Subgame Perfect Implementation and the Walrasian Correspondence*

Olivier Bochet†

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Abstract

Consider a class of exchange economies in which preferences are continuous, convex and strongly monotonic. It is well known that the Walrasian correspondence, defined over such a class of economies, is not implementable in Nash Equilibrium. Monotonicity (Maskin (1999)), a necessary condition for Nash implementation, is violated for allocations at the boundary of the feasible set. However, we know since the seminal work of Moore-Repullo (1988) and Abreu-Sen (1990) that monotonicity is no longer necessary for subgame perfect implementation. We first show that the Walrasian correspondence defined over this class of exchange economies is not implementable in subgame perfect equilibrium. Indeed, the assumption of differentiability cannot be relaxed unless one imposes parametric restrictions on the environment, like assumption EE.3 in Moore-Repullo (1988).

Next, assuming differentiability, we construct a sequential mechanism that fully implements the Walrasian correspondence in subgame perfect and strong subgame perfect equilibrium.. We take care of the boundary problem that was prominent in the Nash implementation literature. Moreover, our mechanism is based on price-allocation announcements and fits the very description of Walrasian Equilibrium.

Keywords: Walrasian equilibrium, double implementation, subgame perfect equilibrium, strong subgame perfect equilibrium.

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†University of Namur and CORE.

1 Introduction

Maskin monotonicity (Maskin (1999)) is a necessary condition for implementation of social choice correspondences in Nash Equilibrium. This condition has been shown to be restrictive in some cases. For instance, for a class of exchange economies in which preferences are continuous, convex and strongly monotone, it is now well-known that the Walrasian correspondence is not monotonic, (see, e.g., Hurwicz-Maskin-Postlewaite (1995)). The violation of monotonicity occurs for Walrasian allocations that are at the boundary of the feasible set[1]. Hurwicz (1979) and Schmeidler (1980) have constructed mechanisms that implement the Walrasian correspondence but in which off equilibrium allocations may award negative quantities to some agents. Postlewaite-Wettstein (1989), Giraud-Rochon (2001), Dutta-Sen-Vohra (1995) or Tian (1992, 2000) among others, construct mechanisms that implements the (Constrained) Walrasian correspondence[2]. These mechanisms are feasible in and out of equilibrium, and (weakly) balanced. Another line of research has been focusing on strategic market games (see, e.g., Shapley-Shubik (1977) or Dubey-Shapley (1994)). Papers on non-cooperative bargaining such as Gale (1986,a and b), or more recently Kunimoto-Serrano(2002) provide full implementation of the Walrasian correspondence for economies with a continuum of agents. A boundary assumption that rules out Walrasian allocation at the boundary of the feasible sets is used. To our knowledge, Yildiz (2002), and the literature on non-cooperative bargaining cited above are the only papers that uses sequential mechanisms. Yildiz's results only cover the two players case, and use assumptions such as uniqueness (and interiority) of Walrasian equilibrium, as well as the single-peakedness of the offer curves in utility space. In our paper, we are interested in trading procedures with a finite number of agents, and with finite length mechanisms.

It is now well understood that the class of implementable social choice correspondences rapidly expands when one considers refinements of Nash Equilibrium as a solution concept. In their seminal papers, Moore-Repullo (1988) (MR in the sequel) and Abreu-Sen (1990) (henceforth, AS) show that

[1] Thomson (1999) shows (figure 3b) that if preferences are not convex, violations of monotonicity can occur also for Walrasian allocations that are in the interior of the feasible set.

[2] For implementation of the Walrasian correspondence in Nash Equilbrium, the class of economies that is considered guarantees that Walrasian allocations are interior. Otherwise, the Walrasian correspondence is substituted by its minimal monotonic extension, namely the Constrained Walrasian correspondence.

monotonicity is no longer necessary for implementation in subgame perfect equilibrium. Suppose that we have two states of the world θ and ϕ, and that outcome a is in the social choice correspondence under θ but not under ϕ. The necessary condition[3] says that there exists a test agent who experiences some preference reversal, when going from one state to the other, between two arbitrary outcomes x and y. Moreover, those two outcomes need only be linked to a, in a particular way, through a string of outcomes. On the other hand, Maskin monotonicity would say that $x = a$. So, all that is really needed for subgame perfect implementation, in exchange economies, is an agent who experiences a preference reversal between two arbitrary outcomes when going from one state to the other. In particular, it is shown in MR (1988) that the Walrasian Correspondence is implementable in subgame perfect equilibrium. They consider a domain of exchange economies in which preferences are continuous, convex and monotonic. We first show that, without further restrictions, the Walrasian correspondence defined over this class of economies is not in general subgame perfect implementable. The necessary sequence of outcomes cannot be constructed: differentiability is a crucial assumption. Hence, unless one imposes restrictions on the environment such as assumption EE.3 in MR[4], implementation of the Walrasian correspondence is not possible. Assumption EE.3 is necessary if one does not impose differentiability, but imposing differentiability does not imply it: assumption EE.3 is sufficient but not necessary in such a case. Differentiability is actually enough to obtain the local information necessary to elicit whether or not a boundary allocation is Walrasian.

Next, imposing differentiability, the construction of an economically appealing mechanism remains. A canonical mechanism is constructed in MR and AS but, as any general mechanism, it fundamentally lacks economic interpretation. Besides, their game form is complicated and has infinite message spaces. It involves each agent reporting the entire preferences profile[5], and announcing an integer at each stage of the mechanism. Parallel to the literature on Nash implementation, we expect that the design of tailor-made sequential mechanisms is more appealing. Our aim is to provide an alternative —and simpler— game form, that is based on the Walrasian notion of allocation and prices. Moreover, it should fit the very description of Walrasian equilibrium. Given the assumptions on preferences we make, Walrasian allocations exist not only in the interior but also at the boundary

[3] See condition C from MR in section 2.

[4] See assumption EE.3 from MR in section 2 below.

[5] That is, each agent reporting her own preferences, as well as the preferences of all the other agents in the economy.

of the feasible sets. Our construction thus takes care (for the first time) of the boundary problem that was prominent in the Nash implementation literature. Moreover, our game form doubly implements the Walrasian correspondence in subgame perfect and strong subgame perfect equilibrium.

The plan of the paper is as follows. Section 2 presents the class of exchange economy we consider and the notations needed for the paper. In section 3, we highlight the problem that may occur if differentiability is not assumed. In section 4, we present the mechanism and the implementation result. Finally, we provide some final comments in section 5.

2 The set-up

2.1 Economic environments

There are L infinitely divisible goods and a set of agents $N = \{1, \ldots, n\}$, with $n \geq 3$. The consumption set of each agent $i \in N$ is $X_i = \mathbb{R}^L_+$. For each agent $i \in N$, R_i is the complete and transitive binary relation on $\mathbb{R}^L_+$ indicating (weak) preferences. The associated strict preference and indifference relations are denoted by P_i and I_i, respectively. The set of possible preferences of each agent $i \in N$ is defined by $\mathcal{R}_i$. Denote by $\mathcal{R} = \prod_i \mathcal{R}_i$ the set of possible preference profiles. A typical preference profile is $R = (R_i)_{i \in N} \in \mathcal{R}$.

Apart from his preferences and consumption set, each agent $i \in N$ is also characterized by his individual endowment[6] $\omega_i > 0$. The aggregate endowment is $\bar{\omega} \gg 0$.

The only characteristics unknown to the planner are the preferences of agents. For each agent $i \in N$, X_i and ω_i are known to the planner and fixed. Only the preferences of agents can vary. An economy is thus a list of preference relation, one for each agent. Formally, an economy is $R = (R_i)_{i \in N} \in \mathcal{R}$. We consider two classes of economies.

Class of economy $\mathcal{T}$: for each agent $i \in N$, every $R_i \in \mathcal{R}_i$ is continuous, convex and strongly monotonic[7].

Class of economy $\mathcal{E}$: for each agent $i \in N$, every $R_i \in \mathcal{R}_i$ is continuous, convex, strongly monotonic and representable by a differentiable utility function.

[6] We order vectors with the usual conventions, $\gg$, $>$, $\geq$.

[7] A preference relation R_i defined over $\mathbb{R}^L_+$ is convex if, for every x_i and $y_i \in \mathbb{R}^L_+$ such that $x_i \; P_i \; y_i$, we have that $\lambda x_i + (1-\lambda) y_i \; P_i \; y_i$ for every $\lambda \in (0, 1]$.

A preference R_i defined over $\mathbb{R}^L_+$ is strongly monotonic if, for each x_i and $y_i \in \mathbb{R}^L_+$, $x_i > y_i$ implies that $x_i \; P_i \; y_i$.

A (feasible) allocation is a list of bundle $(x_i)_{i\in N} \in \mathbb{R}_+^{Ln}$ such that $\sum x_i \leq \bar{\omega}$. Given an agent $i \in N$, $x_{i,l} \in \mathbb{R}_+$ stands for the quantity of good l received by agent i at bundle x_i.

The set of feasible allocations A is,

$$A = \{x \in \mathbb{R}_+^{Ln} : \sum x_i \leq \bar{\omega}\}.$$

Define by F the set of balanced allocations,

$$F = \{x \in \mathbb{R}_+^{Ln} : \sum x_i = \bar{\omega}\}.$$

Define by $P = \mathbb{R}_{++}^L$ the set of strictly positive price vectors.

For each agent $i \in N$, denote by $B_i(p)$ and $B_i(p)|_{x_i \leq \bar{\omega}}$, the budget set and the constrained budget set, respectively, of agent i at a given price $p \in P$,

$$\begin{aligned} B_i(p) &\equiv \{x_i \in X_i \mid p \cdot x_i \leq p \cdot \omega_i\} \\ B_i(p)|_{x_i \leq \bar{\omega}} &\equiv \{x_i \in X_i \mid p \cdot x_i \leq p \cdot \omega_i \text{ and } x_i \leq \bar{\omega}\}. \end{aligned}$$

For each agent $i \in N$, given a bundle $z_i \in \mathbb{R}_+^L$, denote by $\widetilde{B}_i(p, z_i)$ and $\widetilde{B}_i(p, z_i)|_{x_i \leq \bar{\omega}}$, the modified and constrained modified budget sets, respectively, of agent i at price $p \in P$,

$$\begin{aligned} \widetilde{B}_i(p, z_i) &\equiv \{x_i \in X_i \mid p \cdot x_i \leq p \cdot z_i\} \\ \widetilde{B}_i(p, z_i)|_{x_i \leq \bar{\omega}} &\equiv \{x_i \in X_i \mid p \cdot x_i \leq p \cdot z_i \text{ and } x_i \leq \bar{\omega}\}. \end{aligned}$$

Given an agent $i \in N$, a preference $R_i \in \mathcal{R}_i$ and a bundle $x_i \in X_i$, define:

$UC_i(x_i, R_i) = \{y_i \in X_i : y_i \; R_i \; x_i\}$, the upper contour set at x_i.
$LC_i(x_i, R_i) = \{y_i \in X_i : x_i \; R_i \; y_i\}$, the lower contour set at x_i.
$SLC_i(x_i, R_i) = \{y_i \in X_i : x_i \; P_i \; y_i\}$, the strict lower contour set at x_i.
$SUC_i(x_i, R_i) = \{y_i \in X_i : y_i \; P_i \; x_i\}$, the strict upper contour set at x_i.
$I_i(x_i, R_i) = \{y_i \in X_i : x_i \; I_i \; y_i\}$, the indifference curve through x_i.

Finally, given the assumptions on preferences[8], and given a preference profile $R = (R_i)_{i\in N} \in \mathcal{R}$, an allocation $x^* \in F$ is a Walrasian allocation if there exists $p \in P$, such that for each $i \in N$, $x_i^* \in B_i(p)$ and $x_i^* \; R_i \; y_i$, for every $y_i \in B_i(p)$.

The Walrasian correspondence $WE : \mathcal{R} \twoheadrightarrow A$ associates to each economy $R = (R_i)_{i\in N} \in \mathcal{R}$ its set of walrasian allocations $WE(R)$.

[8] In both classes, preferences are strongly montonic. It implies that at a Walrasian equilibrium (x, p), $p \gg 0$.

Before proceeding to the next section, we recall the necessary condition for subgame perfect implementation introduced in MR, as well as assumption EE.3 made on economic environments.

Condition C: For each $R = (R_i)_{i\in N} \in \mathcal{R}$ and $R' = (R'_i)_{i\in N} \in \mathcal{R}$, $R' \neq R$[9], and each allocation $a \in WE(R)$, $a \notin WE(R')$, there exists a finite sequence of allocations $\{a_0 = a, a_1, ..., a_k, a_{k+1}\}$ such that the following is true:

a) For each $l = 0, ..., k-1$, there exists an agent j_l for whom

$$a_l \; R_{j_l} \; a_{l+1}.$$

b) There is some particular agent j_k, with $R_{j_k} \neq R'_{j_k}$, for whom

$$a_k \; R_{j_k} \; a_{k+1} \text{ and } a_{k+1} \; P'_{j_k} \; a_k.$$

EE.3: $R = (R_i)_{i\in N} \in \mathcal{R}$ and $R' = (R'_i)_{i\in N} \in \mathcal{R}$, $R' \neq R$, there exists an agent $i \in N$ and two allocations x and y, $x, y \gg 0$, and such that

$$x_i \; P_i \; y_i \text{ and } y_i \; P'_i \; x_i.$$

Notice that we do not assume EE.3. This assumption ensures that there is at least one agent experiencing a preference reversal inside the feasible sets. We will underline below that, despite its necessity for implementation of the Walrasian correspondence over the class $\mathcal{T}$, it is only sufficient over the class $\mathcal{E}$.

2.2 Game-form: definitions and notations

An extensive game form or mechanism is a game tree with possibly simultaneous moves. More formally, it is defined as an array $\Gamma = (N, T, g)$ where N is the set of players, T a game tree, and g is an outcome function that associates a feasible allocation with each path of play. The set of nodes of the tree T is denoted S. The initial node is s_0. The set of terminal nodes of the tree T is denoted Z. Let M_i be the set of strategies –messages– of player i, and let M_i^s denote the set of strategies available to player i at node s. Denote $M = \prod_i M_i$. Suppose the strategy profile $m \in M$ is played. Let $g(m)^i$ stand for bundle obtained by agent $i \in N$ at the allocation prescribed by the path induced by m, that is, $g(m)$. Let $g(m, s)$ denotes the outcome corresponding to m starting at node s. As is common in the implementation literature, we confine our attention to pure strategies.

[9] That is, $R'_i \neq R_i$ for at least an agent $i \in N$.

Given an economy $R = (R_i)_{i \in N} \in \mathcal{R}$, the mechanism Γ defines an extensive game form (Γ, R). A subgame perfect equilibrium of (Γ, R) is a strategy profile $m^* \in M$ such that for all $s \in S \backslash Z$ and for all $i \in N$,

$$g(m^*, s)^i \ R_i \ g(m_i, m^*_{-i}, s)^i \ \forall m_i \in M_i.$$

For each $R \in \mathcal{R}$, the set of subgame perfect equilibrium outcomes of (Γ, R) is denoted $SPE(\Gamma, R)$.

A strong Nash equilibrium of (Γ, R) is a strategy profile $m^* \in M$ such that no coalition of agents have an incentive to deviate simultaneously. That is, for every $m' \in M$, coalition $S \subseteq N$, if $m_i = m'_i$ for each $i \in N \backslash S$, then there is $j \in S$ such that,

$$g(m^*)^i \ R_i \ g(m')^i.$$

A strong subgame perfect equilibrium of (Γ, R) is a strategy profile m^* such that for each proper subgame, the profile of strategies is a strong Nash equilibrium in that subgame. For each $R \in \mathcal{R}$ the set of strong subgame perfect equilibrium of (Γ, R) is denoted $SSPE\,(\Gamma, R)$.

Given a class of economies $\mathcal{H}$, an extensive game form Γ is said to doubly implement in subgame perfect and strong subgame perfect equilibrium the Walrasian correspondence if

$$SSPE(\Gamma, R) = SPE(\Gamma, R) = WE(R) \ \forall R \in \mathcal{H}.$$

Now, for each agent $i \in N$, select an $\varepsilon_i \in \mathbb{R}^l_+$ such that $\omega_i - \epsilon_i \in \mathbb{R}^l_+ \backslash \{0\}$ (note that, for each agent i, such an ϵ_i exists since we assumed that $\omega_i > 0$ $\forall i \in N$).

We need to introduce one last piece of notation. Define by $\Pi : N \longrightarrow N$ the set of one-to-one functions from the set of agents into itself. Let us define by $f(\pi)$ the composition of the permutations, where $\pi = (\pi_i)$, $\pi_i \in \Pi \ \forall i \in N$. Therefore, $f(\pi) = \pi_1(\pi_2(\ldots(\pi_i \ldots (\pi_n))\ldots)$ stands for the ordered composition of all permutation of π. We call $f(\pi)$ a (endogenously determined) protocol. As we shall confine our attention to pure strategies, notice that any agent $i \in N$, by making a unilateral change from π_i to $\pi_i^{'}$, can induce any protocol from the composition. The use of permutations is in effect quite similar to an integer game or a modulo game. In our case, it captures an idea of anonymity of the mechanism, in the sense that the equilibria should be independent of protocols. Permutations were used first in a different fashion in Thomson (1992). It was then extended by Serrano-Vohra (1997). We make use of this extension.

3 The need for differentiability

Without stressing the obvious, we briefly clarify here a point that was made in MR. In section 6.3 of their paper, they show that if one considers a class of economies in which preferences are continuous, convex and monotonic, the (full) Walrasian correspondence is subgame perfect implementable. Implicitly behind this statement lies the assumptions made on economic environments, in particular assumption EE.3. We show below that without further restrictions (in particular, without EE.3), the Walrasian correspondence is not, in general implementable, in subgame perfect equilibrium[10]. If one allows indifference sets to have kinks, condition C may not be satisfied: the sequence of outcomes identified there cannot be constructed inside the feasible set. Imposing smoothness of indifference curves guarantees that, if an allocation is Walrasian in one state but not in another, local information around that allocation and inside the feasible set can be used to construct this sequence of outcomes; and this without assuming EE.3. After all, having $R \neq R'$ does not necessarily mean that an allocation x that is in the SCC under R should be removed under R'. Preferences could change in such a way that lower contour sets expands outside of the feasible sets, keeping x in the SCC under R', without the need for EE.3 to be satisfied. It seems clear that while with non-differentiability, EE.3 is a necessary assumption for implementation of the Walrasian correspondence, it is not if one adds differentiability. This will be highlighted below in example 1.

It is of importance to note that the problem we underline here also pertains to other competitive concept such as the Lindhal correspondence. Once the reader has understood example 1, it is easy to extend the problem to the Kolm triangle and Lindhal allocations as shown below in figure 3.

We construct an example of an economy with two agents and two goods, for which it is not possible to identify the sequence of outcomes required by condition C in MR, for subgame perfect implementation. The example extends easily to more than two agents economies.

Proposition 1: *The Walrasian correspondence is not implementable in subgame perfect equilibrium in the class of economy* $\mathcal{T}$

Proof: We construct the following example.

Example 1: $n = l = 2$. There are only two different possible preference profiles $R = (R_1, R_2)$ and $R' = (R'_1, R_2)$. The preferences are represented by utility functions as follows.

[10] Or in any other solution concepts.

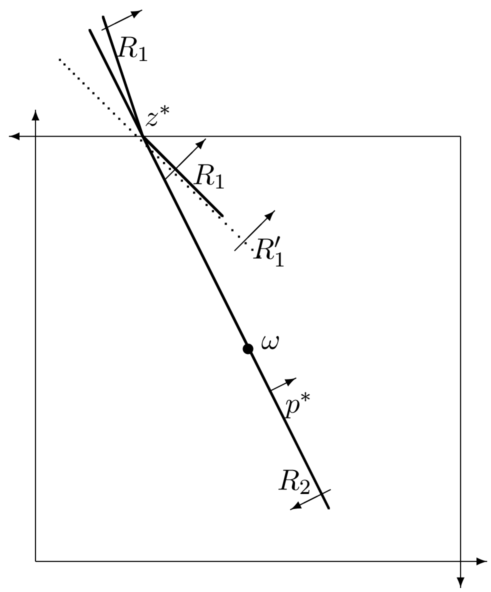

Figure 1

$u_1(\cdot, R_1) = \left\{ \begin{array}{c} x_1 + \frac{1}{3}y_1 \text{ if } y_1 \geq 4 \\ x_1 + y_1 \text{ if } y_1 < 4 \end{array} \right\}$ and $u_1(\cdot, R_1') = x_1 + y_1$.
$u_2(\cdot) = 2x_2 + y_2$
$\omega_1 = \omega_2 = (2, 2)$.

When the profile is R, there exists a Walrasian equilibrium (z^*, p^*) on the boundary of the feasible set.

$z^* = ((1, 4); (3, 0))$ and $p^* = (2, 1)$.

However, (z^*, p^*) is not a Walrasian Equilibrium under R'. Agent 1 would prefer bundles outside of the feasible set. For instance, the bundle $(\frac{1}{2}, 5)$ is affordable at p^* and gives a utility of $5.5 > 5$.

The situation is depicted graphically in figure 1 above.

Since (z^*, p^*) is not a Walrasian Equilibrium under R', implementability in subgame perfect equilibrium requires that there exists a sequence of outcomes $\{z^*, ..., a, b\}$ and a test agent $i \in N$ such that

$$a \; R_i \; b \text{ but } b \; P_i' \; a.$$

Since agent 2 has the same preferences in both profiles, agent 1 has to be the test agent. But notice that the preferences of agent 1 differ only outside of the feasible set. Hence, such a sequence cannot be constructed unless one

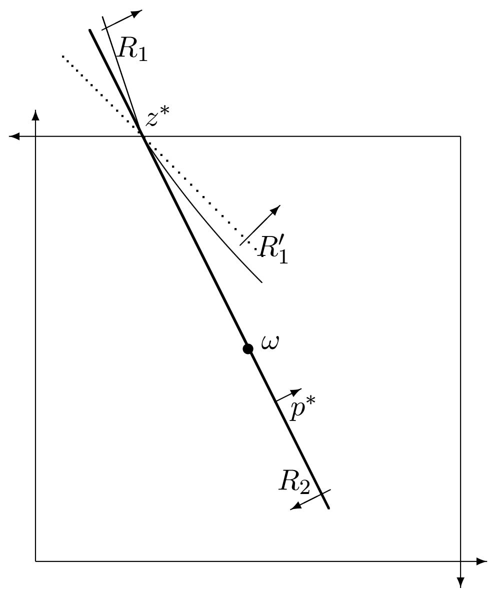

Figure 2

allows for infeasibility, which is of course ruled out in our paper. The problem stems from the non-differentiability of $u_1(\cdot, R_1)$ at z_1^*. Without further restrictions on $\mathcal{T}$, the Walrasian correspondence is not implementable.

Q.E.D.

Assuming differentiability does not imply assumption EE.3. Implementability of the Walrasian correspondence and non-differentiability implies that one has to parametrize the environment in a way similar to EE.3. Excluding kinked indifference curves, we do not need further restriction on the environment. If, given two economies, an allocation is Walrasian in one but not in the other, differentiability guarantees that local information around that allocation can be used to construct a (feasible) sequence of outcomes. This can be seen in figure 2.

The indifference curves of agent 1 under preference R_1 do not have kinks anymore. The allocation z^* is not Walrasian under R' but we can now identify the sequence of outcome $\{z^*, x, y\}$, as shown in the graph. The allocations x and y can be used to show a preference reversal for agent 1 when going from R_1 to R'_1. When differentiability is imposed, the indifference curves going through z^* under R_1 and R'_1 have to be different around z^* (and inside the feasible set), if z^* is no longer Walrasian at R'_1.

In figure 3 above, we see that this problem applies as well to the Lindhal correspondence. Hence, in the class of economy $\mathcal{T}$, the Lindhal correspon-

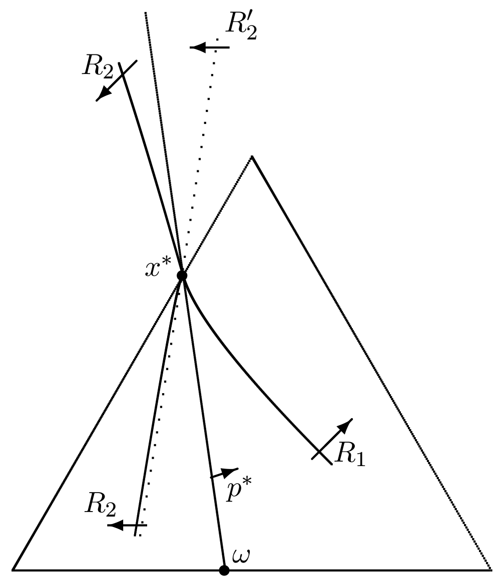

Figure 3

dence is not implementable either.

4 Taking care of the boundary problem

We consider the class of economy $\mathcal{E}$ in which preferences are continuous, convex, strongly monotonic and representable by differentiable utility functions. We construct a mechanism that doubly implements the Walrasian correspondence in subgame perfect and strong subgame perfect equilibrium. As it was shown by Hurwicz and Al. (1995) –and this fact is now well-known–, the Walrasian correspondence defined over our class of economies violates monotonicity for Walrasian allocations that are at the boundary of the feasible set. It is not implementable in Nash Equilibrium. However, since the work of MR and AS, we know that monotonicity is no longer necessary for subgame perfect implementation. As we saw above, without further restrictions on $\mathcal{T}$, we cannot implement the Walrasian correspondence. Having clarified this issue, the reason for constructing an alternative mechanism to the canonical game form constructed in MR and AR is clear. It is of interest to investigate the design of more tailor-made mechanisms. A simple and economically appealing mechanism that solves the boundary problem is absent from the subgame perfect implementation literature.

The mechanism we construct has three stages. It involves price-allocation announcements at the first stage, and is thus more reminiscent of a market process. Moreover, it fits closely the description of Walrasian Equilibrium. First, remember that the Walrasian correspondence is implementable in Nash equilibrium over a class of economies in which Walrasian allocations are always interior. We feel legitimate to make the game stops at stage 1 if allocations are in the interior of the feasible set[11]. If, given a price $p \in P$, an interior allocation is not Walrasian, at least one agent would like to obtain a different feasible bundle at that price. Moving along price hyperplanes fits the Walrasian story: no one should prefer any other affordable bundles. This idea was already used, for instance, in Dutta-Sen-Vohra (1995). Given an interior allocation, the information contained locally in prices is enough to determine whether or not an allocation is Walrasian. When the allocation is on the boundary of the feasible set, this device is not enough anymore. Moves along price hyperplanes can lead to infeasible bundles. Instead, we still rely on the information contained locally in prices, but we use an idea of recontracting. An agent can propose a different price vector to someone receiving a bundle that is in the interior of his consumption set. These prices, along with the (boundary) allocation agreed upon at stage 1, generates new budget sets —hence the idea of recontracting. If a boundary allocation is Walrasian with price p, then a different price p', with this allocation as reference point, will automatically generates agents who would like to retrade. On the other hand, if an allocation is not Walrasian and preferred bundles are infeasible, it is possible to propose a different price vector such that at least one agent –among the agents who receive strictly positive bundles– would want to retrade. To understand this, take a look at figure 4 below.

The graph is as in figure 2. The pair (z^*, p^*) is a Walrasian equilibrium at R but not at R'. The price $p' \neq p$ is such that for every feasible bundles $x_1 \neq z_1^*$ with $p^* \cdot x_1 = p^* \cdot \omega_1$, we have that $p' \cdot x_1 < p' \cdot z_1^*$. When agent 1 has preferences R_1, there exists x_1 such that $x_1 \; P_1 \; z_1^*$ and $p' \cdot x_1 = p' \cdot z_1^*$. However, when agent 1 has preferences R_1', such a feasible and budget-balancing x_1 does not exist. This indicates that (z^*, p^*) is not Walrasian at R'. For if (z^*, p^*) was in fact a Walrasian equilibrium at R', any $p' \neq p$ such that $p^* \cdot x_1 = p^* \cdot \omega_1$ and $p' \cdot x_1 < p' \cdot z_1^*$, for any feasible $x_1 \neq z_1^*$, would create profitable retrading opportunities for agent 1.

The intuition developed with figure 3 is exactly what we use in our mechanism for allocations that are at the boundary of the feasible set. The retrading device is used at stage 2.

We shall now present formally the mechanism we use.

[11] Hence, if the domain of economies is such that Walrasian allocations are interior, our mechanism never goes beyond the first stage.

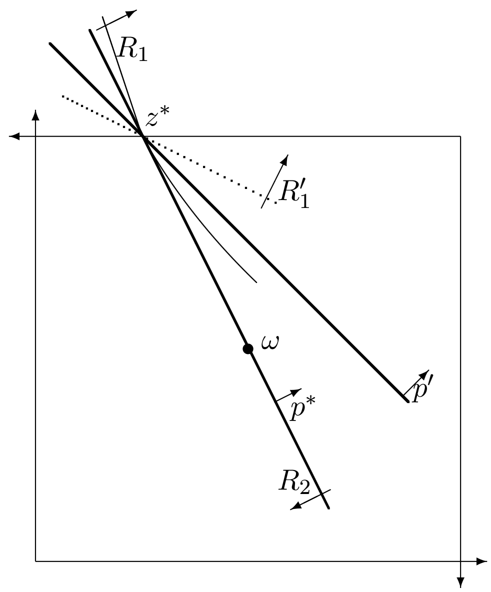

Figure 4

Mechanism Γ:

<u>*Stage 1:*</u> $m^i = (x, p, \pi)^i \in F \times P \times \Pi$ such that $\forall i \in N$, $p^i \cdot x^i_j = p^i \cdot \omega_j$ $\forall j \neq i$. If,

1) $(x, p)^i = (\bar{x}, \bar{p})$ $\forall i \in N$ and $\bar{x}_i \gg 0$ $\forall i \in N$, the game stops and the outcome implemented is $\bar{x}$.

2) $(x, p)^i = (\bar{x}, \bar{p})$ $\forall i \in N$ and $\bar{x}_{j,l} = 0$ for some j and l, then go to stage 2.

3) $(x, p)^j = (\bar{x}, \bar{p})$ $\forall j \neq i$, $i \neq f_n(\pi)$, $m^i = (x', p') \neq (\bar{x}, \bar{p})$. If $\bar{p} \cdot x'_i = \bar{p} \cdot \omega_i$, then agent i gets x'_i. Agent $j = f_n(\pi)$ gets the 0 bundle and the other divide the rest equally.

Otherwise if $\bar{p} \cdot x'_i \neq \bar{p} \cdot \omega_i$, each agent $k \in N$ receives his endowment ω_k.

4) In all other cases, agent $j = f_n(\pi)$ receives $\omega_j - \epsilon_j$. Each agent $i \neq \{f_n(\pi), f_1(\pi)\}$ receives ω_i and agent $k = f_1(\pi)$ receives $\omega_k + \epsilon_j$.

<u>*Stage 2:*</u> Agent $f_1(\pi)$ selects an agent $f_i \neq f_1(\pi)$ and announces $p' \in P$.

1) $p' \neq \bar{p}$, $\bar{x}_{f_i} \gg 0$ and p' is such that there exists feasible bundles $y_{f_i} \neq \bar{x}_{f_i}$, with $\bar{p} \cdot y_{fi} = \bar{p} \cdot \omega_i$ and $p' \cdot y_{f_i} < p' \cdot \bar{x}_{f_i}$. Go to stage 3.

2) In all other cases, the game stops and $\bar{x}$ is implemented.

<u>*Stage 3:*</u> Agent f_i chooses between,

$$q_{f_i} \in \left\{q_{f_i} \leq \bar{\omega},\, q_{f_i} \neq \bar{x}_{f_i} : p' \cdot q_{f_i} = p' \cdot \bar{x}_{f_i},\, \bar{p} \cdot q_{f_i} > \bar{p} \cdot \bar{x}_{f_i}\right\} \text{ and } \bar{x}_{f_i}.$$

If he chooses $\bar{x}_{f_i}$, he gets it. Agent $f_1(\pi)$ gets $\bar{x}_{f_1(\pi)}+\frac{1}{n-2}(\bar{\omega}-\bar{x}_{f_i}-\bar{x}_{f_1(\pi)})$. If agent $f_i \neq f_n(\pi)$, then agent $f_n(\pi)$ receives 0 and the other agents $j \notin \{f_1(\pi), f_i, f_n(\pi)\}$ divide the rest equally. Otherwise, if $f_i = f_n(\pi)$, then agent $f_{n-1}(\pi)$ receives 0 and the other agents $j \notin \{f_1(\pi), f_i, f_{n-1}(\pi)\}$ divide the rest equally.

If he chooses $q_{f_i(\pi)}$, he gets it. Agent $f_1(\pi)$ gets 0. The others divide the rest equally if any.

Theorem 1: *The extensive form mechanism Γ doubly implements in subgame perfect and strong subgame perfect equilibrium the Walrasian correspondence in the class of economies*[12] $\mathcal{E}$.

Proof: We first show that[13] $SPE(\Gamma, R) \subseteq WE(R)$. That is, we proceed to show that if m is a subgame perfect equilibrium of (Γ, R), then $g(m) \in WE(R)$. In order to prove the assertion, consider a subgame perfect equilibrium m, in which $m_i^1 = (x, p, \pi)^i$ and $g(m) = a$.

Lemma 1: $(x,p)^i = (\bar{x}, \bar{p})\ \forall i \in N$

Suppose not. We have two cases to consider.

Case 1: $(x,p)^j = (\bar{x}, \bar{p})\ \forall j \neq i$ and $i \neq f_n(\pi)$. First, if $\bar{p} \cdot x'_{f_1(\pi)} = \bar{p} \cdot \omega_{f_1(\pi)}$, agent $f_n(\pi)$ gets the 0 bundle. We can construct a profitable deviation for this agent. He deviates by appropriately announcing a permutation so as to be first in the protocol, and a different price-allocation pair. A consequence of such a deviation is that he then receives his endowment majored by a positive epsilon. Since $\omega_i > 0$ and preferences are strongly monotonic, this is a profitable deviation for agent $f_n(\pi)$. A contradiction.

Second, if $\bar{p} \cdot x'_{f_1(\pi)} \neq \bar{p} \cdot \omega_{f_1(\pi)}$, then everyone receives his endowment. But notice that any agent $j \neq i$, by modifying his permutation so as to be first in the protocol and announcing $(x', p') \neq (\bar{x}, \bar{p})$ could obtain $\omega_j + \epsilon_{f_n(\pi)}$. Since $\epsilon_k > 0$ for every $k \in N$ and preferences are strongly monotonic, this is a profitable deviation. A contradiction.

Case 2: One agent $i = f_n(\pi)$ disagrees with the other about the price-allocation pair, or more than one agent makes contradictory announcements of a price and an allocation. In such a case, agent $i = f_n(\pi)$ receives $\omega_i - \epsilon_i$ and each agent $j \notin \{f_1(\pi), f_n(\pi)\}$ receives ω_j. Any such agent $j \notin$

[12] Recall that in the class of economies $\mathcal{E}$, for each $i \in N$ and each $R_i \in \mathcal{R}_i$, preferences are continuous, convex, strongly montonic and representable by a differentiable utility functions.

[13] Since $SSPE(\Gamma, R) \subseteq SPE(\Gamma, R)$, it is enough, for the first part of the proof, to show that $SPE(\Gamma, R) \subseteq WE(R)$.

$\{f_1(\pi), f_n(\pi)\}$ could deviate by announcing a different permutation so as to be first in the protocol —modifying his announcement of a price and allocation if necessary— and receive $\omega_j + \epsilon_{f_n}$. Since $\epsilon_k > 0$ for each $k \in N$ and preferences are strongly monotonic, this is a profitable deviation for agent j, a contradiction.

Therefore, both cases lead to a contradiction with m being a subgame perfect equilibrium. A consequence of this proof is that a is individually rational.

Lemma 2: *If for each $i \in N$, $\bar{x}_i \gg 0$, then $(\bar{x}, \bar{p})$ is a Walrasian Equilibrium*

Suppose not. The game stops at stage 1. The allocation $\bar{x}$, with $\bar{x}_i \gg 0$ $\forall i \in N$, is the outcome of the game but is not Walrasian given the price $\bar{p}$. By definition of a Walrasian equilibrium, convexity of preferences and the fact that $\bar{x}$ is an interior allocation, there exists an agent i with preferences say, $R_i \in \mathcal{R}_i$, and a feasible bundle x'_i with $p' \cdot x'_i = p' \cdot \omega_i$, and such that $x'_i \ P_i \ \bar{x}_i$. Agent i can deviate at stage 1 by appropriately announcing a permutation $\pi'_i \neq \pi_i$ so as to be, say, first in the protocol, as well as $(x', \bar{p})$ with x'_i as identified above. In x', agent i assigns, say, $x'_j = \frac{\bar{\omega} - x'_i}{n-1}$ to each agent $j \neq i$. Agent i is awarded x'_i, which is strictly preferred. This is a profitable deviation, a contradiction.

As a consequence, if $\bar{x}$ is an interior allocation, it is the outcome of the game and it is Walrasian given $\bar{p}$.

Lemma 3: *If $\bar{x}$ is a boundary allocation, it is the outcome of the game*

Suppose not. There exists an agent $i \in N$ for whom $\bar{x}_{i,l} = 0$ for some l, and the game goes beyond stage 2. By the rules of the game, one agent $k \in \{f_1(\pi), f_n(\pi), f_{n-1}(\pi)\}$ receives the 0 bundle. Consider such an agent k. Agent k modifies his permutation, if necessary, so as to be first in the protocol. At stage 2, agent k announces $p' = \bar{p}$ and whatever name in the remaining protocol. The game stops with $\bar{x}$ as outcome. Remember that for each agent $j \in N$, $\omega_j > 0$, $\bar{p} \cdot \bar{x}_j = \bar{p} \cdot \omega_j$ and $\bar{p} \gg 0$. Thus, it is the case that $\bar{x}_j > 0$ for each agent $j \in N$. Hence, by deviating, agent k can obtain $\bar{x}_k > 0$. By strong monotonicity of preferences, this is a profitable deviation. A contradiction.

Lemma 4: *If $\bar{x}$ is a boundary allocation, $(\bar{x}, \bar{p})$ is a Walrasian Equilibrium*

Suppose not. $\bar{x}$ is the outcome of the game at stage 2 but $(\bar{x}, \bar{p})$ is not Walrasian. We have two cases to consider.

1) $\bar{x}$ is the outcome of the game but there exists an agent $i \in N$, with preferences $R_i \in \mathcal{R}_i$, for whom

$$B_i(\bar{p})|_{x_i \leq \bar{\omega}} \cap SUC_i(\bar{x}_i, R_i) \neq \emptyset.$$

Agent i has a profitable deviation. He deviates at the first stage and modifies his permutation, if necessary, so as to be last in the protocol. At stage 2, he announces $(x', \bar{p})$ with $x'_i \in B_i(\bar{p})|_{x_i \leq \bar{\omega}} \cap SUC_i(\bar{x}_i, R_i)$ such that $\bar{p} \cdot x'_i = \bar{p} \cdot \omega_i$. For each agent $j \neq i$, $x'_j = \frac{\bar{\omega} - x'_i}{n-2}$. In consequence, the game stops at stage 1 and agent i receives x'_i which is strictly preferred to $\bar{x}_i$ by construction. This case is therefore not possible in equilibrium. The only case left is when there exists an agent $i \in N$, for whom $B_i(\bar{p}) \cap SUC_i(\bar{x}_i, R_i) \neq \emptyset$.

2) $\bar{x}$ is the outcome of the game but there exists an agent $i \in N$, with preferences $R_i \in \mathcal{R}_i$, for whom

$$B_i(\bar{p}) \cap SUC_i(\bar{x}_i, R_i) \neq \emptyset.$$

Since the previous case is ruled out, we have that if $x'_i \in B_i(\bar{p}) \cap SUC_i(\bar{x}_i, R_i)$, then $x'_{i,l} > \bar{\omega}_l$ for some good l. Notice that for this agent i, $\bar{x}_i \gg 0$. Consider an agent $j \neq i$. Agent j has a profitable deviation. He announces a different permutation at stage 1, if necessary, so as to be first in the protocol. At stage 2, he announces $p' \neq p$ such that $\widetilde{B}_i(p', \bar{x}_i)|_{x_i \leq \bar{\omega}} \cap UC_i(\bar{x}_i, R_i) = \{\bar{x}_i\}$, —where $\widetilde{B}_i(p', \bar{x}_i)|_{x_i \leq \bar{\omega}} = \{x_i \leq \bar{\omega} : p' \cdot x_i \leq p' \cdot \bar{x}_i\}$—, and calls agent i. The best response of agent i at stage 3 is to choose $\bar{x}_i$. Hence, agent j will be awarded $\bar{x}_j + \frac{1}{n-2}(\bar{\omega} - \bar{x}_i - \bar{x}_j)$. Since $\bar{x}_k > 0$ for each $k \in N$ and preferences are strongly monotonic, this is a profitable deviation.

Thus, both cases lead to the construction of a profitable deviation, a contradiction with m being a subgame perfect equilibrium. This concludes the first part of the proof.

We have showed that every SPE outcome should be a Walrasian allocation (on the boundary or inside the feasible set). To complete the proof, we prove the opposite direction. That is, we prove that $WE(R) \subseteq SSPE(\Gamma, R)$. Suppose (x^*, p^*) is a Walrasian equilibrium and that the preference profile is $R = (R_i)_{i \in N} \in \mathcal{R}$. Then the following strategies support x^* as SSPE outcome of (Γ, R).

(i) Every agent i announces $(x^*, p^*, \pi^I)^i$, where π^I is the identity permutation.

(ii) Let $(\bar{p}, \bar{x})$ be the unanimously agreed price-allocation pair and $f(\pi)$ the composition of permutations at stage 1.

Agent $f_1(\pi)$:

a) There exists an agent $f_i \neq f_1(\pi)$ with preferences, say, $R_{fi} \in \mathcal{R}_{f_i}$, and $\bar{x}_{f_i} \gg 0$, such that $B_{f_i}(\bar{p})|_{x_{f_i} \leq \bar{\omega}} \cap SUC_{f_i}(\bar{x}_{f_i}, R_{f_i}) = \emptyset$ and there exists $x'_{f_i} \in B_{f_i}(\bar{p}) \cap SUC_{f_i}(\bar{x}_{f_i}, R_{f_i})$ with $x'_{f_i,l} > \bar{\omega}_l$ for some l.

Agent $f_1(\pi)$ calls agent f_i and announces an appropriate p' (given the rules of the game) such that $\bar{x}_{f_i} \in \widetilde{B}_{f_i}(p', \bar{x}_{f_i})|_{x_{f_i} \leq \bar{\omega}} \cap UC_{f_i}(\bar{x}_{f_i}, R_{f_i})$ and $\widetilde{B}_{f_i}(p', \bar{x}_{f_i})|_{x_{f_i} \leq \bar{\omega}} \cap SUC_{f_i}(\bar{x}_{f_i}, R_{f_i}) = \emptyset$.

b) Otherwise, agent $f_1(\pi)$ announces $p' = \bar{p}$ and calls agent $f_2(\pi)$.

(iii) Agent f_i chooses the bundle she prefers between $\bar{x}_{f_i}$ and $q_{f_i} \in \{q_{f_i} \leq \bar{\omega} : p' \cdot q_{f_i} = p' \cdot \bar{x}_{f_i}, \bar{p} \cdot q_{f_i} > \bar{p} \cdot \omega_{f_i}\}$. If she is indifferent between $\bar{x}_{f_i}$ and any such q_{f_i}, then she announces $\bar{x}_{f_i}$.

The optimality of part (iii) is clear. Agent f_i chooses the bundle she prefers between the two that are proposed to her. If she is indifferent between the two[14], she agrees with agent $f_1(\pi)$. Now, notice that agent $f_1(\pi)$ is playing a best response at stage 2. He announces $p' \neq \bar{p}$ only if there exists an agent $f_i \neq f_1(\pi)$ for whom $\bar{x}_{f_i} \gg 0$, $B_{f_i}(\bar{p})|_{x_{f_i} \leq \bar{\omega}} \cap SUC_{f_i}(\bar{x}_{f_i}, R_{f_i}) = \{\bar{x}_{f_i}\}$ and there is a $x'_{f_i} \in B_{f_i}(\bar{p}) \cap SUC_{f_i}(\bar{x}_{f_i}, R_{f_i})$ with $x'_{f_i,l} > \bar{\omega}_l$ for some l. By doing so, agent $f_1(\pi)$ can obtain $\bar{x}_{f_1(\pi)} + \frac{1}{n-2}(\bar{\omega} - \bar{x}_{f_i} - \bar{x}_{f_1(\pi)}) > \bar{x}_{f_1(\pi)}$ (by appropriately choosing a price $p' \neq \bar{p}$, and calling agent f_i). Whenever this condition is not satisfied, one of the best response of agent $f_1(\pi)$ is to announce $p' = \bar{p}$. If agent $f_1(\pi)$ does not make the game to stop at stage 2 in such a case, given the strategies of agent f_i, agent $f_1(\pi)$ would receive at best $\bar{x}_{f_1(\pi)}$ or the 0 bundle. Can both these agents gain by deviating? Given the rules of the game, they cannot be made both better off at stage 3.

Finally, since every Walrasian allocation is individually rational and such that for each $i \in N$, $B_i(\bar{p}) \cap SUC_i(\bar{x}_i, R_i) = \emptyset$, the behavior in (i) is also optimal for each individual agents. Any deviation by a coalition will result in the same outcome (if agents just modify their permutation) or in an outcome that is weakly dominated by any Walrasian allocations (obtaining individual endowments in which it is not possible to make coalitions strictly better off). This profile of strategies is a strong subgame perfect equilibrium.

Hence, on the equilibrium path, each agent $i \in N$ announces $(x, p)^i = (x^*, p^*)$. If for each agent $i \in N$ we have that $x^*_i \gg 0$, the game stops at stage

[14] This situation could happen off the equilibrium path.

1 and x^*, an interior Walrasian allocation, is implemented. Otherwise, it goes to stage 2 where agent $f_1(\pi)$ confirms the status-quo coming from stage 1. The game stops and x^*, a boundary Walrasian allocation, is implemented.

Q.E.D.

Remark 1 *In the previous section, we underlined that the issue raised in example 1 pertains to the Lindhal correspondence. The above mechanism can be adapted to that case. Lindhal equilibrium, being a competitive concept, is well-adapted to the use of prices and allocations as part of the message spaces of agents. In our case, the message space of agents should be enlarged at stage 1. It should incorporate the quantities of public good consumed and personalized price vectors associated to them*[15]*. An example of such a mechanism is provided in the appendix.*

5 Conclusion

We have shown that, without additional parametric restrictions on the environment, the Walrasian correspondence may not be implementable in subgame perfect equilibrium. Differentiability can be a crucial assumption. Moreover, this issue also applies to another well-known competitive concept, namely Lindhal Equilibrium. Condition C, a necessary condition for implementation in subgame perfect equilibrium, may not be satisfied in such a case, as shown in example 1. Taking into account this observation, we added differentiability and constructed a sequential game form that takes care of the boundary problem. It doubly implements the Walrasian correspondence in subgame perfect and strong subgame perfect equilibrium. It is thus robust to coalitional deviations. Our mechanism is based on price-allocation announcements and fits the Walrasian story. In accordance with Nash implementability of interior Walrasian allocations, the game stops at stage 1 when allocations are interior. Beyond stage 1, only boundary allocations are allowed, and a retrading device is used to elicit whether or not the price-allocation pair agents agreed on is a Walrasian equilibrium. Moves along price hyperplanes are at the heart of the Walrasian equilibrium concept. In addition, the mechanism could be modified so as to cover the case of the (full) Lindhal correspondence. Our mechanism is thus well-adapted to competitive concepts. Price-taking behavior is a consequence of the rules of the game. Competition can be maintained even with a small number of agents.

[15] Remember that Lindhal Equilibrium are based on a system of personalized price vectors, one for each agent. The prices of the private goods are the same for all agents. However, agents typically face different pices for the public goods.

A further extension is to incorporate the important case of two agents. We would then obtain a unified construction. Moreover, having a continuous outcome function, incorporating production, or simply to allow for a more general model in which individual endowments are not known to the designer and can vary across states would constitute possible avenues. We leave these questions open for future research.

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Appendix

We construct a mechanism to implement the Lindhal correspondence in subgame perfect equilibrium. For simplicity, suppose there are the same number of public goods and private goods. Each agent $i \in N$ has a preference relation defined over $\mathbb{R}_+^{2L}$. As before, the only characteristics of agents unknown to the planner are the preferences of agents. An economy is simply $R = (R_i)_{i \in N} \in \mathcal{R}$. We consider the class of economies $\mathcal{E}$ defined earlier. Each agent $i \in N$ is endowed with an amount $\omega_{ix} > 0$ of private good. The aggregate endowment is $\bar{\omega}_x \gg 0$.

Each public good y_l is produced using private good x_l as input. Public goods are produced using a constant returns to scale technology. Formally, for each public good l, $y_l = (\frac{1}{\beta^l})x_l$.

A (feasible) allocations is a list of bundles $z = (x_i, y)_{i \in N} \in \mathbb{R}_+^{2L}$ such that $\sum x_i + \beta \cdot y \leq \bar{\omega}_x$, where $\beta = (\beta^1, ..., \beta^K)$

Given an agent $i \in N$, $x_{i,l} \in \mathbb{R}_+$ stands for the quantity of good l received by agent i at bundle of private goods x_i.

Define by F the set of balanced allocations,

$$F = \left\{ (x, y) \in \mathbb{R}_+^{2Ln} : y_i = y_j = y \text{ for each } i, j \in N \text{ and} \sum x_i + \beta \cdot y = \bar{\omega}_x \right\}.$$

Define by $P = \mathbb{R}_{++}^L$ the set of strictly positive price vectors for private goods. For each agent $i \in N$, define by $Q_i = \mathbb{R}_{++}^L$ the set of personalized strictly positive price vectors for public goods. Denote by $Q = \prod Q_i$.

For each agent $i \in N$, denote by $B_i(p)$ and $B_i(p)|_{x_i \leq \bar{\omega}}$, the budget set and the constrained budget set, respectively, of agent i at a given price $p \in P$ and $q_i \in Q_i$,

$$\begin{aligned} B_i(p, q_i) &\equiv \left\{ (x_i, y) \in \mathbb{R}_+^{2L} \mid p \cdot x_i + q_i \cdot y \leq p \cdot \omega_{ix} \right\} \\ B_i(p, q_i)|_{x_i + \beta \cdot y \leq \bar{\omega}_x} &\equiv \left\{ (x_i, y) \in \mathbb{R}_+^{2L} \mid p \cdot x_i + q_i \cdot y \leq p \cdot \omega_{ix} \text{ and } x_i + \beta \cdot y \leq \bar{\omega}_x \right\}. \end{aligned}$$

For each agent $i \in N$, given a bundle $z_i = (x_i', y') \in \mathbb{R}_+^{2L}$, denote by $\widetilde{B}_i(p, q_i, z_i)$ and $\widetilde{B}_i(p, q_i, z_i)|_{x_i + \beta \cdot y \leq \bar{\omega}_x}$, the modified and constrained modified budget sets, respectively, of agent i at price $p \in P$ and $q_i \in Q_i$,

$$\begin{aligned} \widetilde{B}_i(p, , q_i, z_i) &\equiv \left\{ (x_i, y) \in \mathbb{R}_+^{2L} \mid p \cdot x_i + q_i \cdot y \leq p \cdot x_i' + q_i \cdot y' \right\} \\ \widetilde{B}_i(p, q_i, z_i)|_{x_i + \beta \cdot y \leq \bar{\omega}_x} &\equiv \left\{ (x_i, y) \in \mathbb{R}_+^{2L} \mid p \cdot x_i + q_i \cdot y \leq p \cdot x_i' + q_i \cdot y' \text{ and } x_i + \beta \cdot y \leq \bar{\omega}_x \right\}. \end{aligned}$$

Finally, given the assumptions on preferences, an $(x^*, y^*) \in F$ is a Lindhal allocation if there exists $p^* \in P$ and personalized price vectors $q_i^* \in Q_i$, one for each agent, such that $\sum q_i^* = \beta$, and for each agent $i \in N$, $(x_i^*, y_i^*) \in B_i(p, q_i)$ and $(x_i^*, y_i^*)\ R_i\ (x_i, y_i)$ for every $(x_i, y_i) \in B_i(p, q_i)$.

For each $R \in \mathcal{R}$, denote by $LE(R)$ the set of Lindhal allocations of that economy.

The modified version of our mechanism is the following.
Mechanism Γ:

Stage 1: $m^i = ((x, y), p, q, \pi)^i \in F \times P \times Q \times \Pi$ such that for each $i \in N$, $p^i \cdot x_j^i + q_j^i \cdot y = p^i \cdot \omega_{jx}$ and $\sum q_j^i = \beta\ \forall j \neq i$.

1) $((x, y), p, q)^i = ((\bar{x}, \bar{y}), \bar{p}, \bar{q})\ \forall i \in N$ and $\bar{x}_i \gg 0\ \forall i \in N$, the game stops and the outcome implemented is $(\bar{x}, \bar{y})$.

2) $((x, y), p, q)^i = ((\bar{x}, \bar{y}), \bar{p}, \bar{q})\ \forall i \in N$ and $\bar{x}_{j,l} = 0$ for some j and l, then go to stage 2.

3) $((x,y),p,q)^i = ((\bar{x},\bar{y}),\bar{p},\bar{q})\ \forall j \neq i,\ i \neq f_n(\pi),\ m^i = ((x',y'),p',q') \neq ((\bar{x},\bar{y}),\bar{p},\bar{q})$. If $\bar{p}\cdot x'_i + \bar{q}\cdot y' = \bar{p}\cdot\omega_{ix}$, then agent i gets (x'_i, y'). Agent $j = f_n(\pi)$ gets the 0 bundle and each agent $k \notin \{i, f_n(\pi)\}$receives $\left(\frac{1}{n-2}(\bar{\omega}_x - x'_i - \beta\cdot y'), y'\right)$.

Otherwise, if $\bar{p}\cdot x'_i + \bar{q}\cdot y' \neq \bar{p}\cdot\omega_{ix}$, each agent $k \in N$ receives his endowment ω_{kx}.

4) In all other cases, agent $j = f_n(\pi)$ receives $\omega_{jx} - \epsilon_j$. Each agent $i \neq \{f_n(\pi), f_1(\pi)\}$ receives ω_{ix} and agent $k = f_1(\pi)$ receives $\omega_{kx} + \epsilon_j$.

Stage 2: Agent $f_1(\pi)$ selects an agent $f_i \neq f_1(\pi)$ and announces $(p', q'_{f_i}) \in P \times Q_{f_i}$.

1) $(p', q'_{f_i}) \neq (\bar{p}, \bar{q}_{f_i})$, $\bar{x}_{f_i} \gg 0$ and (p', q'_{f_i}) is such that there exists feasible bundles $(x'_{f_i}, y') \neq (\bar{x}_{f_i}, \bar{y})$, with $\bar{p}\cdot x'_{fi} + \bar{q}_{f_i}\cdot y' = \bar{p}\cdot\omega_{ix}$ and $p'\cdot x'_{f_i} + q'_{f_i}\cdot y' < p'\cdot\bar{x}_{f_i} + q'\cdot\bar{y}$. Go to stage 3.

2) In all other cases, the game stops and $(\bar{x},\bar{y})$ is implemented.

Stage 3: Agent f_i chooses between $(\bar{x}_{f_i}, \bar{y}_{f_i})$ and

$$(x_{f_i}, y) \in \left\{x_{f_i} + \beta\cdot y \leq \bar{\omega}_x : p'\cdot x_{f_i} + q'_{f_i}\cdot y = p'\cdot\bar{x}_{f_i} + q'_{f_i}\cdot\bar{y},\ \bar{p}\cdot x_{f_i} + \bar{q}_{f_i}\cdot y > \bar{p}\cdot\omega_{f_i x}\right\}.$$

1) If he chooses $(\bar{x}_{f_i}, \bar{y}_{f_i})$, he gets it. Agent $f_1(\pi)$ gets $(\bar{x}_{f_1(\pi)} + \frac{1}{n-2}(\bar{\omega}_x - \bar{x}_{f_i} - \bar{x}_{f_1(\pi)} - \beta\cdot\bar{y}), \bar{y})$. If agent $f_i \neq f_n(\pi)$, then agent $f_n(\pi)$ receives 0 and each agent $j \notin \{f_1(\pi), f_i, f_n(\pi)\}$ receives $(\frac{1}{n-2}(\bar{\omega}_x - \bar{x}_{f_i} - \bar{x}_{f_1(\pi)} - \beta\cdot\bar{y}), \bar{y})$. Otherwise, if $f_i = f_n(\pi)$, agent $f_{n-1}(\pi)$ receives 0 and each agent $j \notin \{f_1(\pi), f_i, f_{n-1}(\pi)\}$ receives $(\frac{1}{n-2}(\bar{\omega}_x - \bar{x}_{f_i} - \bar{x}_{f_1(\pi)} - \beta\cdot\bar{y}), \bar{y})$.

2) If he chooses (x_{f_i}, y), he gets it. Agent $f_1(\pi)$ gets 0. Each agent $j \notin \{f_1(\pi), f_i\}$ receives $\left(\frac{1}{n-2}(\bar{\omega}_x - x_{f_i} - \beta\cdot y), y\right)$.

Theorem 2: *The extensive form mechanism Γ implements in subgame perfect equilibrium the Lindhal correspondence in the class of economies $\mathcal{E}$.*

Proof: The proof is very similar to the one of theorem 1. It is therefore omitted.

Q.E.D.