be reduced. We only need $M_i \equiv A \times \Theta \times \mathbb{N}$ for each $i \in N$. We keep it larger because this is the message spaces we will use in the discussion of the relaxation of the assumptions of *unanimity* and *strictness*.

profitable. Therefore, m is a Nash Equilibrium. Second, Suppose the true state is $\theta \in \Theta$. We show that for each $\gamma \in \Sigma(\theta)$, $g(NE(G,\theta)) \subseteq f(\theta)$. We have different cases to consider.

1) $g(m)$ is given by *Rule 3*

If $g(m) = x \in g\left(NE\left(G,\theta\right)\right)$ for each $\gamma \in \Sigma(\theta)$, then $LC_i(\theta, x) = A$ for each $i \in N$. Otherwise, any agent $j \neq i^*$ could announce $m'_j = (a, \cdot, \cdot, n'_j)$ with $n'_j > n_{i^*}$ and $aP_i(\theta)x$. Therefore, if there are no deviations, by *unanimity*, $a \in f(\theta)$.

2) $g(m)$ is given by *Rule 2*

If $g(m)$ is obtained by *Rule 2*, note that any agent $j \neq i$ could deviate by announcing $m'_j \neq m_j$ with $n'_j > n_k$ for all $k \neq i$, and obtain his top alternative under θ. We have two cases to consider. If $g(m) \in \mathcal{L}$ and $\text{supp}(g(m)) = \{a \in A : x_a > 0\}$ is not a singleton, $g(m)$ is not a Nash equilibrium. Since preferences are strict, for each agent $i \in N$, any non-degenerate lottery is dominated by the top alternative obtained with probability one. On the other hand, suppose that $g(m) = \bar{x} \in A$. Given that $m_j = \left(\bar{x}, \bar{\theta}, \cdot\right)\ \forall j \neq i$, we know that $\bar{x} \in f(\bar{\theta})$. If x is a Nash equilibrium outcome, then $LC_j(\theta, x) = A$ for each $j \neq i$. Moreover, $LC_i(\bar{\theta}, x) \subseteq LC_i(\theta, x)$. By *Maskin monotonicity*, $x \in f(\theta)$.

3) $g(m)$ is given by *Rule 1.*

If θ is announced truthfully, then the outcome lies in $f(\theta)$ by *Rule 1.* Suppose instead that each $i \in N$ announces $m_i = (x, \phi, \cdot)$ with $\phi \neq \theta$ and $x \in f(\phi)$. Any agent $i \in N$ could deviate and obtain $(1-\epsilon)\,x + \epsilon b$ for any $b \in A$ such that $xP_i(\phi)b$. If there are no such deviations, it is the case that $LC_i(\phi, x) \subseteq LC_i(\theta, x)$ for each $i \in N$. By *Maskin monotonicity*, $x \in f(\theta)$.

Q.E.D.

Discussion:

We discuss now the relaxation of two of our assumptions: *strictness* and *unanimity.* These two assumptions are in fact not necessary for the theorem to hold.

1) Suppose *strictness* is satisfied. It is possible to drop the assumption of *unanimity.* We need the SCC to be *unanimous* only when message $m \in M$ reported is such that the outcome is given by *Rule 3.* In such a case, if the alternative $g(m)$ given by the mechanism is top ranked for everyone, there

are no profitable deviations from m. The modified version of *Rule 3* is the following:

Rule 3.1: In all other cases, the outcome is $\left(1-\frac{1}{n_{i^*}}a\right)+\frac{1}{n_{i^*}}b$ if $m_{i^*}=(a,\cdot,b,n_{i^*})$,where $i^*=\min\{i\in N : n^i\geq n^j\ \forall j\in N\}$, and $a\neq b$. Otherwise, the outcome is the lottery that assigns equal weights on all the alternatives in A.

If the outcome is a lottery on a and b, since preferences are strict, even if agents agree on the ranking of a and b –with a being top ranked– by *monotonicity in probabilities*, any agent $i\in N$ would like to announce a higher integer in order to give more weight to a and relatively less weight to b. Thus, we exploit the fact that the set of such profitable deviations is open to guarantee that $\left(1-\frac{1}{n_{i^*}}a\right)+\frac{1}{n_{i^*}}b$ is not an equilibrium outcome. Next, if $g(m)=\frac{1}{\ell}a+...$ is the outcome, again by *monotonicity in probabilities*, any agent $i\in N$ would deviate to obtain a lottery restricted on two alternatives.

2) What happens if *strictness* is not satisfied? Suppose that *no-total indifference* is satisfied[8] (see Serrano-Vohra, 2004). Indeed, this is not enough. Suppose that in *Rule 3*, the outcome is $\left(1-\frac{1}{n_{i^*}}a\right)+\frac{1}{n_{i^*}}b$ and those two allocations are top ranked for every agent (i.e. each $i\in N$ is indifferent between a and b). Therefore, $\left(1-\frac{1}{n_{i^*}}a\right)+\frac{1}{n_{i^*}}b$ would be an equilibrium outcome. This is an undesirable equilibrium outcome: even if we assume *unanimity*, the SCC f is deterministic. The same thing happens if the outcome is $(1-\epsilon)\bar{x}+\epsilon b$ determined by *Rule 2*. Therefore, *unanimity* is enough only if f is such that for every profile $\theta\in\Theta$ and every alternative $a,b\in f(\theta)$ we have that,

$$(1-\varepsilon)a+\varepsilon b\in f(\theta)\ \forall\varepsilon\in(0,1)\,.$$

Arguably, this is a defendable position. If more than one alternative is selected by the planner for a given preference profile, it means that she does not discriminate between these alternatives. They should be seen as equivalent from society's point of view, for that profile. Obtaining one of them with probability one or each with positive probability should be seen as equivalent by the planner. However, this reasoning has not been followed in the

[8]No-total indifference states that for each $\theta\in\Theta$, and each $i\in N$, there exists a,$b\in A$ such that $aP_i(\theta)b$. That is, no agent is indifferent between all alternatives.

literature so far[9]. If we want to stick with the initial assumption of deterministic SCC, *no-total indifference* is clearly not enough. If we consider the mechanism of theorem 2, then *unanimity* is necessary. We need to introduce an additional definition:

For each $\theta \in \Theta$, and each $i \in N$, let $TOP_i(\theta) \equiv \{a \in A : aR_i(\theta)b \text{ for each } b \in A\}$. The following assumption should be added:

Top strict difference: For each $\theta \in \Theta$, and for each $a \in A$ such that $LC_i(\theta, a) = A$ for at least $(n-1)$ agents $i \in N$, there exist $j, k \in N$ for whom $TOP_j(\theta) = TOP_k(\theta) = \{a\}$.

Combined with *unanimity*, this rules out undesirable equilibrium outcome in *Rule 2* of the mechanism. To see this, suppose that $\theta \in \Theta$ is the true state, and that $(1-\epsilon)\bar{x} + \epsilon b$ is the outcome of the game following the report of the message profile $m \in M$. Remember that only $(n-1)$ agents, say j, can trigger the integer game. In case $LC_j(\theta, b) = A$ for each j but possibly $i \in N$, *top strict difference* guarantees that at least one j is such that $TOP_j(\theta) = \{b\}$, and therefore guarantees that a profitable deviation exists: agent j can trigger the integer game by announcing the highest integer such that $\frac{1}{n_j} < \epsilon$, and obtain b with probability one. Moreover, if an outcome is determined by *Rule 3*, again *top strict difference* guarantees that there are in fact no undesirable equilibria in that region of the message space.

To conclude, *unanimity* can again be dropped by using the modified *Rule 3.1* suggested above. If outcomes come from *Rule 3.1*, *top strict difference* guarantees that there are in fact no equilibria in that region of the message space.

[9] Suppose that there is an SCC f that the planner wants to implement. The traditionnal views are to perform exact implementation or virtual implementation. However, consider the following alternative notion of implementation. A restricted version of the class of lottery mechanisms is used in which every outcome is a non-degenerate lottery. How does it differ from Abreu-Sen (1991)'s paper on virtual implementation? The difference is that equilibrium outcomes should be lotteries on alternatives in the original SCC f. Based on the discussion concerning the equivalence of alternatives in the SCC at a given preference profile, we see our suggestion as a modified exact implementation problem. Moreover, it could solve the coordination problem that is proeminent in the literature. By considering social choice functions that contain, for each preference profile, a lottery on alternatives initially selected at that profile, one may not need integer game anymore.

4 Examples and Extensions

First, we provide examples of *Maskin monotonic* SCCs that violate *no-veto power*. We then apply our method (example 3) to voluntary implementation (Jackson-Palfrey, 2001). Among others, the core correspondence, the individually rational correspondence (henceforth IR correspondence) in voting problems and assignment of indivisible goods, the stable rule in matching problems fail to satisfy *no-veto power*. For all these rules, Maskin's theorem does not allow to check wether or not they are Nash implementable. The message of Theorem 2 and the discussion that followed is that, in the environment we consider, any *Maskin monotonic* SCC is Nash implementable by a *lottery mechanism*. However, it is important to note that a lottery mechanism in the way we have defined it will not implement "multidimensional" rules in which each agent gets a different alternative. This is the case for assigning indivisible goods or matching problems. For instance, the stable rule in the context of marriage problems[10]is an example in which each man is assigned to a woman and each woman is assigned to a man. An individual will be indifferent between two different matching in which he gets his most preferred mate. If *strictness* and *unanimity* are satisfied, it is enough to require that an agent obtains a lottery on two different matching only if her mate is different for the two matching. That is, suppose μ is a matching. Let Λ be the set of matching.

Rule 2.1: If $m_j = \left(\bar{\mu}, \bar{\theta}, \bar{\mu}', .\right)\ \forall j \neq i$ and $m_i = (\mu, \phi, \mu', \cdot) \neq m_j$, then the outcome is

$$g(m) = \begin{cases} (1-\epsilon)\bar{\mu} + \epsilon\mu' \text{ if } \mu'_i \neq \bar{\mu}_i \text{ and } \mu'_i \in LC_i(\bar{\theta}, \bar{\mu}_j) \\ \bar{\mu} \text{ otherwise} \end{cases}$$

If indifferences are allowed, then *top strict difference* should be redefined accordingly. Moreover, the mechanism has to be further complicated. In *Rule 2*, if $g(m) \in \mathcal{L}$, then among the $(n-1)$ agent $j \neq i$, it should be the case that at least one is not indifferent between both matching. Hence, this requirement should be added to *Rule 2.1*.

Top strict difference for marriage problems: For each $\theta \in \Theta$, and each matching $\mu \in \Lambda$ such that $\mu_i \in TOP_i(\theta)$ for each $i \in N$, there exists $j, k \in N$ for whom $TOP_j(\theta) = \left\{\mu' : \mu'_j = \mu_j\right\}$ and $TOP_k(\theta) = \{\mu' : \mu'_k = \mu_k\}$.

[10]See Kara-Sonmez (1995) for a precise defintion of the stable rule.

Rule 2.2: If $m_j = \left(\bar{\mu}, \bar{\theta}, \bar{\mu}', .\right) \; \forall j \neq i$ and $m_i = (\mu, \phi, \mu', \cdot) \neq m_j$, then the outcome is $g(m) = (1-\epsilon)\bar{\mu} + \epsilon\mu'$ if,

$$\begin{cases} \text{1) } \mu'_i \neq \bar{\mu}_i \text{ and } \mu'_i \in LC_i(\bar{\theta}, \bar{\mu}_j) \text{ and,} \\ \text{2) if } \mu'_j \in TOP_j(\phi) \text{ for each } j \neq i \text{, then there exists } k \neq i \text{ for whom } \bar{\mu}_k \notin TOP_k(\phi); \\ \text{or if } \bar{\mu}_j \in TOP_j(\phi) \text{ for each } j \neq i \text{, then there exists } k \neq i \text{ for whom } \mu'_k \notin TOP_k(\phi). \end{cases}$$

Otherwise, $g(m) = \bar{\mu}$.

Lottery mechanisms may loose their appeal for that type of problems once specific forms of indifferences are allowed. However, note that allowing indifferences but requiring that the top be strict for at least $(n-1)$ agent simplifies the problem and we would not need to have a complicated version of *Rule 2*. Allowing indifferences in the way of *top strict difference for marriage problems*, entails in fact to construct personalized attainable sets like in Moore-Repullo (1990).

We now consider an example of the IR rule in models of assignments of indivisible objects –without monetary transfers– in which preferences of agents are strict[11].

Example 2: Assignment of (indivisible) objects and the individually rational rule.

$N \equiv \{1,2,3\}$ and $A \equiv \{a_1, a_2, a_3\}$. Each agent is endowed with only one object. Let ω_i be the endowment of agent $i \in N$, and Ω be the endowment set. We have $\Omega = \{a_1, a_2, a_3\}$. An allocation is an assignment of indivisible objects, one to each agent. Formally, an assignment is a bijection $\sigma : N \to A$. Let Z be the set of assignments. The IR rule is a correspondence such that for each $\theta \in \Theta$, $f(\theta) = \{\sigma \in Z : \sigma_i R_i(\theta) \omega_i \text{ for each } i \in N\}$. Consider the following (strict) preferences over objects.

θ			ϕ		
1	2	3	1	2	3
a_2	a_3	a_3	a_2	a_3	a_1
a_1	a_1	a_1	a_1	a_1	a_3
a_3	a_2	a_2	a_3	a_2	a_2

[11] The original example in an abstract setting is due to Maskin (1985).

The selection operated by f is $f(\theta) = \{(a_1, a_2, a_3), (a_2, a_1, a_3)\}$, and $f(\phi) = \{(a_1, a_2, a_3), (a_2, a_1, a_3), (a_2, a_3, a_1)\}$. If the SCC satisfies *no-veto power*, the allocation (a_2, a_3, a_1) should be selected in θ since $LC_1(\theta, a_2) = LC_2(\theta, a_3) = A$. But a_1 is not individually rational for agent 3. Thus, the IR correspondence violates *no-veto power*. This correspondence satisfies *unanimity* and is therefore implementable by the mechanism of Theorem 2 in which *Rule 2* is replaced by *Rule 2.1*.

Example 3: Voluntary implementation

Our approach is not restricted only to Maskin's theorem. In a recent paper, Jackson-Palfrey (2001) consider voluntary implementation. The problem related to the enforceability of the outcome function out of equilibrium is studied. Agents are not forced to accept the outcome of the mechanism. In particular, agents are allowed to veto some subset of the set of alternatives. For instance, a state-contingent participation constraint defines a mapping from outcomes vetoed by agents into individually rational outcomes. First, we need to introduce some additional definitions[12]. Let $\mathcal{F}$ be the set of all social choice correspondences –possibly single-valued– over A. A reversion function is a mapping $h : \Theta \to A$ that indicates what the outcome is in the case of a veto by at least one individual. A reversion function h induces a mapping $H : A \times \Theta \times \mathcal{F}$ by

$$H(a, \theta, h) = \begin{cases} a \text{ if } aR_i(\theta)h(\theta) \text{ for each } i \in N \\ h(\theta) \text{ otherwise.} \end{cases}$$

Given a game form (M, g), a message profile m is an h-Nash equilibrium of (M, g) at θ if for each agent $i \in N$,

$$H(g(m), \theta, h)R_i(\theta)H(g(m_i', m_{-i}), \theta, h) \ \forall m_i' \in M_i.$$

A SCC f is h-Nash implementable if there exists a mechanism (M, g) such that, for all $\theta \in \Theta$:

(1) For each $a \in f(\theta)$ there exists an h-Nash equilibrium, $m \in M$, such that $H(g(m), \theta, h) = a$.

(2) If $m \in M$ is an h-Nash equilibrium at θ, then $H(g(m), \theta, h) \in f(\theta)$.

An analog to Maskin monotonicity is derived. A SCC f is *reversion-monotonic relative to h* if for each $\theta \in \Theta$, and each $a \in f(\theta)$, there exists $z \in A$ such that

[12] We follow the notations introduced by Jackson-Palfrey (2001).

1. $H(z,\theta,h) = a$.

2. For all $\phi \in \Theta$ such that $H(z,\phi,h) \notin F(\phi)$, there exists $y \in A$ and $i \in N$ such that $H(z,\theta,h)R_i(\theta)H(y,\theta,h)$ and $H(y,\phi,h)P_i(\phi)H(z,\phi,h)$.

Reversion monotonicity is indeed necessary for h-Nash implementation. Coupled with $h-$*no-veto power*, it is also sufficient, provided $n \geq 3$. Fix $i \in N$. A SCC f satisfies $h-$*no-veto power* if for each pair $(a,\theta) \in A \times \Theta$,

$$[H(a,\theta,h)R_j(\theta)H(b,\theta,h) \text{ for each } j \neq i, \text{ each } b \in A] \implies [H(a,\theta,h) \in f(\theta)].$$

Again, $h-$*no-veto power* is not necessary for h-Nash implementation. The planner cannot know whether a *reversion-monotonic relative to h* SCC that violates *no-veto power* is h-Nash implementable. An interesting feature of the voluntary implementation approach is that it is possible to construct *non-Maskin monotonic* SCCs that are, given h, *reversion-monotonic relative to h*. We construct an example of such a SCC that also violates *no-veto power*. It is a variant of an example of Jackson-Palfrey (2001)[13]. The example is as follows:

$N \equiv \{1,2,3\}$, $A \equiv \{a,b,c,d\}$ and $\Theta \equiv \{\theta,\phi\}$. The status-quo is c. The reversion function is constant across states and equal to the status-quo, that is $h(\theta) = c$ for each $\theta \in \Theta$. The function H maps outcomes that are not individually rational to the status-quo. Hence, we want to perform *IR*-Nash implementation. The preferences over alternatives are described as follows.

θ			ϕ		
1	2	3	1	2	3
a	b	d	d	b	d
d	d	a	a	d	a
b	a	c	b	a	b
c	c	b	c	c	c

Let $f(\theta) = \{a,d\}$ and $f(\phi) = \{a\}$. First, observe that this correspondence is not *Maskin monotonic*. Alternative d has not gone worse in the ranking of any agent, but d is excluded from the alternatives chosen by f at ϕ. Second,

[13] The SCC in their example satisfies $h - NVP$.

this SCC is *reversion-monotonic relative to h*. We show that agent 2 experiences a *preference reversal relative to h* when going from θ to ϕ. Since alternative d is individually rational for every agent, $H(d,\theta,h) = H(d,\phi,h) = d$. However, for alternative b, by $cP_3(\theta)b$, we obtain that $H(b,\theta,h) = c$. Moreover, $H(b,\phi,h) = b$. Thus,

$$H(d,\theta,h)R_2(\theta)H(b,\theta,h) \text{ and } H(b,\phi,h)P_2(\phi)H(d,\phi,h); \text{ or}$$
$$dR_2(\theta)c \text{ and } bP_2(\phi)d.$$

Therefore, f is *reversion-monotonic relative to h*. Finally, it is easy to see that f does not satisfy $h-$*no-veto power*. For agent 1 and 3, $LC_1(\phi, H(d,\phi,h)) = LC_3(\phi, H(d,\phi,h)) = A$ but $H(d,\phi,h) = d \notin f(\phi)$. We can state the following theorem.

Theorem 3 *Suppose strictness is satisfied. Then any SCC f satisfying unanimity is $h-$implementable by a (ordinal) lottery mechanism if and only if it is reversion monotonic relative to h.*

Proof: The necessity part is omitted and can be adapted from Jackson-Palfrey (2001).

The message space of each agent $i \in N$ is as before, that is $M_i \equiv A \times \Theta \times A \times \mathbb{N}$. A typical message will be denoted $m_i \equiv (m_i^1, m_i^2, m_i^3, m_i^4) \equiv (x, \theta, x', n_i)$. Fix a number $\varepsilon \in (0,1)$. The outcome function is described as follows:

Rule 1: If $m_i = (\bar{x}, \bar{\theta}, \bar{a}, .)$ $\forall i \in N$ and $H(\bar{x}, \theta, h) \in f(\bar{\theta})$ then $g(m) = H(\bar{x}, \theta, h)$.

Rule 2: If $m_j = (\bar{x}, \bar{\theta}, \bar{a}, .)$ $\forall j \neq i$ and $m_i = (c, \phi, b, n_i) \neq m_j$, then the outcome is

$$g(m) = \begin{cases} (1-\epsilon)H(\bar{x},\bar{\theta},h) + \epsilon H(b,\bar{\theta},h) \text{ if } H(b,\bar{\theta},h) \in LC_i(\bar{\theta}, H(\bar{x},\bar{\theta},h)) \\ H(\bar{x},\bar{\theta},h) \text{ otherwise} \end{cases}$$

Rule 3: In all other cases, $g(m) = m_{i^*}^1$, where $i^* = \min\{i \in N : n^i \geq n^j \ \forall j \in N\}$.

The proof is similar to the proof of theorem 2 and is thus omitted.
Q.E.D.

The remarks of the previous section apply again here. *Unanimity* can be dropped by complexifying *Rule 3.* Moreover, the assumption of strict preferences can also be relaxed, as before. If the SCC is on assignment of indivisible goods or on matching problems, the discussion preceding example 2 applies once more.

5 Concluding remarks

1) Extending the class of admissible mechanisms is useful for (exact) Nash implementation. Under some assumptions, it allows the planner to check only whether f is *Maskin monotonic* for its (exact) Nash implementability. Moreover, no complicated restriction on the attainable sets has to be constructed, unlike in Moore-repullo (1990).

2) It is possible to relax the *strictness* assumption. Moreover, *unanimity* of f can also be dropped by modifying *Rule 3* to *Rule 3.1.* In order to relax *strictness*, we needed to replace it by *Top strict difference*. In fact, this assumption can be further relaxed as well. The following modification of *Rule 2* is one possibility.

Rule 2.3: If $m_j = \left(\bar{x}, \bar{\theta}, \bar{a}, \cdot\right)\ \forall j \neq i$ and $m_i = (c, \phi, b, \cdot) \neq m_j$, then the outcome is

$$z' = \begin{cases} \frac{1}{n_i}\bar{x} + (1 - \frac{1}{n_i})b \text{ if } b \in LC_i(\bar{\theta}, \bar{x}) \\ \bar{x} \text{ otherwise} \end{cases}$$

The only assumption that is needed is then the following:

Unilateral top strict difference: For each $\theta \in \Theta$, for each $a \in A$ such that $LC_i(\theta, a) = A$ for at least $(n-1)$ agent $i \in N$, there exists $j \in N$ for whom $TOP_j(\theta) = \{a\}$.

However, one could argue that dropping *unanimity* and relaxing *strictness* to *unilateral top strict difference* makes the mechanism questionable. Equilibria only come from *Rule 1* and the set of profitable deviations in the other regions of the message space is open.

3) *Lottery mechanisms* can also be useful for alternative implementation approach. This is the case for voluntary implementation of Jackson-Palfrey (2001). Moreover, the same method could be applied, for instance, to subgame perfect implementation. It is easy to see now that by using a *lottery mechanism*, one can drop the assumption of *no-veto power* from Abreu-Sen's (1990) theorem on subgame perfect implementation.

4) When the problem is not "unidimensional", unlike in voting settings, the mechanism has to be modified accordingly. Moreover, relaxing indifferences imposes strong constraints. Assuming that the top of the ranking be

strict for at least $(n-1)$ agents does not reduce the power of *lottery mechanisms*. However, relaxing the assumption of strict preferences to *top strict difference* entails restrictions like in Moore-Repullo (1990): the planner needs to construct personalized attainable sets.

4) Possible extensions of this work would be to design simple lottery mechanisms to implement SCC violating *no-veto power*.

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