Pricing in Social Networks under Limited Information

Elias CARRONI and Simone RIGHI

CERPE – April 2015

Department of Economics
Working Papers
2015/03

UNIVERSITÉ DE NAMUR

Pricing in Social Networks under Limited Information

Elias Carroni[*]   Simone Righi[†‡]

Abstract

We model the choices of a monopolist who faces a partially uninformed population of consumers. She aims at expanding demand by exploiting his (limited) knowledge about consumers' social network. She offers rewards to current clients in order to induce them to activate their social network and to convince peers to buy the product sold by the company. The program is profitable provided that the monopolist faces a serious enough informational problem and that the cost of investment in the social network is not prohibitively high. Price for informed consumers is lowered by the introduction of the reward compared to the benchmark where no program is run. There are no effects on the price charged to uninformed consumers. The offer of bonuses affects individual incentives of informed people to share information, determining a minimal degree condition for the costly investment in the social network. The level of such threshold strongly depends on the distribution of connections in the social network. In random networks, roughly the most popular half of informed consumers invests, regardless of network density. On the contrary, in scale-free networks the monopolist faces a clear-cut decision between maximising margins (running a small referral program) and maximising demand (motivating many informed agents to communicate). The optimal choice depends on the probability of observing highly-connected individuals. In empirically observed scale-free networks, the first alternative would be preferred, in line with real-world markets.

Keywords: social networks, monopoly pricing, network-based pricing.

[*]Department of Economics, CERPE, University of Namur. Mailing address: 8 Rempart de la Vierge, 5000 Namur, Belgium. Email: elias.carroni@unamur.be.

[†]Department of Economics "Marco Biagi", University of Modena and Reggio Emilia. Mailing address: Via Berengario 51, 41121 Modena, Italy Email: simone.righi@unimore.it

[‡]MTA TK "Lendület" Research Center for Educational and Network Studies (RECENS), Hungarian Academy of Sciences.

1 Introduction

Programs that attribute referral bonuses to customers are an established marketing strategy through which companies attempt to increase their market penetration. This strategy is effective since consumers are part of a network of acquaintances and thus can be incentivised to use their social relationships to diffuse the knowledge about the existence of the company's product. These programs can be particularly advantageous when a new product is launched as the market is not well covered and some potential clients are unaware of its existence (Sernovitz and Clester 2009).

In the typical referral bonus program, the company offers rewards to its established customers, provided that they are able to convince some peers to become new clients. In order to obtain rewards old customers need to invest in their existing social network by informing their peers about the existence of the product. Depending on their willingness to pay, newly-informed agents will then decide upon purchase.

Referral bonuses are generally used in markets for subscription goods and services, in both on- and off-line services. In the market for online storage services, Dropbox offers free storage space to clients that convince their friends to subscribe. According to Huston (2010), founder and CEO of Dropbox, their referral program extended their client basis by 60% in 2009 and referral was responsible for 35% of new daily signups. Besides online services, banks offer advantageous conditions to old customers introducing new ones. Better conditions are provided both in the form of higher interest rates on the deposit and that of lowered service fee. Another form of reward used by banks is to embed the rewarding mechanism in established customer loyalization schemes[1] awarding points in exchange for referrals. Such points can then be used to claim prizes (mobile phones, televisions etc). Other well-known examples can be found in markets for massively multiplayer on-line games, payment systems, touristic accommodation, online content providers and enterprise software solutions.

In all these examples the company provides incentives that target current popular and informed clients, proposing them to sustain a costly investment with uncertain return. The uncertainty of the investment in social network follows from two considerations. Firstly, some of the peers contacted may not be willing to buy the product even once aware of its existence. Secondly, uninformed consumers may get information about the service from multiple sources while in most referral programs only one person can receive the resulting bonus. We take into account both these issues when modeling the expectations of customers considering to activate their social network.

The current literature mainly focuses on pricing in online social networks (Bloch and Quérou 2013) and telecommunication services (Shi 2003), where sellers can directly observe the precise structure of the consumers' network. Referral programs are a simpler but widely used strategy which only requires very limited information about consumers' social interactions. Indeed, in our setup the seller needs only to be aware of the distribution of the number of connections (*degree distribution* in the language of graph theory) of current and prospective clients. Under such a condition, the company cannot price-discriminate according to the precise position of the consumer in the social network. However, it can still influence clients' decisions by setting prices and bonuses so that some of them will have incentive to

[1]For example UBS at the time of writing this paper.

diffuse information.

The power of referrals follows from the fact that each player in the market has incentives that favor the success of this strategy. The producer wants to extend his client base, old customers are motivated by expected rewards and potential new buyers are given the opportunity to learn about the existence of a potentially valuable service. Consequently, the structure of incentives of referral marketing strategies makes them an effective tool in the presence of significant informational problems on the consumers' side, i.e., when some of the potential customers are unaware of the existence of the product. This situation is typical when a product or service is relatively new on a market and when the existence of many specialized markets leads the consumers to a situation of information overload (Zandt 2004). Mass media advertisement can provide a partial solution to this informational problem. However, it is well-known (Lazarsfeld and Katz 1955) that information coming from the mass-media is not fully trusted by consumers, who tend to be more influenced by social neighbours. The strategy we study is thus an effective and relatively cheap alternative for companies to expand their client base as it allows to harness the power of customers' social networks.

In this paper, a monopolist decides upon the introduction of a referral program in the presence of uninformed consumers. Our setup allows for results along three lines. Firstly, we characterise the conditions for a rewarding program to be optimal from the producer's point of view, showing that it is profitable in almost every reasonable situation (i.e., assuming a significant informational problem and a limited cost of activating the social network).

Secondly, when the referral program is run, informed buyers see the price they are charged increasing with time. This increase determines a transfer going from agents receiving many bonuses to agents receiving fewer or no bonuses. The probability to be on either side will depend on one's popularity (the degree) in the consumers' network. Uninformed consumers are always better off as the program may provide them with information about potentially valuable goods.

Finally, we characterise the impact of the structure of network interactions among consumers on the model's outcomes. We study in particular two broad classes of networks: one where the number of connections in the population is distributed around an average value (random networks, Erdős and Rényi 1959) and one in which the distribution is power law (scale-free networks, Barabási and Albert 1999). Notably, we show that in random networks roughly all people with an above-average degree will be incentivised to spread information regardless of the density of social interactions or the severity of the informational problem. On the other hand, scale-free networks give the monopolist a clear-cut choice: either maximising demand by setting incentives to motivate many informed consumers to invest, or maximising margins by offering a very small referral reward.

The remaining part of this paper is divided as follows. After discussing the related literature in the following section, we outline the mathematical aspects of the model in Section 3. Then, in Section 4, we provide the equilibrium of the model and we analyze the general implications that can be derived when the form of the consumers' network is left implicit. We follow up by providing numerical evidence of the impact of network topology on the players' choices (Section 5), and finally we draw the conclusions (Section 6).

2 Related literature

The solipsistic view of the consumer, which characterised the economic discipline in the past, can be relaxed considering the single agent as a member of a social group. Indeed, individuals influence and are influenced by social behavior through local interactions. The concept of network has been introduced and applied in a variety of fields. As pointed out in the comprehensive review of Jackson (2005) networks influence agents' economic behavior in fields such as decentralized financial markets, labor markets, criminal behavior and the spread of information and diseases.

In recent years, the attention of industrial economists shifted from the network externalities approach, following the tradition of Katz and Shapiro (1985), to a new focus on the direct study of the effects of social interactions on the behavior of economic agents. The new tendency comprises the consideration of a subset of neighbours rather than the population overall as the main driving force influencing individual choices (Sundararajan 2006; Banerji and Dutta 2009). The concept of network locality has been used by Banerji and Dutta (2009) to show the emergence of local monopolies with homogeneous firms competing in prices, by Bloch and Quérou (2013) to study the optimal monopoly pricing in on-line social networks and by Shi (2003) to study pricing in the presence of weak and strong ties in telecommunication markets. The main concern of the last two papers is price discrimination based on network centrality and on the strength of social ties respectively. While their models assume a full knowledge of the links among consumers, the strategy we discuss requires only very limited information. Instead of gathering detailed information about individuals in order to directly price discriminate, the company offers incentives that motivate buyers to become channels of information diffusion.

An alternative approach is to assume that consumers discuss with peers the products they buy and that this can be taken as given by the sellers when defining their strategies. Along these lines Campbell (2013) studies optimal pricing when few consumers are initially informed and engage in word-of-mouth (WOM); Galeotti and Goyal (2009) discuss the optimal target to maximise market penetration with WOM; and Galeotti (2010) investigates the relationship between interpersonal communication and consumer investments in search. When WOM is taken as given, the key issue for the seller is to assess how the consumers' network reacts to any marketing strategy in terms of percolation of information. Instead of focusing on WOM *per-se*, our paper discusses communication resulting from a deliberate incentive scheme predisposed by the monopolist. In other words, the strategy we analyze generates communication which would not exist otherwise.

The empirical literature has long considered the word-of-mouth. The seminal work of Lazarsfeld and Katz (1955) formulated the general theory that when people speak with each other and are exposed to information from media, their decisions are based on what peers say rather than on what media communicate. They showed that an effective way for companies and governments to reach their goals is to influence a small minority of opinion leaders, who then tend to spread the message. Arndt (1967) is among the first scholars to study empirically the short-term sales effects of product-related conversations, showing that favorable comments lead to an increase in the adoption of new products and vice-versa. Van den Bulte and Joshi (2007) and Iyengar et al. (2011) point out the importance of opinion leaders or influential agents in the diffusion of product adoptions. Our paper essentially is in concord with this empirical observation as the company targets relatively more popular agents in order to increase profits.

The paper is also related to the marketing literature studying referral bonuses. Two papers are worth mentioning from a theoretical point of view: Biyalogorsky et al. (2001) and Kornish and Li (2010). The first one shows that rewards are positively correlated with the share of delighted consumers. Kornish and Li (2010) argue that the more agents value friends' utility, the higher is the bonus set by the company, as long as recommendations cannot be induced with a lower price but with a higher quality product. Both these papers, however, focus on peculiar consumers' preferences as drivers for the sellers' strategy, disregarding the effects of the existence of a social network among consumers on strategic interactions in the market.

3 The model

We consider a two-period model where a monopolist sells a non-durable product to a large but finite population $N = \{1, 2, \ldots, i, \ldots, n\}$ of myopic agents.[2] Consumers differ according to their willingness to pay and the information they have. The utility function of an agent i buying the product at price p is defined as:

$$u_i = r_i - p.$$

The reservation price r is uniformly distributed on the support $[0, 1]$ and there are no externalities from others' consumption.[3] A proportion $1 - \beta$ of consumers is informed about the existence of the product, while the remaining β is not. Uninformed consumers would never buy the product unless they passively receive the information from their informed peers. This modelling choice captures the existence of a difference in consumers' skills to access and use the available informational tools.

Interactions and communication among consumers are restricted by an existing social network structure, which we consider as given. In particular, each agent i has a finite number of neighbours $K_i \subseteq N$ to interact with. The degree k_i (the number of neighbours) is the cardinality of K_i. We further assume the consumer's social network to be undirected, in the sense that if node i is linked to node j, then j is in turn linked to i. The degree of the agents is distributed according to some p.d.f. $f(k)$, which has to be interpreted as the fraction of agents having k neighbours. In other terms, upon selecting a random agent from the social network, the probability that she has exactly k neighbours is $f(k)$. This general formulation allows for results valid in any interaction structure. Moreover, it is possible to substitute $f(k)$ with specific networks and to compare results across different topologies (see Section 5).

In a two-period model, the monopolist aims at maximising the sum of inter-temporal profits.[4] Defining $D_1(p_1)$ as the demand coming from first period buyers and assuming a

[2] Consumers are myopic in the sense that they are unable to forecast, in the first period, the monopolist's decisions in the second period.

[3] We make this assumption in order to simplify the presentation of the results. Indeed, network externalities could be introduced in the utility function of the consumers without qualitatively changing the results.

[4] For the sake of simplicity, we normalise the inter-temporal discount to 1.

marginal cost normalized to 0, the expected profit in that period, obtained charging price p_1, will be given by:

$$\pi_1 = p_1 \mathbb{E}[D_1(p_1)]. \tag{1}$$

In the second period of our model, we allow the monopolist to offer rewards to old customers through a referral program. Namely, the monopolist knows the distribution of degrees in the social network and, accordingly, offers a gift to the old consumers who inform their friends about the existence of the product and who convince them to buy it. The rationale for this offer is to eliminate the lack of information which prevents some of the potential consumers from buying the product. This gift takes the form of a unitary amount b for each successful referral.

Since each new consumer corresponds to one reward b given to an old customer, the margin on the new second-period buyers is given by $(p_2 - b)$, where p_2 is the price set for the consumers who buy the product in the second period. Thus, defining $D_2^2(p_2, b)$ as the demand in the second period coming from new consumers and D_2^1 the one coming from previously informed consumers, the expected profit π_2 turns out to be:

$$\pi_2 = p_2 \mathbb{E}\left[D_2^1(p_2)\right] + (p_2 - b)\mathbb{E}\left[D_2^2(p_2, b)\right]. \tag{2}$$

Function $D_2^2(p_2, b)$ takes into account both the indirect positive effect of the bonus on the probability of new consumers to get informed and the direct negative effect of p_2 on their decision to buy the good. To enjoy rewards, old buyers need to contact their social network, which implies a costly investment of a fixed amount C.[5]

It is important to discuss the informational structure of the model as it constitutes a peculiar feature of our study. Specifically, the information available to agents about the idiosyncratic characteristics of all the others is summarized in Assumption 1.

Assumption 1. *Each agent has perfect private information about his own characteristics. Moreover, the distributions of r_i and k_i as well as the proportion β are common knowledge and independent from each other.*

Assumption 1 implies that consumers cannot condition their decisions on their local social neighborhood and the monopolist is not able to base his choice upon individual characteristics of consumers.

Our game is played in two periods and it is solved by backward induction. Each time period, in itself, is a sequential game where the monopolist chooses first, and the consumers react. The timing of the model is reported in Figure 1. In period 1 the monopolist sets a price p_1, and the consumers, after having observed it, decide whether to purchase the good. In the second period, the monopolist sets a new price p_2 and she introduces the reward b, while the first period buyers decide whether to buy the good again and contact their friends. Given the total investment of old consumers, information about the existence of the product may reach some potential new buyers, who in turn buy the product if their reservation prices are sufficiently high.

[5]The choice of studying the case of fixed cost has been made in order to capture the idea that the emergence of online social networks and the use of e-mails tend to make the difference in the number of people contacted negligible in terms of total cost.

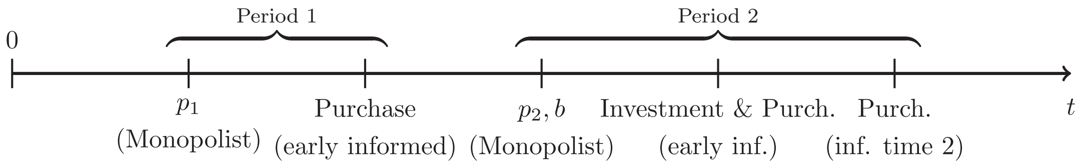

Figure 1: Timing of the model.

4 Results

We now proceed to solve our model by studying the decisions of the agents, from the last to the first, and assuming that what happened before is taken as given.

Purchase decisions of uninformed consumers. In the last step, uninformed consumers may receive the information through old buyers. We define ρ as the probability for an agent to receive the information at least once. From the point of view of the single agent, ρ is function of the number k of social ties and of the number of first period consumers that invest in the social network, which we define as D_1^{Inv}. Indeed, the more friends one person has, the more likely it is that at least one of them decides to invest and to speak with him about the product. Moreover, as the number of investors increases, the odds for each single neighbor to be an investor are higher. The new second-period demand D_2^2 is composed of the fraction of newly-informed agents exhibiting reservation price $r_i > p$. Given the degree distribution $f(k)$ and the probability of receiving the information $\rho(k, D_1^{Inv})$ we can derive the new expected demand in the second period as:

$$\mathbb{E}[D_2^2] = \beta(1 - p_2)\bar{\rho}n, \tag{3}$$

where $\bar{\rho} = \sum_{k=1}^{n-1} \rho(k, D_1^{Inv}) f(k)$ represents the average probability of receiving the information about the existence of the product. It follows that $\bar{\rho}n$ is the total expected number of receivers in the population.

The probability of receiving the information for each k can be easily specified by considering the case of an uninformed agent with degree k. In expected terms, the probability of receiving the information from each single friend turns out to be equal to the share of investors in the total population $\frac{D_1^{inv}}{n}$. Thus, the probability of receiving the information from at least one among k friends is:

$$\rho(k, D_1^{inv}) = 1 - \left[1 - \frac{D_1^{inv}}{n}\right]^k. \tag{4}$$

Summing the expression in Equation (4) over all ks, we find explicitly $\bar{\rho}$. This can be plugged in Equation (3), obtaining the expected number of new consumers buying the product in period 2. The average spread of information and the expected competition will depend on the incentive to speak of informed agents.

Investment decisions of old buyers. After having observed the second-period price and the reward offered by the monopolist, old buyers decide upon purchase, confronting the new price with their reservation value. Moreover, they decide about the investment in their social network, considering the expected purchase behavior of the agents they inform. The two alternatives are either to bear a cost and inform their friends (thus possibly getting rewards) or to give up the benefit, enjoying no extra utility. The expected utility of an informed agent i with connectivity k_i is thus defined as follows.

$$\mathbb{E}\left[U(k_i)\right] = (r_i - p_2) + \begin{cases} \phi_b(p_2, \beta, D_1^{Inv})k_i b - C & \text{if } i \text{ invests} \\ \\ 0 & \text{if } i \text{ does not invests.} \end{cases} \tag{5}$$

According to Equation (5), each agent invests if the amount she expects to receive is bigger than the cost C. While the cost of activating the social network is assumed to be fixed, the expected benefit requires a more precise analysis. Indeed, this amount is composed of three elements. ϕ_b is the probability of getting a bonus for each person contacted. As it will be discussed in the next paragraph, the odds depends on the social contact being a potential buyer and referring i as the source of information. Clearly, a higher number of investors D_1^{Inv} implies a lower likelihood to be indicated as the recommender, as the competition for each single bonus becomes stronger. In order to obtain the total expected benefit, ϕ_b is then multiplied by the agent connectivity and the unitary bonus.

Given the presence of a fixed cost C, the actual investors will be those for which the expected benefits are higher than C. Since this amount is monotonically increasing with the degree, there exists some $\underline{k}$ such that all agents with $k_i \geq \underline{k}$ will invest. Simply by equating benefits and cost, we find the critical degree such that the net expected benefit from investing is exactly equal to 0 (the net utility obtained by abstaining from investing).

Given this cutoff for investment, we can directly pin down the expected proportion of investors in the population. In particular, among the informed agents, only the ones with $k \geq \underline{k}$ invest, i.e., a proportion $\sum_{k \geq \underline{k}} f(k)$, bringing us to conclude $D_1^{Inv} = n(1-\beta)(1-p_1)\sum_{k \geq \underline{k}} f(k)$. Accordingly, the probability of getting a bonus can be represented by an increasing function of this cut-off instead of a decreasing function of the number of investors. Since the number of investors is monotonically decreasing in the threshold $\underline{k}$, the infra-marginal informed agent has degree $\underline{k}$ such that:

$$\phi_b(p_2, \beta, \underline{k})k > \phi_b(p_2, \beta, \underline{k})\underline{k} \geq \frac{C}{b}. \tag{6}$$

As we will show in the analysis of the monopolist's optimal decisions, the bonus will be set in such a way that the least connected investors ($\underline{k}$) are just willing to invest, i.e., they do not receive positive utility. For this reason, we will consider hereafter the second (weak) inequality in (6) as an equality, ruling out situations in which the infra-marginal investors receive a positive utility.

The interpretation of the cutoff is very straightforward. It will indeed depend on the individual economic incentives to invest, which are clearly weaker for higher cost of investment and stronger when the unitary bonus is higher. Assume a marginal increase in b or, equivalently, a fall in C. Since the LHS in the inequalities in (6) is clearly increasing in $\underline{k}$, then the threshold needs to adjusts to a lower level to maintain the inequality above. [6]

Competition among buyers and expected benefit. To explicitly find out what function ϕ_b is composed of, consider an uninformed agent with degree $\hat{k}$, willing to buy at price p_2. For given cutoff $\underline{k}$ and corresponding number of investors $D_1^{Inv}(\underline{k})$, the probability for this agent of receiving the information x times is given by the probability of x among his friends to be informed investors, $\left(\frac{D_1^{Inv}(\underline{k})}{n}\right)^x$. This can be interpreted as the likelihood for a given informed friend to have $x-1$ competitors when speaking with him. Thus, in expected terms, for the single uninformed agent with connectivity $\hat{k}$ it holds that:

$$\sum_{x=2}^{\hat{k}} \left(\frac{D_1^{Inv}(\underline{k})}{n}\right)^x (x-1)$$

is the expected number of competitors from the point of view of an informed agent speaking with him. Since agents do not know the specific characteristics of their friends (degree, information and willingness to buy), they also have expectations according to the corresponding distributions. Thus the function $\phi_b(p_2, \beta, D_1^{Inv})$ can be defined as follows:

$$\phi_b(p_2, \beta, \underline{k}) = \sum_{k=1}^{n-1} f(k) \left[\frac{\beta(1-p_2)}{1 + \sum_{x=1}^{k} \left(\frac{D_1^{Inv}(\underline{k})}{n}\right)^x (x-1)} \right].$$

Unsurprisingly, an increase in β or a decrease in p_2 result in a higher number of bonuses making the inequality (6) more likely to be satisfied for all degrees. These variations will induce higher incentives to investments pushing downward the corresponding cutoff $\underline{k}$. On the other hand, ϕ_b is decreasing in $\underline{k}$, as to a higher cutoff corresponds a lower share of investors and, thus, of competitors for the single bonus. This entails clear effects on the expected number of bonuses each informed individual expects to get.

These effects are different among degrees. Assume that some variable affecting the individual incentive changes in such a way that the cutoff increases from $\underline{k}$ to $\underline{k}' = \underline{k} + 1$. This in turns would imply the expected number of competitors to drop and consequently the probability to increase from $\phi_b(\underline{k})$ to $\phi_b(\underline{k}')$. We can divide informed individuals into three categories. Agents with a degree higher than $\underline{k}$ would enjoy a greater expected benefit from communication as competition is less intense. Conversely, agents with degree strictly below $\underline{k}$ would find it even less profitable to invest if the cutoff is $\underline{k}'$. Agents belonging to one of these two categories would confirm *a fortiori* their decision when moving to a higher cutoff for investment, albeit for exactly opposite reasons. The only people that would change their decisions are the infra-marginal with degree $\underline{k}$, who are just willing to invest when the cutoff is $\underline{k}$ but they would receive a negative utility from investing once we move to a greater cutoff.

[6] For the sake of completeness, since the degree k is a discrete variable, if we consider the case in which infra-marginal people are left with positive even small utility, $\underline{k}$ may not vary if the changes are not strong enough to make the cutoff move. All the results remain intuitively unaltered, but we should speak about the cutoff non-increasing (non-decreasing) instead of decreasing (increasing) in the variable in question.

With all these considerations in mind, we can summarize the results concerning consumer interactions and their economic consequences in the following proposition.

Proposition 2. *The function $\bar{\rho}$ is decreasing in C, p_1, p_2 and increasing in b. The opposite is true for ϕ_b. Moving from β to a higher β' has an ambiguous effect, depending on the balance between the increase of individual incentives $(1-\beta')f(\underline{k}-1)$ and the fall of the share of potential investors $(\beta' - \beta)$.*

Proof. See Appendix. ■

Proposition 2 has a very intuitive interpretation. The share of investors in the population of buyers determines how much information about the product is available in the second period as well as how strongly old consumers compete for each single bonus. In particular, the more people invest the more information and competition there are. In turn, the number of investors depends on the incentives to communicate as well as on the total number of agents who are eligible to receive a bonus (i.e., first-period buyers). Incentives are negatively and directly affected by the cost of the investment, and negatively and indirectly by the second-period price. Indeed, an increase p_2 reduces the likelihood for a friend to buy the product. The bonus is instead the tool the monopolist uses to stimulate communication, so a rise in b results in an increased investment.

Unlike the second-period price, the first-period price does not affect the incentives but the number of potential investors. Indeed, charging a higher price in the first period lessens the number of first-period buyers, with the consequence of a fall in the number of investors in the second period, keeping incentives for each of them unchanged. For this reason, a higher p_1 implies a decrease in the number of investors and consequently $\bar{\rho}$ decreases and ϕ_b increases.

Varying instead the share of uninformed agents β entails both effects on the number of potential investors and on their communication incentives. From the point of view of the single informed agent, who decides whether to speak or not, a higher β results in an increased likelihood of speaking with an uninformed friend, so that incentives to communicate are stronger. However, the number of potential investors is obviously lower, with a consequent negative effect on the actual number of investors. If the first effect prevails, the number of investors increases, involving a higher diffusion of information but a lower probability of obtaining the bonus for each investing person. The opposite is true otherwise.

The results shown in Proposition 2 are important in order to understand how the second-period expected demand reacts to changes in bonus, prices, cost of investment and proportion of uninformed consumers.

Lemma 3. *The demand faced by the monopolist in the second period is increasing with the unitary reward b, decreasing with both prices p_1 and p_2 and decreasing with the cost of investing in the social network. The effect of the proportion of uninformed consumers is ambiguous.*

Proof. See Appendix. ■

Second period demand is composed by two parts, D_2^1 and D_2^2. The first one, coming from early-informed consumers, reacts only to changes in the second period price whereas the second one, coming from newly-informed people, can be split into two different components.

The first component is the *potential* new market created by the bonus, which depends on the amount of information circulating thanks to the second-period word-of-mouth communication, and also on the proportion of uninformed people. The second component concerns the *actual* response in terms of purchase at price p_2 of this potential market.

Second-period price and bonus setting. At the beginning of the second period, the monopolist sets the bonus and the second-period price in order to maximise expected profits. In particular, the monopolist anticipates investment decisions of old buyers who are stimulated by the bonus and purchase decisions of newly informed consumers, whose choice depends on the price they are asked to pay. Formally, the monopolist solves the following maximisation problem:

$$\max_{p_2, b}(p_2 - b)\mathbb{E}\left[D_2^2(\underline{k})\right] + p_2\mathbb{E}\left[D_2^1(p_2)\right], \tag{7}$$

subject to the investment condition defined in inequality (6). Intuitively, the problem of the monopolist is the following. A rise in the price increases per-consumer margins, but reduces the number of individuals willing to buy the product. This is the well-known trade-off of a price setting monopolist. Peculiarly, the introduction of the referral reward entails another, different, trade-off. Indeed, the bonus represents an additional cost but has the role of creating new demand by inducing referrals. On the one hand, a higher bonus clearly reduces the margins that the monopolist can attain on the single new buyer as it works as a cost: for each new buyer, the monopolist gives an amount b to one old buyer. On the other hand, the dimension of the unitary reward has a positive effect on the demand for the good, as it helps reducing the informational problem. The solution to this trade-off is summarized in Proposition 4.

Proposition 4. *The maximisation problem in Equation* (7) *is solved by setting the price* $p_2^* = \frac{1}{2}$ *and* $b^* = \frac{2C}{\beta\phi_b(\underline{k}^*)\underline{k}^*}$, *where* $\underline{k}^*$ *is the* arg max *of the following maximisation problem:*

$$\max_{\underline{k}}\left[\frac{\beta}{4} - \frac{C}{\phi_b(\underline{k})\underline{k}}\right] n\bar{\rho}(\underline{k}). \tag{8}$$

Proof. See Appendix. ■

The intuition behind the result in Proposition 4 is crucially linked to the number of old consumers the monopolist finds optimal to induce to invest in their social network. Let us assume the monopolist decides to target $\underline{k}$, i.e., that she finds optimal to attract all old buyers with degrees at least equal to this cutoff. To reach the desired level of investment (and thus information), the monopolist sets the price for new consumers p_2 and the bonus b. In particular, since the bonus represents a cost for the monopolist, it is always optimal to choose the smallest b compatible with having a given level of investment and, for a given cutoff, the price turns out to be equal to 1/2. Once the profit is maximised for each possible cutoff $\underline{k}$, then the monopolist's choice trivially falls on the cutoff $\underline{k}^*$ resulting in the highest profit. This optimal cutoff balances two different effects that derive from the results of Proposition 2. Indeed, a higher cutoff results in higher margins but smaller demand. The first result derives from the fact that generating a higher $\underline{k}$ requires a smaller bonus and, since the price is always 1/2, the second-period per-consumer margin is clearly larger. However, a higher cutoff also means less information for early uninformed people, with the consequent squeeze on the demand.

In order to understand the conditions under which the program is optimal to be run, a natural corollary of Proposition 4 is:

Corollary 5. *When the informational problem is not sufficiently strong (small β) or the cost of the investment in the social networks is too high (large C), the bonus program is not run. Otherwise the program is run with $b > 0$.*

Proof. Assume that $\beta \leq \frac{4C}{\phi_b(\underline{k})\underline{k}}$ or $C \geq \frac{\beta\phi_b(\underline{k})\underline{k}}{4}$ for all $\underline{k}$s. Then, from the maximisation problem in Equation (8), the monopolist cannot find any $\underline{k}$ compatible with obtaining positive second period profits. ■

What this corollary expresses is simply that the program is optimal to be run, except for very specific cases in which it would generate negative profits. This would be the case when either the informational problem is not severe enough or the cost of spreading the information is too high. Indeed, in the first case, the program is not worth implementing because the potential new demand is very small. When β approaches the limit of 0, the potential new demand to be reached thanks to the referral program disappears. On the other hand, the cost influences the incentives of informed people. If this is very high, then the bonus required to incentivize the word-of-mouth communication is so high that the program would lead to negative per-consumer margins.

Given the diffusion of online social networks and of ICT (which reduces investment costs) and the presence of a large variety of new products on many markets (which makes the informational problems more substantial), we expect the conditions of Corollary 5 to be unlikely to be met in the contemporary world.

First-period purchase decisions. In the first period, the monopolist sets the price and the consumers willing to buy at that price purchase. In particular, after having observed the price p_1 charged by the monopolist, agent i decides whether to buy the product. The utility that she enjoys is $u_i = r_i - p_1$ if the good is bought, and 0 otherwise. According to our informational assumptions, only a proportion $1 - \beta$ of the population is aware of the existence of the product and can then, in principle, buy it. Our assumption that the reservation prices are uniformly distributed implies the probability of buying the good to be equal to $(1 - p_1)$. Accordingly, the total number of buyers at price p_1 is:

$$\mathbb{E}\left[D_1(p_1)\right] = (1-\beta)(1-p_1)n.$$

The remaining part of the population is composed of βn agents who are uninformed and $(1-\beta)np_1$ who are informed but not interested in buying the product at price p_1.

First-period price setting. Anticipating what will occur in the second period and having expectations about the purchase decisions of the present period, the monopolist sets the price to maximise its inter-temporal profits as defined in Equations (1) and (2):

$$\pi = \pi_1 + \pi_2(b^*, k^*) = n\left(1-p_1\right)\left(1-\beta\right)p_1 + \left[\frac{\beta}{4} - \frac{C}{\phi_b^*\left(D_1^{Inv}(p_1)\right)\underline{k}^*}\right] n\bar{\rho}^*\left(D_1^{Inv}(p_1)\right).$$

The optimal price p_1^* follow from the balance between the *margins vs. demand* trade-offs of the first period and of the second period. The first trade-off is direct, as a higher price entails higher margins but smaller demand from period one consumers. The second one is indirect, as a higher first period price reduces the number of potential speakers, who are the means through which information about the product circulates in the second. This reduction results in stronger individual incentives to speak, as there is less potential competition for each bonus; consequently, the bonus needed to generate a given level of information turns out to be lower, with an increase in margins. This is captured by function $\phi_b^*(D_1^{Inv}(p_1))$. However, at the same time, fewer potential investors also mean less circulation of information in the second period, with a consequent reduction in second period demand represented by $\bar{\rho}^*(D_1^{Inv}(p_1))$. Formally, the specific first period price follows from the first order condition of the maximisation of profit and will depend on the second-period spread of information, which in turn is network-specific:

$$p_1^* = \frac{1}{2} - \frac{\sum_{k \geq \underline{k}^*} f(k)}{2} \left[\left(\frac{\beta}{4} - \frac{C}{\phi_b^* \underline{k}^*} \right) \frac{\partial \bar{\rho}^*}{\partial D_1^{Inv}} - \bar{\rho}^* \frac{C \frac{\partial \phi_b^*}{\partial D_1^{Inv}}}{(\phi_b^* \underline{k}^*)^2} \right].$$

By simple comparisons of prices, the following proposition gives some results concerning how prices change over time in our setting and compares our prices with a benchmark case of no reward.

Proposition 6. *The second-period price p_2^* is always higher than the price paid by earlier consumers p_1^*. Moreover, early-informed consumers pay in the first period a price lower than the one that would be paid without the introduction of the bonus. The presence of the bonus does not change the second-period price.*

Proof. *Increasing Prices.* Assume that the bonus is introduced. As stated in Corollary 5, this requires that the monopolist realizes positive profits for some $\underline{k}$ and more specifically for $\underline{k}^*$. Thus, C and β must be such that $\frac{\beta}{4} - \frac{C}{\phi_b^* \underline{k}^*} > 0$. Since $\frac{\partial \bar{\rho}^*}{\partial D_1^{Inv}} > 0$ and $\frac{\partial \phi_b^*}{\partial D_1^{Inv}} < 0$, p_1^* has the highest bound at $1/2 = p_2^*$.

Comparisons with the no-bonus regime. Assume the case in which the reward is not introduced. This implies that only the first-period informed agents can be attracted and the maximisation problem reduces to the choice of a price p that solves:

$$\max_p np(1-\beta)(1-p),$$

which yields the optimal price again equal to $1/2$. ■

Proposition 6 implies some remarkable effects on the welfare of informed consumers. On the one hand, they see their price increase from period one to period two. From the point of view of the monopolist this increase in price works as a partial source to cover the expense in terms of bonuses to be paid. Moreover, it creates among old buyers some transfers from agents obtaining few bonuses to those receiving many. Indeed, only some informed consumers will obtain enough referral bonuses to cover the increase in price. Typically, the final position (winner or loser) of one agent depends on his popularity in the consumer network, as people with higher degrees expect to receive more bonuses.

Compared to the case in which no referral bonus program is run, all consumers are better off as the prices are never above the benchmark case. This price-setting behaviour should be read as follows. In the first period, a lower-than-the-benchmark price is offered so as to attract more potential investors in the second period, sowing seeds for the development of the second-period market. Once a sizeable amount of informed consumers buy the product, the monopolist can offer them the bonus, and the price comes back to the one that would have been charged without the bonus, with the usual price equalizing marginal revenue with marginal cost (0 in our case). Without bonus, the need for seeding in the first period is not there, as consumers would only be buyers in the strict sense and not channels to enlarge demand.

5 Making Social Network Structure Explicit

The results drawn in the previous section are general, being valid for any conceivable structure of social interactions. Generality, however, comes at the price of being unable to define explicitly the dynamics relative to the choice of the minimal connectivity cutoff, which is simply defined as the one that maximises profits once the other variables are set optimally.

Refining these results requires to make assumptions about the consumer's social network and to make the degree distribution $f(k)$ explicit. This allows to perform a numerical comparative statics analysis of results in different setups, which clarifies the changing weights of the different incentives. Besides, some network structures are more likely to be realistic than others as human social networks tend to have specific topological characteristics. Thus studying specific classes of networks increases the empirical relevance of our results.

In this section, we discuss precise numerical solutions of our model for specific classes of degree distributions, typical of the theoretical literature on social networks. The first type is the random networks (Erdős and Rényi 1959; Gilbert 1959). Following the construction mechanism of Gilbert (1959) these graphs are characterised by a given number of nodes (n) and a given probability $0 \leq \lambda \leq 1$, which describes the chance of each link between pairs of nodes to exist. Since λ is assumed to be equal for each pair of nodes, these networks are characterised by a binomial distribution of degrees, i.e., $\forall 1 \leq k \leq n-1$:

$$f(k) = \binom{n-1}{k} \lambda^k (1-\lambda)^{n-1-k}, \tag{9}$$

where λn approximates the characteristic degree of nodes in the networks. In other terms, λ (alternatively called p_{link}) can be considered as a measure of network density. While random networks with this type of distribution cannot be considered as good fit for most empirical human networks, they constitute an established benchmark upon which to discuss other topologies.

The second type of degree distribution we discuss characterises networks defined as scale-free due to the tendency of the standard deviation of the degrees to diverge. This type of construction does fit many of the characteristics of empirical social networks, in particular the observation that a lot of them approximately follow a power law degree distribution (for specific examples see Ugander et al. 2011; Ebel et al. 2002; Liljeros et al. 2001; Barabási

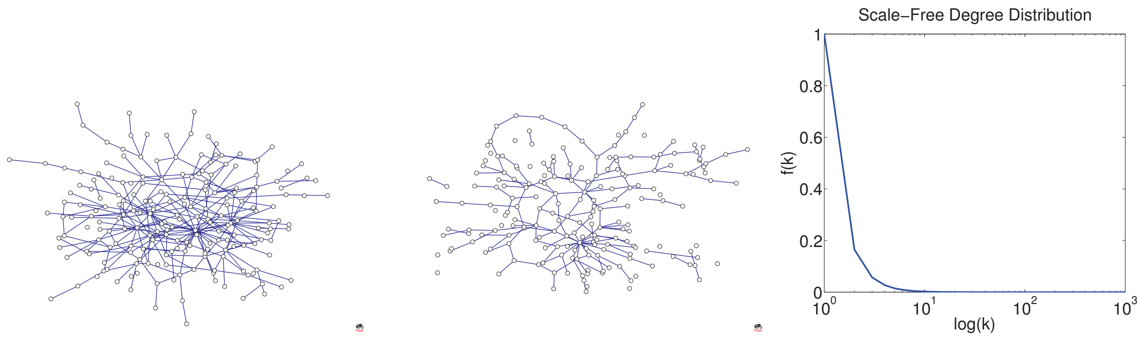

Figure 2: Two examples of scale-free networks with 400 agents and the p.d.f. from which the second one is drawn (Right Panel). The network in the Left Panel is an example of a scale-free network with a slope $\gamma = 2.2$, while the one in the Central Panel with a slope $\gamma = 2.6$.

et al. 2002; Yu and Van de Sompel 1965; Albert et al. 1999). Formally, we study networks with:

$$f(k) = \frac{1/k^{\gamma}}{\sum_{k \in N} (1/n^{\gamma})},$$

where $1 < \gamma \leq 3$ represents the slope of the power law. Two characteristics of this class of networks need to be emphasised to understand our results. The first is that increasing the parameter γ implies lowering the probabilities to observe highly connected individuals, thus leading to sparser networks (see Figure 2 for a graphical exemplification). The second is that this type of network reproduces the empirically observed fact that the probability of having individuals that are much more connected than the population's average is significantly higher than what random networks would suggest.

5.1 Random networks

Random networks, with their bell-shaped degree distributions, represent a good benchmark upon which to compare results obtained from other network topologies. In the following we discuss how our outcomes depend on both network and non-network parameters of the model. On the one hand, as noted in the previous section, the proportion of uniformed people in the initial population β has an impact on both the incentives to invest (positive) and on the number of potential investors in the second period (negative). Which of these two effects dominates depends strongly on the degree distribution $f(k)$ considered. On the other hand, the cost of investing C affects (negatively) only the personal incentives to diffuse information. Thus, the sign of its effect does not depend strictly on the structure of the social network (see Proposition 2). For this reason, we chose to fix the value of the latter parameter and to focus our attention on the interaction between β and the network structure. Results for different costs would be simply shifted.

In Figure 3, we numerically solve the model for specific binomial degree distributions and we study the results for each possible combination of the proportion of uninformed agents

$0.05 \leq \beta \leq 0.95$ (with steps of 0.05) and of $0.05 \leq \lambda \leq 1$ (again, with steps of 0.05)[7]. To obtain the displayed results we fix $C = 0.002$ and $n = 400$ and we compute the profit maximising threshold for investing k^* as well as the optimal referral bonus b^* and first-period price p_1^*. The optimal values are found, for each combination of parameters, by exhaustive search. The model is solved for all possible values of the monopolist's decision variables and the one offering the highest profit is selected. This method guarantees that the values found are the global maxima. This allows for obtaining a numerical comparative statics of our model.

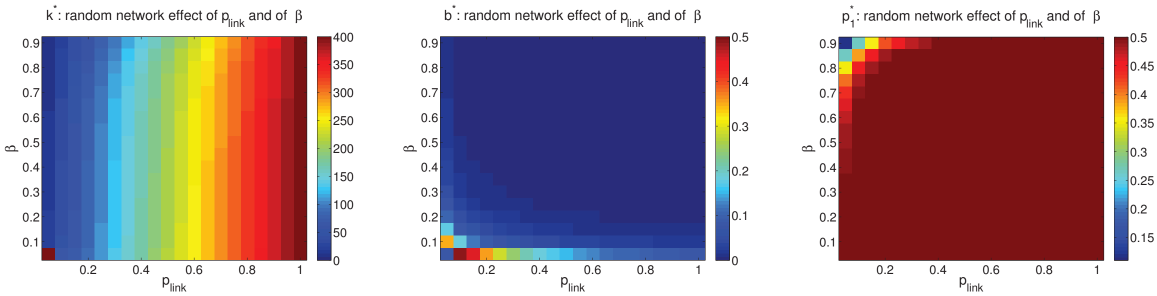

Figure 3: Numerical solution of the model using a random network of 400 individuals as $f(k)$. Results are reported for different combinations of the proportion of uninformed individuals β (vertical axis) and of the probability of links between each pair of nodes to exist λ (horizontal axis). Left Panel shows the results concerning the optimal setting of the cutoff (k^*), Central Panel shows the setting of the optimal bonus awarded for each referral (b^*), Right Panel shows the optimal value of the first period price p_1^*. In all simulations $C = 0.002$.

Lemma 7. *In random networks, k^* is always monotonically increasing in both λ and β (Left Panel of Figure 3).*

The Left Panel of Figure 3 reports the results the results concerning the optimal cutoff. In other terms, these results tell us where the infra-marginal agent (the one with just enough connections to invest given the incentives) is positioned, given the incentives set optimally by the monopolist. It is immediately evident that in a random network the incentives are such that only the most-connected half of the population is motivated to invest in his social networks, regardless of the network density λ. Correspondingly, increasing the density increases the targeted degree and the first-period prices (Right Panel of Figure 3) and reduces the optimal bonus required (Central Panel of Figure 3).

Concerning the impact of β, at any level of λ, the optimal cutoff is decreasing with the proportion of the uninformed. Indeed, when there are fewer informed agents, the monopolist needs to extend the proportion of those who invest in communication. We know that she can do this by increasing b and/or reducing p_1. Figure 3, allows us to conclude that, in the presence of a significant informational problem, the monopolist prefers to lower the first period price so as to extend the number of potential investors, while the relative level of b^*

[7]As discussed above λn approximates the characteristic degree of agents in the population. The degrees in the population are centered around this value when considering a random network (Equation 9).

is decreasing with β. In other words, in a random network, the lowered cutoff for high β's derives from decreased competition among the informed rather than from higher bonuses.

The choice of the monopolist to set incentives so that the cutoff is around the central value of the binomial distribution can be explained in a relatively simple way. Profit maximisation results from a trade-off between reducing prices (or increasing bonuses), thus extending the new demand due to information circulation, or doing the opposite, increasing margins made on a smaller group of consumers. By setting the cutoff around the average degree in the population, she manages to have the information diffused by a large proportion of his current customers (approximately 1/2 of them) while still providing contained bonus. Further efforts beyond this level would allow him to involve a decreasing additional share of individuals at a progressively higher cost in terms of reduced margins.

Lemma 8. *In random networks, b^* is monotonically always decreasing in the network density and in the proportion of the uninformed people (Central Panel of Figure 3).*

Indeed, the less consumers are connected with each other, the smaller will be their expected benefit for each given level of b and thus the higher the bonus demanded in equilibrium will be. Similarly, when increasing the proportion of the uninformed, the optimal k^* changes only slightly, thus each investor will face fewer expected competitors and thus will require lower bonus.

Lemma 9. *In random networks, p_1^* is always monotonically decreasing in both network density and proportion of uninformed agents (Right Panel of Figure 3).*

The relationship with the share of unaware people follows from the need of the monopolist to create a sufficiently large population of buyers in the first period in order to sustain information diffusion in the second period. As the number of informed agents dwindles, the monopolist needs to capture more of them to run an effective referral program. Moreover, at any level of β, the sparser the network, the more difficult it is to spread information. Therefore, the monopolist faces stronger incentives to extend the number of potential investors, even at the cost of reducing his margins on informed buyers.

5.2 Scale-free networks

Let us now consider the more realistic power law degree distribution, characteristic of scale-free networks. As before, we again propose a numerical solution of our model for all different combinations of power law's slopes $1.1 \leq \gamma \leq 3$ (in steps of 0.1)[8] and of the proportion of the uninformed changing $0.05 \leq \beta \leq 0.95$ (with steps of 0.05). For the different combinations of these two parameters, in Figure 4 we report the results concerning the profit maximising threshold for investing k^* (Left Panel), the optimal referral bonus b^* (Central Panel) and the first period price p_1^* (Right Panel). All values are computed fixing $C = 0.002$ and $n = 400$ with the aim to provide results comparable across parameter combinations.

The results in Figure 4 are quite different from those proposed in the previous subsection for random networks. Indeed, for scale-free networks the optimal choice of the monopolist

[8] Barabási and Albert (1999) note that most empirically studied social networks have slopes $2 \leq \gamma \leq 3$. As we will show below, when we take γ in such a interval, our model will give very clear-cut conclusions on the targeted cutoff. We prefer however to provide theoretical results for the whole spectrum of possible γs.

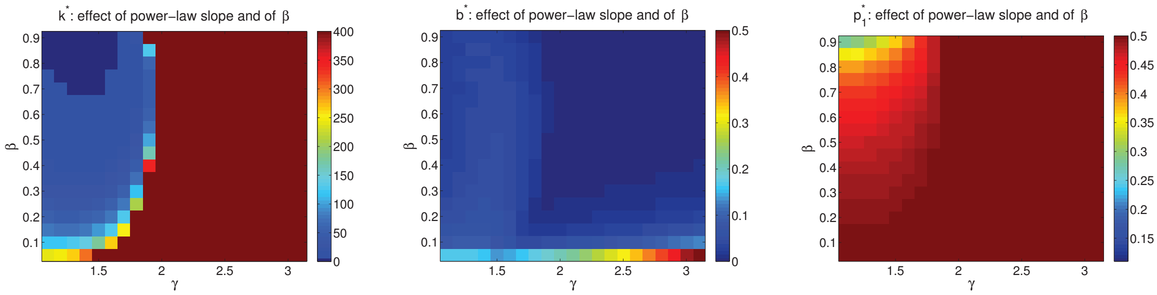

Figure 4: Numerical solution of the model using scale-free networks of 400 individuals as a social network. Results are reported for different combinations of the proportion of the uninformed β (vertical axis) and of the slope of the power law γ (horizontal axis). Left Panel shows the results concerning the optimal setting of the cutoff (k^*), Central Panel shows the setting of the optimal bonus awarded for each referral (b^*), Right panel shows the optimal value of the first-period price p_1^*. In all simulations C=0.002.

suddenly changes at a certain level of $\gamma = \bar{\gamma}(\beta)$, characterising a sharp transition in the monopolist's decisions. At each level of β, above the critical slope, incentives are set so that only the most connected consumers are induced to communicate. Below $\bar{\gamma}(\beta)$, incentives are instead set so that many individuals are willing to invest and to diffuse information. Notably, the targeted degree experiences a big jump downwards and then responds to variations of β, as increasing the latter reduces the competition among informed buyers. Fixing a low γ (say 1.7) and progressively increasing β, the monopolist optimal offer will incentivise lower degree consumers, since the bonus needed to induce their investment gets progressively lower.

The value of the optimal bonus (Central Panel of Figure 4) is very small above $\bar{\gamma}(\beta)$ as highly connected individuals have very strong incentives to invest both in terms of the expected benefits and in terms of the lack of competition when they are the only ones targeted. Below the critical value of γ, the bonus required is suddenly higher because competition becomes tougher.

Lemma 10. *In Scale-free networks, p_1^* is decreasing in network density and weakly decreasing in the proportion of uninformed agents (Right Panel of Figure 4).*

The relationship between β and the optimal first period price p_1^* is consistent with the observations made for random networks. The price is lower when fewer people are aware of the existence of the monopolist's product as the the latter attempts to maximise information diffusion. However, considering that network density is decreasing with γ for scale-free networks, the relationship between price and network density is reversed with respect to the random network case. Indeed, fixing β and increasing γ increases the optimal prices as the leverage effect of the number of potential investors becomes smaller. Once such leverage effect becomes too small, the monopolist chooses p_1^* almost equal to 0.5 and focuses on maximising margins on current customers.

In order to understand the reasons for the abrupt change in the producer's choice observed for the scale-free networks and to generally characterise the monopolist's choice in this setup, we need to focus our attention on the differences between the area below and above the critical value $\bar{\gamma}(\beta)$. In order to simplify the analysis, let us fix the proportion of the uninformed to

$\beta = 0.65$ and observe the optimal choice of k^* before and after the phase transition. Figure 5 represents the profits that the monopolist obtains when she selects her decision variables optimally so as to obtain a given $\underline{k}$ cutoff for his investment.

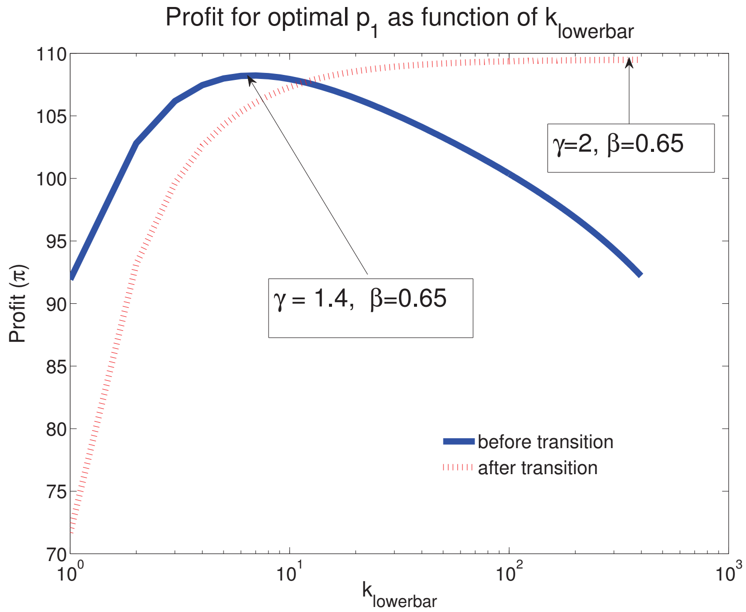

Figure 5: Profit of the monopolist as function of the chosen level of cutoff for the investment. The blue line represents profits from the case in which $\gamma = 1.4$ while the red dotted line represents the case in which $\gamma = 2$. In both cases $\beta = 0.65$, $C = 0.002$ and the size of the population $n = 400$.

Lemma 11. *For $\gamma > \bar{\gamma}(\beta)$ the monopolist maximizes margins. For $\gamma < \bar{\gamma}(\beta)$, the monopolist trades off a reduction in her margins in exchange for a larger additional demand.*

One can immediately observe that the structure of the monopolist's trade-off differs significantly before and after the phase transition. This difference is grounded in the interaction between economic incentives to invest and the topology of scale-free social networks. A larger γ decreases the average degree in the population and makes the network sparser. Therefore, the competition decreases and targeting individuals of relatively higher degrees requires less incentive and produces larger margins. Conversely, lowering γ produces a denser network where informed consumers require comparatively higher bonuses due to increased competition. This reduces the seller's margins on the new demand. However, higher degrees in the network also imply that investing agents are more effective in spreading information and in increasing the monopolist's demand.

Above the phase transition, profits are monotonically increasing in the $\underline{\text{k}}$ chosen. Indeed, with sufficiently sparse networks the vast majority of individuals have few connections and it is expensive for the monopolist to motivate them to invest. In such setups the monopolist has strong incentives to reduce the costs involved in the referral program by increasing the targeted $\underline{\text{k}}$. Thus, in scale-free networks with high γ the role of the few individuals that are much more connected than the average (hubs) becomes pivotal. They can be motivated with very little incentives given their degree and the fact that they suffer little competition while guaranteeing a very efficient diffusion of the information. The monopolist prefers to obtain high margins on his current clients and on a small additional demand, thus running a small referral program (which involves only the most connected agents). The alternative would not be cost-effective since the gains in information diffusion from the investment of agents with few connections are small.

Below the critical value $\bar{\gamma}(\beta)$, profits become non monotone in $\underline{\text{k}}$ as a proper trade-off between diffusion of information and costs is introduced. From the topological point of view with lower γs the difference in popularity between the hubs and the average individuals comparatively becomes smaller as the network becomes denser. Thus it becomes cost effective for the monopolist to induce agents with somewhat lower connectivity to invest in their social networks. On the other side, motivating agents with very few connections would be still too expensive. The monopolist will thus settle on an intermediate value of connectivity, similarly to what observed for random networks.

6 Conclusion

We considered the strategies of a monopolist facing a partially uninformed population of consumers. Having some knowledge about the social network that interconnects his current and potential clients (limited to the distribution of the number of contacts in the population), she runs a referral program offering to his current customers bonuses in exchange for the introduction of new clients. These rewards incentivise part of the current customer base in order to invest in communication, thus creating a flow of information that generates new buyers and extends the demand of the monopolist.

From the point of view of the monopolist, introducing the referral program is convenient as long as the informational problem that she faces is significant and as long as the cost of the investment is not prohibitively high. These conditions for the optimality of the referral program tend to be more easily met in contemporary markets. On the one hand, the diffusion of ICT and of online social networks makes diffusing information cheaper for consumers, both in terms of time and money. On the other hand, the presence of a large multiplicity of goods and services and the frequent launch of new products imply that companies frequently face significant informational problems. We thus expect referral programs to become more extensively used in a large variety of markets in the future.

The introduction of the rewards in the second period has different effects on the utility of different agents. Earlier uninformed potential buyers who receive the information about the product are clearly better off, since they can now buy a potentially valuable good. Informed consumers see prices increase in the second period when the bonus is introduced, while only some of them obtain referral bonuses. This means that some informed consumers are worse off and some others are better off in the second period. Indeed, agents obtaining many bonuses are compensated for the increased price, while those receiving fewer or no bonuses are not. The probability to be on each side of this division depends on one's popularity (the degree) in the consumer network.

Comparing our results to the case in which no referral bonus program is run, it is clear that all consumers are better off as the prices are never above the benchmark case. This is a further testimony of the power of referral marketing, which can thus be considered a very effective way to solve significant informational problems.

Referral bonus programs work by introducing incentives so that clients spread the information about the existence of the company's product to their social neighbours. Such incentives are clearly stronger in the case of more central (or popular) individuals. This leads to the emergence of a minimal degree, above which an agent invests and communicates with peers. The level at which this critical degree settles depends strongly on the network structure. For this reason we analysed specific types of network structures, that is, random and scale-free networks. In the first case, we showed that roughly the most-connected half of informed consumers are incentivised to invest in equilibrium, regardless of the network density. On the contrary, using the more realistic scale-free network topology, which fits better the characteristics observed in empirical networks, the proportion of investing customers takes relatively more extreme values. The monopolist faces a choice between two very different options: to maximise demand by running an expensive referral program that induces many informed agents to invest, or to maximise margins by running a very limited program that is only adopted by the most-connected individuals. His choice depends essentially on how likely it is to observe agents with unusually high degrees. In many empirically observed social networks, we expect the second option to be preferred.

Our results confirm that centrality matters when pricing is done in social networks as in Bloch and Quérou (2013). The main difference is that, in our setup, the monopolist has only limited information about the topology of the network while they assume that the producer is fully aware of all nodal characteristics of each single agent. Moreover, the results we provide are very different. Bloch and Quérou (2013) find that central agents are charged more, unless consumption generates some positive externality on other consumers. In our model being central is always advantageous as it allows to receive "discounted" prices (due to the presence of the rewards).

While it is reasonable to assume that the reservation price and the degree are independent, one could challenge our assumption in that the probability of being informed is independent of centrality and reservation prices. For example, one could consider the case in which a more central node may have a higher probability of being informed. However, the only channel through which an agent may become informed is by receiving the information through its social network. The communication among agents is indeed the core of this paper and captures the fact that highly-connected people who are initially uninformed will be more likely to receive the information in the second period.

Studying the monopoly case is a necessary starting point in order to understand the effects of limited information about the consumer's network on pricing. However, most markets where such programs are run are, up to some degree, oligopolistic. Consequently, our current research endeavours focus on extending this setup to an imperfect competition environment, where firms compete in terms of prices. We expect that increasing the competitive pressure would push producers to offer higher rewards (thus extending the share of consumers interested in activating their social network). In such models, the informational problem described here could be accompanied by a problem of switching costs, which may induce producers to offer rewards to switchers as wells as to those who convince them to buy the good. Referral bonuses are also relevant in the context of entry models. Here, the challenge would be to understand the conditions under which the referral program is a way to prevent entrance for the incumbent or a way to gain part of the market for the entrant.

Finally, a future line of research would be to understand how informational problems can be solved using mixed marketing strategies involving both mass-media and the social network of consumers as channels of informational spread. Specifically, the strategy based on social networks discussed in the present paper can be seen both as an alternative and a complement to the one based on mass-media advertisement, which offers a uniform probability of reaching any potential customer.

Acknowledgments

We wish to thank Paul Belleflamme, Jean Marie Baland, Ennio Bilancini, Timoteo Carletti, Jacques Crémer, Marc Bourreau, Rosa Branca Esteves, Mathias Hungerbuhler, Paolo Pin, Stefano Galavotti, Károly Takács and Eric Toulemonde for their suggestions. We wish to thanks the audience to the seminars in Central European University, Cergy-Pontoise, Corvinus University, FUSL, Modena, MTA TK "Lëndulet" RECENS and Sassari for helping us improve our article. We thank the participants of the AFSE 2014, ASSET 2013 and ASSET 2014 Conferences as well as those at Ecore Summer School 2013 (Leuven), Bomopav 2015 (Modena), IO in the digital Economy Workshops (Liège) for their useful comments and critics. Elias Carroni acknowledges the "Programma Master & Back - Regione Autonoma della Sardegna" for financial support. Simone Righi acknowledges the "International Mobility Fund" and the "International Publication Fund" of the Hungarian Academy of Sciences for financial Support.

7 Appendix

7.1 Proof of Proposition 2

$\bar{\rho}(D_1^{Inv})$ is increasing and $\phi_b(D_1^{Inv})$ is decreasing in D_1^{Inv}. All the results of Proposition 2 depend on the effect of all the variables on the number of investors $D_1^{Inv} = n(1-\beta)(1-p_1)\sum_{k\geq\underline{k}} f(k)$. We clearly have the following partial derivatives:

$$\frac{\partial D_1^{Inv}}{\partial p_1} = -n(1-\beta)\sum_{k\geq\underline{k}} f(k) < 0, \tag{10}$$

$$\frac{\partial D_1^{Inv}}{\partial \sum_{k\geq\underline{k}} f(k)} = n(1-\beta)(1-p_1) > 0. \tag{11}$$

From Equation (11) we can conclude that any variable affecting $\underline{k}$ affects the number of investors in the opposite way. According to the inequality in (6), $\underline{k}$ changes in response to a change in b, C, β or p_2. An increase in b or β makes people with a lower degree willing to invest, since the incentives become higher for each degree level. This implies that the minimal degree for investment decreases. Conversely, $\underline{k}$ increases in response to a rise in C or p_2. A consequence is that the number of investors D_1^{Inv} increases in response to a rise in

b and decreases when C, p_2 or p_1 become lower. From the same reasoning follow the signs of the effects on the diffusion of information $\bar{\rho}$ and on the probability of getting the bonus ϕ_b.

β instead has a non monotonic effect on the number of investors. Let us assume to move the proportion of investors from β to β', with $\beta' > \beta$. This will make the cutoff move from $\underline{k}$ to $\underline{k}-1$, as expectations about the number of bonuses are revised upward (more potential buyers and fewer potential competitors). By computing the variation in the number of investors, we get:

$$D_1^{Inv}(\beta') - D_1^{Inv}(\beta) = n(1-\beta')(1-p_1)\sum_{k\geq \underline{k}-1} f(k) - n(1-\beta)(1-p_1)\sum_{k\geq \underline{k}} f(k). \tag{12}$$

The difference in 12 is positive if the following inequality holds:

$$(1-\beta')f(\underline{k}-1) \geq \beta' - \beta. \tag{13}$$

and it is negative if this is reversed. The sign of the inequality above will determine the sign of the effect of an increase in β on the two functions $\bar{\rho}$ and ϕ_b.

7.2 Proof of Lemma 3

The demand in the second period is composed of two parts. D_2^2 is the demand coming from newly informed people while D_2^1 represents the number of early informed people buying the good in time 2. We have:

$$D_2 = D_2^2 + D_2^1 = \beta(1-p_2)\bar{\rho}n + (1-\beta)(1-p_2)n. \tag{14}$$

Computing the partial derivatives yields the result:

$$\frac{\partial D_2}{\partial b} = \beta(1-p_2)n\frac{\partial \bar{\rho}}{\partial b} > 0 \text{ since } \frac{\partial \bar{\rho}}{\partial b} > 0, \tag{15}$$

$$\frac{\partial D_2}{\partial p_1} = \beta(1-p_2)n\frac{\partial \bar{\rho}}{\partial p_1} > 0 \text{ since } \frac{\partial \bar{\rho}}{\partial p_1} < 0, \tag{16}$$

$$\frac{\partial D_2}{\partial p_2} = -\beta\bar{\rho}n - (1-\beta)n < 0, \tag{17}$$

$$\frac{\partial D_2}{\partial \beta} = \underbrace{(1-p_2)\bar{\rho}n\frac{\partial \bar{\rho}}{\partial \beta}}_{\text{ambiguous}} + \underbrace{(1-p_2)n(\bar{\rho}-1)}_{<0}. \tag{18}$$

7.3 Proof of Proposition 4

Proof. Let us define $\pi_2^*(\underline{k})$ as the maximised second period profits for a given cutoff $\underline{k}$. These profits are maximised in the sense that p_2 and b are chosen optimally under the constraint that the monopolist wants a share $\sum_{k\geq \underline{k}} f(k)$ of old buyers to invest.

To explicitly find out this function, let us maximise the profit for given $\underline{k}$. We will do it in three steps:

(i) *Choice of the bonus.* Since the bonus b represents a cost for the monopolist, the optimal b to obtain a given $\underline{k}$ is the one such that the constraint above is binding, i.e.,

$$b_{(\underline{k})} = \frac{C}{(1-p_2)\beta\phi_b(\underline{k})\underline{k}}, \tag{19}$$

where the subscript $(\underline{k})$ means that $b_{\underline{k}}$ generates $\underline{k}$. Consequently, choosing such a b, we obtain

$$\pi\left(\underline{k}, p_2\right) = \left(p_2 - \frac{C}{(1-p_2)\beta\phi_b(\underline{k})\underline{k}}\right)\beta(1-p_2)\bar{\rho}(\underline{k})n + p_2(1-\beta)(1-p_2)n, \tag{20}$$

which is the profit that would be obtained by setting a bonus generating cutoff $\underline{k}$.

(ii) *Choice of the price.* The price that maximises $\pi\left(\underline{k}, p_2\right)$ is the solution to:

$$\max_{p_2} \pi\left(\underline{k}, p_2\right). \tag{21}$$

The first order condition of the problem above requires that $(1-2p_2)\left(\frac{\beta\bar{\rho}+1-\beta}{\beta\bar{\rho}}\right) = 0$ or simply $p_2^* = \frac{1}{2}$.

(iii) *maximised profit.* Plugging p_2^* into equation (19) and into (21) we find:

$$\pi^*(\underline{k}) = \left[\frac{\beta}{4} - \frac{C}{\phi_b(\underline{k})\underline{k}}\right] n\bar{\rho}(\underline{k}) + \frac{(1-\beta)n}{4} \tag{22}$$

and

$$b^* = \frac{2C}{\beta\phi_b(\underline{k})\underline{k}}. \tag{23}$$

To conclude the proof, since the function $\pi^*(\underline{k})$ can take a finite number of values as $\underline{k} \in \{1, 2, ..., n-1\}$ one of them is the maximum. We call $\underline{k}^*$ the cutoff leading to this maximal value. ■

References

Albert, R., Jeong, H., and Barabási, A.-L. (1999). Internet: Diameter of the world-wide web. *Nature*, 401(6749):130–131.

Arndt, J. (1967). Role of product-related conversations in the diffusion of a new product. *Journal of Marketing Research (JMR)*, 4(3).

Banerji, A. and Dutta, B. (2009). Local network externalities and market segmentation. *International Journal of Industrial Organization*, 27(5):605–614.

Barabási, A.-L. and Albert, R. (1999). Emergence of scaling in random networks. *Science*, 286(5439):509–512.

Barabási, A.-L., Jeong, H., Néda, Z., Ravasz, E., Schubert, A., and Vicsek, T. (2002). Evolution of the social network of scientific collaborations. *Physica A: Statistical Mechanics and its Applications*, 311(3):590–614.

Biyalogorsky, E., Gerstner, E., and Libai, B. (2001). Customer referral management: Optimal reward programs. *Marketing Science*, 20(1):82–95.

Bloch, F. and Quérou, N. (2013). Pricing in social networks. *Games and Economic Behavior*, 80(C):243–261.

Campbell, A. (2013). Word of mouth and percolation in social networks. *American Economic Review*, 103(6):2466–2498.

Ebel, H., Mielsch, L.-I., and Bornholdt, S. (2002). Scale-free topology of e-mail networks. *Phys. Rev. E*, 66:035103.

Erdős, P. and Rényi, A. (1959). On random graphs. *Publicationes Mathematicae Debrecen*, 6:290–297.

Galeotti, A. (2010). Talking, searching, and pricing. *International Economic Review*, 51(4):1159–1174.

Galeotti, A. and Goyal, S. (2009). Influencing the influencers: a theory of strategic diffusion. *The RAND Journal of Economics*, 40(3):509–532.

Gilbert, E. N. (1959). Random graphs. *The Annals of Mathematical Statistics*, 30(4):1141–1144.

Huston, D. (2010). http://www.slideshare.net/gueste94e4c/dropbox-startup-lessons-learned-3836587.

Iyengar, R., Van den Bulte, C., and Valente, T. W. (2011). Opinion leadership and social contagion in new product diffusion. *Marketing Science*, 30(2):195–212.

Jackson, M. (2005). The economics of social networks. *California Institute of Technology.*

Katz, M. L. and Shapiro, C. (1985). Network externalities, competition, and compatibility. *The American economic review*, 75(3):424–440.

Kornish, L. J. and Li, Q. (2010). Optimal referral bonuses with asymmetric information: Firm-offered and interpersonal incentives. *Marketing Science*, 29(1):108–121.

Lazarsfeld, P. and Katz, E. (1955). Personal influence: the part played by people in the flow of mass communications. *Glencoe, Illinois*.

Liljeros, F., Edling, C. R., Amaral, L. A. N., Stanley, H. E., and Åberg, Y. (2001). The web of human sexual contacts. *Nature*, 411(6840):907–908.

Sernovitz, A. and Clester, S. (2009). *Word of mouth marketing: How smart companies get people talking*. Kaplan Austin.

Shi, M. (2003). Social network-based discriminatory pricing strategy. *Marketing Letters*, 14(4):239–256.

Sundararajan, A. (2006). Local network effects and network structure. Industrial Organization 0412011, EconWPA.

Ugander, J., Karrer, B., Backstrom, L., and Marlow, C. (2011). The anatomy of the facebook social graph. *arXiv preprint arXiv:1111.4503*.

Van den Bulte, C. and Joshi, Y. V. (2007). New product diffusion with influentials and imitators. *Marketing Science*, 26(3):400–421.

Yu, P. and Van de Sompel, H. (1965). Networks of scientific papers. *Science*, 169:510–515.

Zandt, T. V. (2004). Information overload in a network of targeted communication. *RAND Journal of Economics*, 35(3):542–560.