Game Theory, Population Dynamics, Social Aggregation

Daniele Vilone (CSDC - Firenze)
Namur - 18.12.2008

Summary

- Introduction
- General concepts of Game Theory (GT)
- Game Theory and Social Dynamics
- Application: “Game of Social Norms”
- A microscopic approach to the GSN
- Conclusions

Hystorical overview of GT

- GT was developped to modelize decisional processes where rational agents interact in order to get some income (Von Neumann, Zermelo, Borel)
- Then, it was used for the study of the behaviour of stock markets and financial processes (Von Neumann, Nash, Sen *et al.*)
- GT is very useful also to understand evolutionary phenomena (populations under natural selection: Smith, Sigmund *et al.*)
- Recently, it has been applied to social dynamics
- In general, GT can be used when a pay-off matrix can be written for the process under scrutiny

FUNDAMENTAL CONCEPTS

- *N* agents (*i=1,...,N*), M possible (pure) strategies for each player

$$\vec{s}_i \equiv \{ s_1 , \dots , s_{M_i} \}_i$$

  - Pay-off functions

$$u_i = u_i \left(\{ \vec{s}_j \}_{j=1,\dots,N} \right)$$

  - Nash Equilibrium:

$$u_i \left(s_1^E , \dots , s_i^E , \dots , s_N^E \right) \geq u_i \left(s_1^E , \dots , s_i , \dots , s_N^E \right)$$

$$\forall i , \forall s_i \neq s_i^E$$

GAMES 2X2

- N=2 players with only M=2 pure strategies, C (cooperation) and N (no cooperation, or defection)
- Pay-off matrix

$$\begin{pmatrix} u_i(C,C) & u_i(C,N) \\ u_i(N,C) & u_i(N,N) \end{pmatrix} \quad i=1,2$$

Nash Equilibria in 2x2 games

- In this case, finding pure N. eq. is easy by means of pay-off bimatix:

$1/2$	C	N
C	$u_{1,CC};u_{2,CC}$	$u_{1,CN};u_{2,NC}$
N	$u_{1,NC};u_{2,CN}$	$u_{1,NN};u_{2,NN}$

  - Positions reached by two arrows are the Nash equilibria. Arrows go towards the biggest value of the pay-off.

Mixed strategies games

- For iterated games analysis, it is convenient to allow players to adopt mixed strategies. For a 2x2 game, a mixed strategy is the vector

$$\vec{p} = (p, 1 - p) \qquad [1]$$

  - In eq. [1] *p* is the probability to adopt the strategy *C* by a player, *1-p* the probability to adopt *N*.
  - From the *Nash theorem*, we know that a finite game admits always at least one equilibrium in mixed strategies ("finite game": game with finite number of players each one with a finite number of available strategies)

Replicator equation

- In Evolutionary Game Theory (*N* players interacting with each other), *p* is the fraction of cooperators in the population
- The dynamics is given by the replicator equation

$$\frac{1}{p}\cdot\frac{dp}{dt}=f_C-\Phi=p(1-p)(f_C-f_N)$$

  - f_C = averaged payoff of a cooperator
  - f_N = averaged payoff of a defector
  - Φ = averaged payoff of a generic player

“Game of Social Norms”

- The fitness of an individual increases with the number of other individuals sharing his social norms
- The fitness decreases according to an “economic criterium”, due to the effort to share the norms of a foreigner
- Said M_i the number of agents sharing the same norms of player *i*, and *2x* the mean distance between individuals with different norms, we have

1/2	C	N
C	$M_2 - x ; M_1 - x$	$M_2 - M_1 + 1 - 2x ; 1$
N	$1 ; M_1 - M_2 + 1 - 2x$	$0 ; 0$

Static Homogeneous Model

- If we assume in the pay-off matrix $M_1 = M_2 = M$, we get

1/2	C	N
C	$1+\epsilon;1+\epsilon$	$-2x;1$
N	$1;-2x$	$0;0$

$$(\epsilon = M - x - 1)$$

  - The quantity ϵ is crucial for the number and the nature of the Nash equilibria

Pure equilibria in homogeneous social norms game

1/2	C	N
C	$1+\epsilon;1+\epsilon$	$-2x;1$
N	$1;-2x$	$0;0$

1/2	C	N
C	$1-\lvert\epsilon\rvert;1-\lvert\epsilon\rvert$	$-2x;1$
N	$1;-2x$	$0;0$

- $\varepsilon>0$: (C,C) and (N,N)
  - $\varepsilon<0$: (N,N) only

Now, what happens in a large population of players?

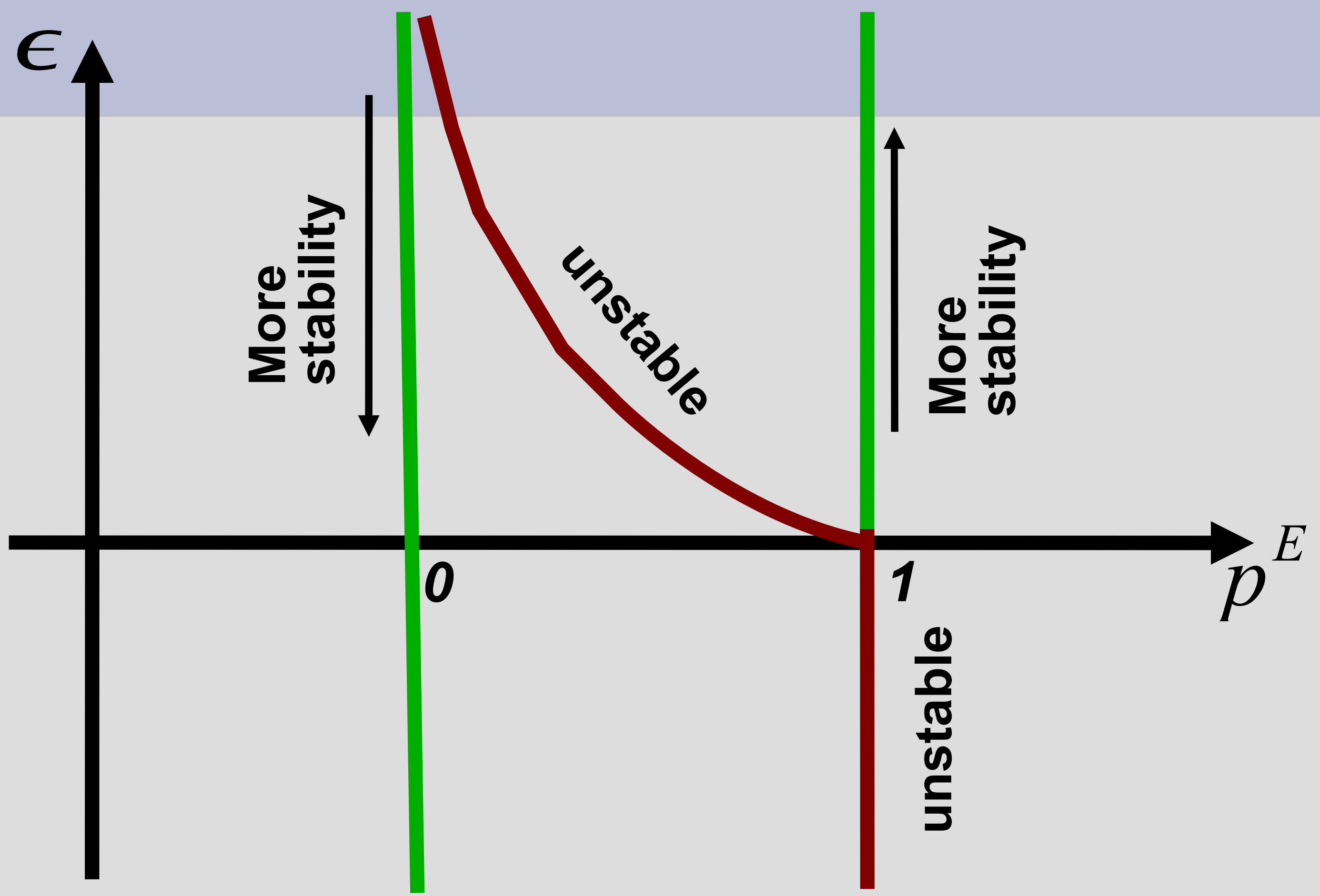
Mixed equilibria in homogeneous social norms game

- By solving the replicator equation for this case, we can find all the Nash equilibria
- For *ε<0* there is ***<u>one</u>*** stable equilibrium for *p*=0* (being *p* the fraction of the population adopting the strategy *C*)
- For *ε>0* we have ***<u>three</u>*** equilibria, *p*=0*, *p*=1*, **<u>and</u>** the mixed equilibrium in

$$p_M^E = \frac{2x}{2x+\epsilon}$$

  - To catch the physical meaning of this picture, a phase diagram of the equilibria is needed

Phase Diagram

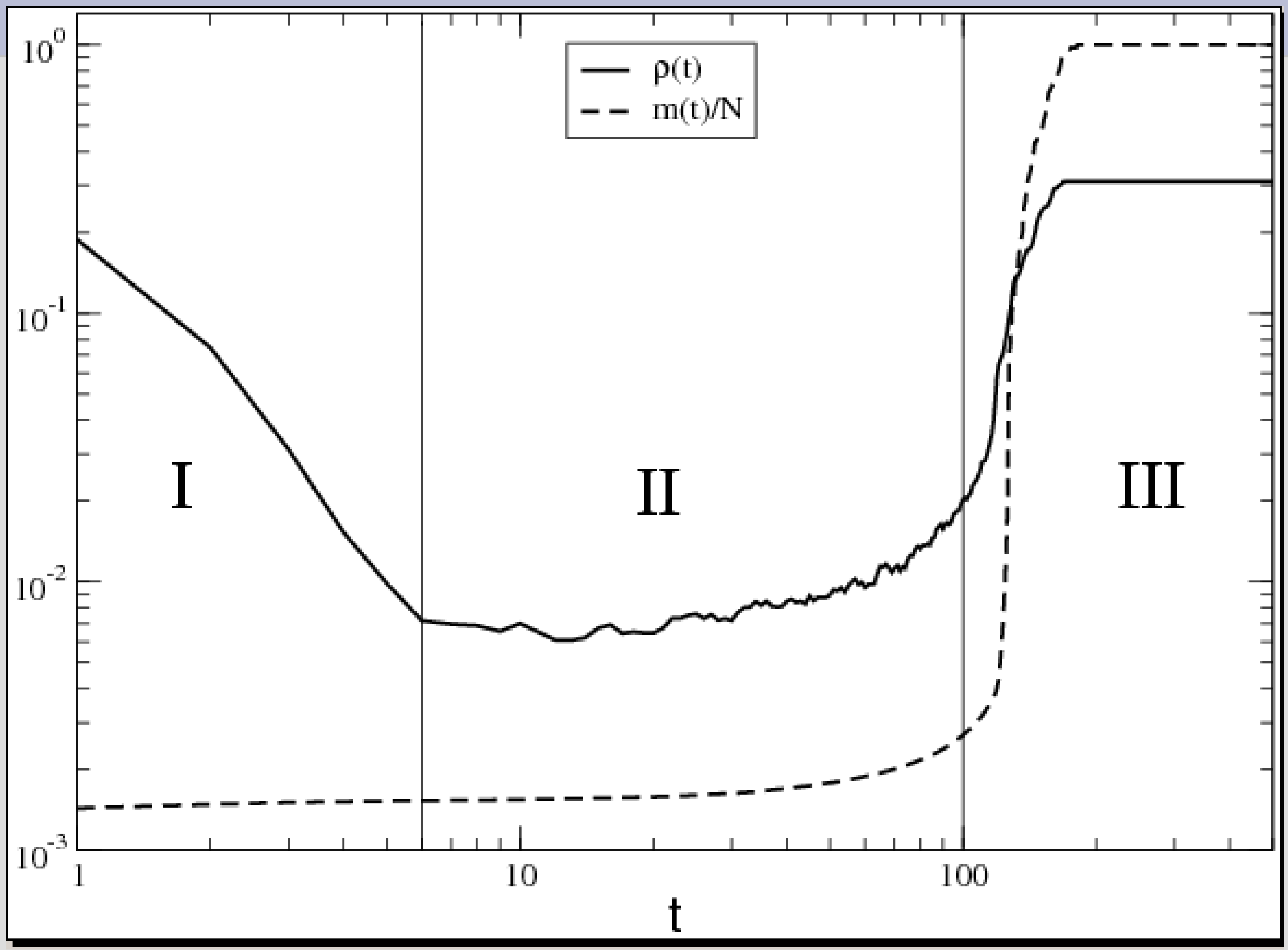


- The cooperation becomes an equilibrium more and more stable as ε (that is M) increases.
- Then, a rational agent will cooperate only if its fitness is already high, *i.e.* when the clusters' size is big.

Dynamical Homogeneous Model

- Initial configuration: system of *N* agents divided into N/m_0 clusters of the same size m_0 . The distance between two agents belonging to the clusters *i* and *j*, respectively, is $|i-j|$.
- Initial strategy of agents can be *C* or *N* with equal probability.
- At each elementary step two agents are randomly extracted. If they are in the same cluster, nothing happens, otherwise they “play the game”, according the usual payoff matrix.
- After the game, each agent computes what they would have gained if it had adopted the other strategy (being fixed the opponent's one).
- If the virtual payoff is bigger than the real one, agent changes its strategy.

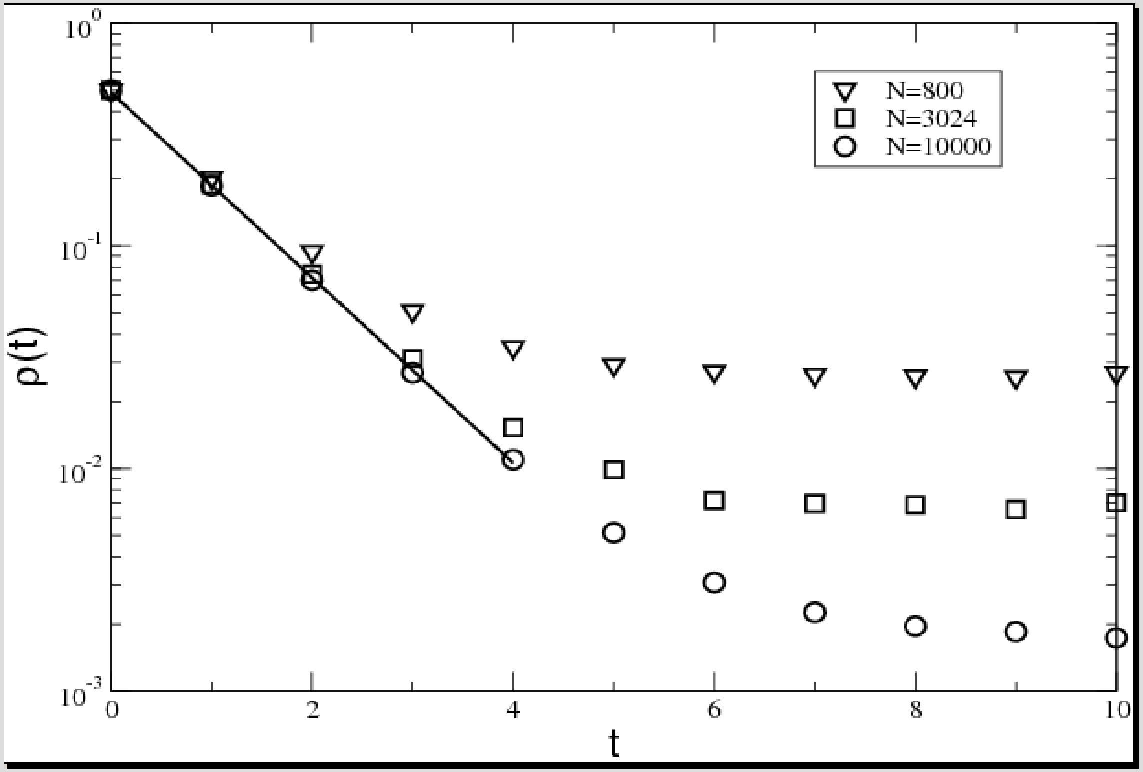
Dynamical behaviour



$$N = 3024\,;\, m_0 = 4\,;\, S_{iter} = 25\,;\, \rho_0 = 0.5$$

Three different regimes

I. In the first regime, cooperators' density decreases exponentially, averaged clusters' size is almost constant



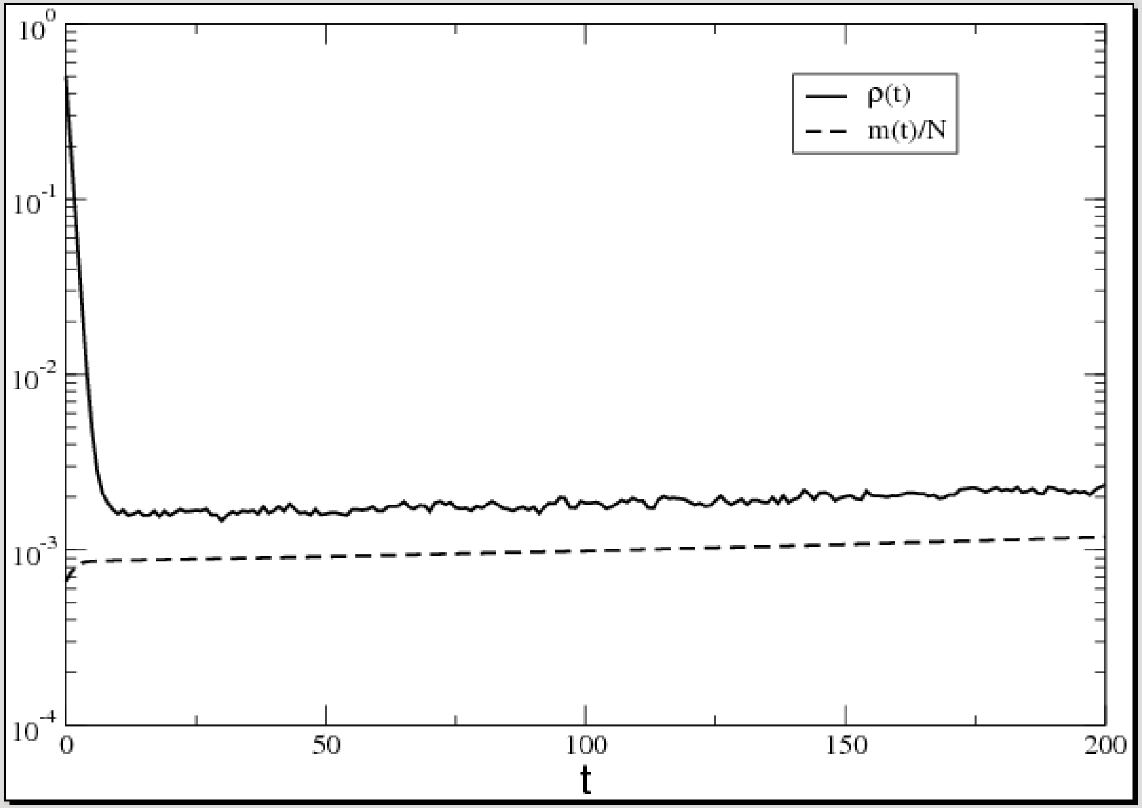
$$m_0 = 4 \, ; S_{iter} = 100 \, ;$$

$$\rho_0 = 0.5$$

$$\rho_t \sim \exp(-\beta \, t) \, ; \beta = 1$$

Dynamical regimes - II

I. In the second regime, cooperators' density and averaged clusters' size remain almost constant



$$N = 3024\,;\, m_0 = 2\,;\, S_{iter} = 100\,;\, \rho_0 = 0.5$$

Dynamical regimes - III

I. In the final regime, system reaches quickly a frozen state

II. For small values of m_0, the frozen state is disordered:

$$\rho^F = 0\,;\, m^F < N$$

I. For higher values of m_0, the frozen state is instead ordered:

$$\rho^F > 0\,;\, m^F = N$$

Equations of the dynamics

$$\frac{d\varrho}{dt} = \left\langle \frac{N-m_i}{N}\left[-2\beta_{ij}\varrho^2 + (1-\beta_{ij}-\alpha_{ij})\varrho(1-\varrho) + 2(1-\alpha_{ij})(1-\varrho)^2\right]\right\rangle_{i,j}$$

$$\frac{dm}{dt} = \left\langle \frac{m_i(N-m_i)}{N^2}\cdot m_j\varrho^2\right\rangle_{i,j}$$

$$[\alpha_{ij} = Pr(m_j - m_i + 1 - d_{ij} < 0)]$$

$$[\beta_{ij} = Pr(m_j - d_{ij}/2 < 1)]$$

– Previous equations are quite complicated but can be approximatedly solved, giving (at least qualitatively, but in regime I exactly) the correct behaviour of the model.

Conclusions

- GT can be usefully applied to social problems
- Game of Social Norms: modelizing the tendency of individuals to share social norms with others
- There is competition between the benefit of sharing norms and the effort to meet the others
- <u>The cooperation becomes a convenient strategy when agents have already high fitness.</u>
- In the microscopical version (DHM) we see how it is not necessary to have 100% of cooperators to get a complete ordering.
- Improved and more realistic models are needed.

Useful References

- R. Gibbons, “A primer in Game Theory”, Harvester-Wheatsheaf (1992).
- J. Hofbauer and K.Sigmund, “Evolutionary Games and Population Dynamics”, Cambridge (2004).
- M. A. Nowak, “Evolutionary Dynamics”, Harvard (2006).
- C. Castellano, S. Fortunato and V. Loreto, “Statistical Physics of social dynamics”, arXiv:physics.soc_ph/07103256v1 (2007).
- **D. Vilone, A. Guazzini, “Social Aggregation as a Cooperative Game”, arXiv:physics.soc_ph/0810.4316v2**