# Skew-Gaussian Term Structure Models

U. Namur Seminar

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# Outline

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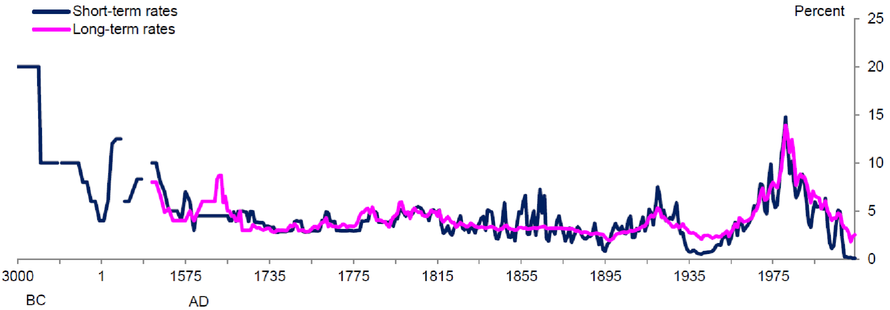
# Motivation

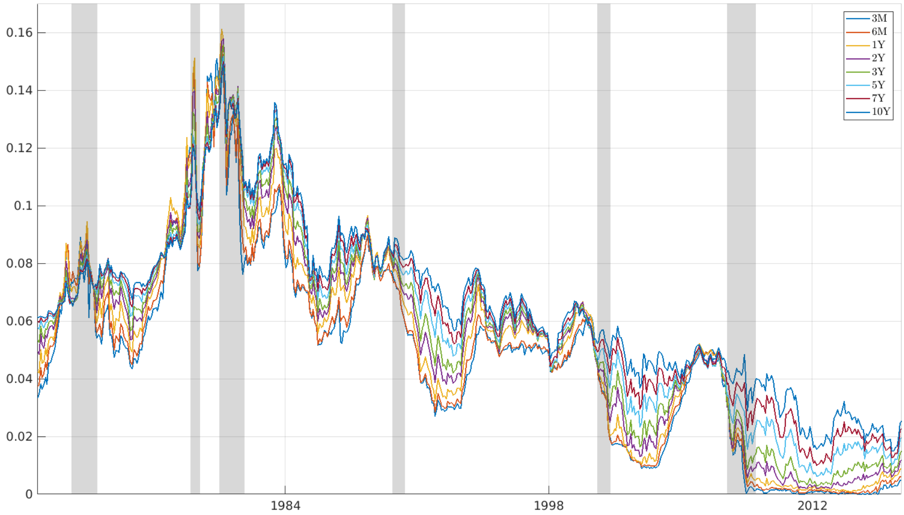
*'just as the laws of physics imply strange and surprising consequences as an object approaches a black hole, the laws of economics can yield some strange and surprising results as an economy gets too near the zero-lower bound on interest rates.'*, Rogoff (2017)

# Interest rates at historical low level




Sources: Homer and Sylla (1991); Heim and Mirowski (1987); Weiller and Mirowski (1990); Hills, Thomas and Dimsdale (2015); Bank of England; Historical Statistics of the United States Millenial Edition, Volume 3; Federal Reserve Economic Database. Notes: the intervals on the x-axis change through time up to 1715. From 1715 onwards the intervals are every twenty years. Prior to the C18th the rates reflect the country with the lowest rate reported for each type of credit: 3000BC to 6th century BC - Babylonian empire; 6th century BC to 2nd century BC - Greece; 2nd century BC to 5th century AD - Roman Empire; 6th century BC to 10th century AD - Byzantium (legal limit); 12th century AD to 13th century AD - Netherlands ;13th century AD to 16th century AD - Italian states. From the C18th the interest rates are of an annual frequency and reflect those of the most dominant money market: 1694 to 1918 this is assumed to be the UK; from 1919-2015 this is assumed to be the US. Rates used are as follows: Short rates: 1694-1717- Bank of England Discount rate;1717-1823 rate on 6 month East India bonds; 1824-1919 rate on 3 month prime or first class bills; 1919-1996 rate on 4-6 month prime US commercial paper ; 1997-2014 rate on 3month AA US commercial paper to non-financials. Long rates: 1702-1919 - rate on long-term government UK annuities and consols; 1919-1953, yield on long-term US government bond yields; 1954-2014 yield on 10 year US treasuries.

# Higher-order moments in macro-finance (1)

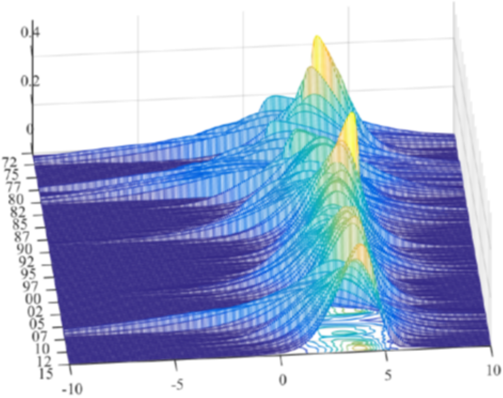


# Take-away (1)

- Periods of ultra-low interest rates might be more common in the future.
- This would lead to asymmetric distribution of shocks.
- Important to consider higher order moments if we want to model interest rates

# Higher-order moments in macro-finance (2)

Figure 2: Conditional Distribution of U.S. GDP Growth




Source: Adrian et al. (2017)

# Take-away (2)

- The conditional distribution of GDP Growth exhibits time-varying higher-order moments.
- Important to consider higher order moments if we want to model interest rates
- This phenomenon is not limited to the zero lower bound period

# Is asymmetry in interest rates dynamics only a zero lower bound phenomenon?

- We construct robust measures of asymmetry for interest rates
- We check if asymmetry is important for bond risk premia
- We check if asymmetry exhibits business cycle variation

# Extracting conditional asymmetry from conditional quantile (1)

Let $\Delta\mathcal{P}_t \equiv \mathcal{P}_t - \mathcal{P}_{t-1}$, be the $1^{\text{st}}$ difference for a linear combination of yields[1]. The frequency is monthly (end-of-month observations).

Compute quantile-based measure of conditional skewness as in Ghysels, Plazzi and Valkanov (2016):

$$\textsc{sk}_{\textsc{int},t-1}(\Delta\mathcal{P}_t) = 6\ \textsc{ra}_{\textsc{int},t-1}(\Delta\mathcal{P}_t)\frac{\int_0^{0.5} q_\alpha(u)d\theta}{\int_0^{0.5} q_\alpha^2(u)d\alpha}$$

$$\textsc{ra}_{\alpha,t-1}(\Delta\mathcal{P}_t) = \frac{(q_{\alpha,t-1}(\Delta\mathcal{P}_t) - q_{0.50,t-1}(\Delta\mathcal{P}_t)) - (q_{0.50,t-1}(\Delta\mathcal{P}_t) - q_{1-\alpha,t-1}(\Delta\mathcal{P}_t))}{q_{\alpha,t-1}(\Delta\mathcal{P}_t) - q_{1-\alpha,t-1}(\Delta\mathcal{P}_t)}$$

[1] We consider in our analysis PCs, portfolios of yields or degenerate portfolios with only one maturity.

The following model is estimated for quantiles $\alpha \in \{0.05,\ 0.25,\ 0.5,\ 0.75,\ 0.95\}$:

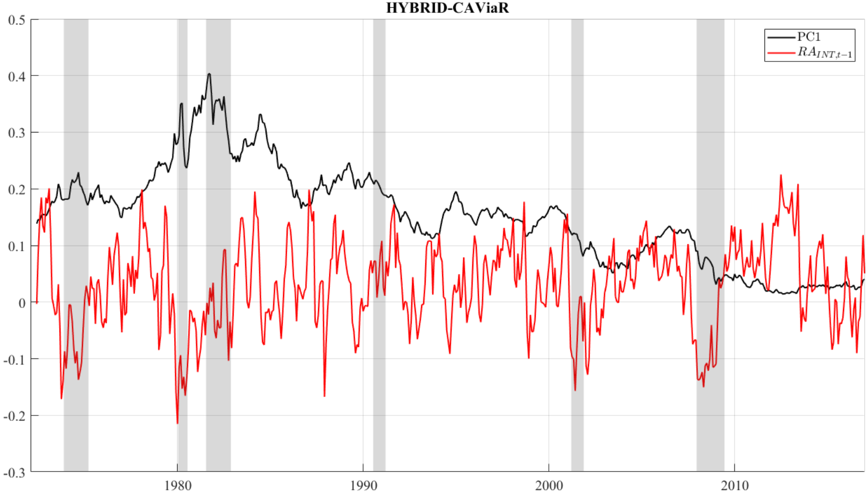
HYBRID-CAViaR:

$$q_{\alpha,t-1}(\Delta\mathcal{P}_t;\theta_\alpha) = \beta_\alpha^1 + \beta_\alpha^2\ q_{\alpha,t-2}(\Delta\mathcal{P}_{t-1};\theta_\alpha) + \beta_\alpha^3 \sum_{d=0}^{D} w(\kappa_\alpha)\ |\Delta\mathcal{P}_{t-1-d}|$$

- Low-frequency information: monthly changes in target variable.
- High-frequency information: absolute value of daily changes in target variable (66 lags).

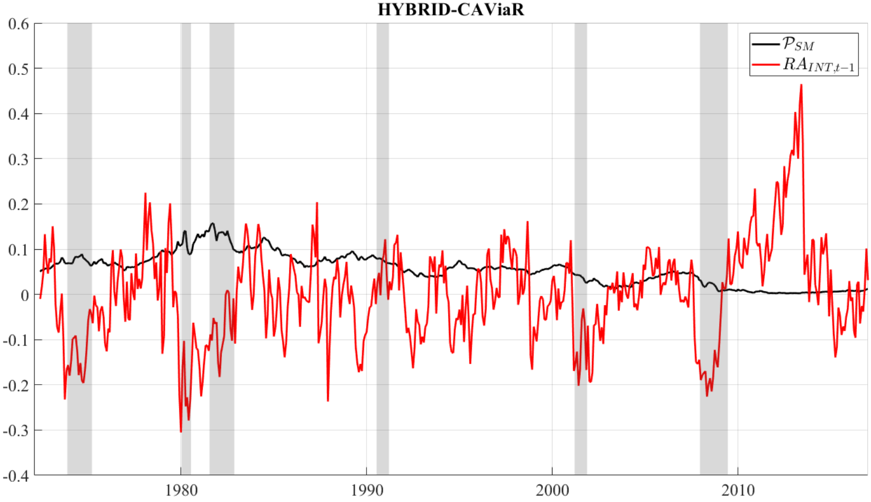
# First PC of the yield curve: business cycle variations

Figure 1: Comparison of $\text{PC}_1$ and its conditional asymmetry ($\text{RA}_{\text{INT},t-1}$)



# Short maturity portfolio: impact of the ZLB

Figure 2: Comparison of $\mathcal{P}_{\text{SM}}$ and its conditional asymmetry ($\text{RA}_{\text{INT},t-1}$)



# The impact of conditional asymmetry on bond risk premia

| Maturity | Intercept | $\textsc{pc}_1$ | $\textsc{pc}_2$ | $\textsc{pc}_3$ | $\textsc{sk}_{\textsc{int},t-1}(\textsc{pc}_2)$ |
|---|---|---|---|---|---|
| 24-month | -0,004 | 0,022 | -0,465 | -1,168 | -0,014 |
| | (-0,541) | (0,890) | (-2,710) | (-2,474) | (-3,530) |
| 36-month | -0,006 | 0,016 | -0,947 | -2,071 | -0,024 |
| | (-0,451) | (0,347) | (-2,975) | (-2,446) | (-3,374) |
| 60-month | -0,013 | -0,018 | -1,939 | -3,208 | -0,037 |
| | (-0,535) | (-0,221) | (-3,489) | (-2,070) | (-3,043) |
| 84-month | -0,024 | -0,051 | -2,928 | -3,796 | -0,045 |
| | (-0,742) | (-0,451) | (-3,913) | (-1,710) | (-2,752) |
| 120-month | -0,046 | -0,088 | -4,364 | -4,481 | -0,055 |
| | (-1,025) | (-0,540) | (-4,339) | (-1,419) | (-2,378) |

Forecasting regressions of bond excess returns on yield curve PCs and on the conditional skewness of the $\textsc{pc}_2$. The model estimated is :

$$\textsc{xhpr}^h_{t,n} = \delta^n_1 + \delta^{n\top}_{2:4}\textsc{pc}_t + \delta^n_5\textsc{sk}_{\textsc{int},t-1}(\textsc{pc}_2) + v^n_t$$

The holding period, $h$, is one year and excess returns are computed using overlapping data over the period from 1973 to 2015 (US Treasury zc). Standard errors correct for overlap with the Newey-West correction (using 13 lags). t-statistics are reported below their corresponding parameter estimate between brackets.

# Comparison of conditional moments for short maturity portfolio, $\mathcal{P}_{\text{SM}}$

| | $\mathcal{P}_{\text{SM}}$ | $\Delta\mathcal{P}_{\text{SM}}$ | $\sigma_{\text{GARCH},t-1}$ | $\sigma_{\text{IQR},t-1}$ | $\text{SK}_{\text{INT},t-1}$ |
|---|---|---|---|---|---|
| $\mathcal{P}_{\text{SM}}$ | 1 | | | | |
| $\Delta\mathcal{P}_{\text{SM}}$ | 0,0888 | 1 | | | |
| $\sigma_{\text{GARCH},t-1}$ | 0,5373 | -0,1116 | 1 | | |
| $\sigma_{\text{IQR},t-1}$ | 0,5651 | -0,0207 | 0,8266 | 1 | |
| $\text{SK}_{\text{INT},t-1}$ | -0,2472 | 0,1054 | -0,6657 | -0,4323 | 1 |

Correlation matrix between the monthly short-maturity portfolio ($\mathcal{P}_{\text{SM}}$), its monthly changes ($\Delta\mathcal{P}_{\text{SM}}$), the conditional volatility measure derived from the interquartile range (HYBRID-CAViaR model), $\sigma_{\text{IQR},t-1}$, the conditional volatility measure derived from a GARCH(1,1) on $\Delta\mathcal{P}_{\text{SM}}$ , $\sigma_{\text{GARCH},t-1}$ and the quantile-based measure of conditional skewness, $\text{SK}_{\text{INT},t-1}$

# Take-away (3)

- Interest rates dynamics show time varying / business cycle related asymmetry
- This phenomenon is coupled with conditional asymmetry embedded in macroeconomics variable
- Need to account for these empirical observations in macro-finance modelling

# Outline

Want:

- Model retaining flexibility of Gaussian-DTSM (fit, deviation from expectation hypothesis)
- Account for conditional asymmetry in a non-trivial way
- To model adequately interactions between conditional moments
- Capture the impact of (PCs) conditional asymmetry on bond risk premia

# Risk-Neutral Pricing and No-Arbitrage: Gaussian case (1)

The price of a risk-free zero-coupon bond maturing in $n$ period is defined as:

$$P_{t,n} = \mathbb{E}_t \left[ M_t P_{t+1,n-1} \right] \tag{1}$$

In absence of arbitrage opportunities, $M_t > 0$, and under the risk-neutral measure ($\mathbb{Q}$)

$$P_{t,n} = \mathbb{E}_t^{\mathbb{Q}} \left[ \exp\left(-r_t\right) P_{t+1,n-1} \right] \tag{2}$$

We assume the nominal short rate, $r_t$, to be affine in the $\mathrm{N_X}$ risk factors, $\mathbf{x}_t$:

$$r_t = \rho_{0\mathrm{x}} + \boldsymbol{\rho}_{1\mathrm{x}}^{\top} \mathbf{x}_t \tag{3}$$

where the risk factors follow a VAR(1) under the $\mathbb{Q}$:

$$\mathbf{x}_t = \boldsymbol{\mu}_{\mathbf{x}} + \boldsymbol{\Phi}_{\mathbf{x}} \ \mathbf{x}_{t-1} + \boldsymbol{\Sigma}_{\mathrm{x}}^{1/2} \boldsymbol{\varepsilon}_t \tag{4}$$

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Where $\boldsymbol{\varepsilon}_t \overset{\mathbb{Q}}{\sim} \mathcal{N}(\mathbf{0}_{\mathrm{N_X}}, \boldsymbol{I}_{\mathrm{N_X}})$.

# Risk-Neutral Pricing and No-Arbitrage: Gaussian case (2)

In an exponential affine setting, we want to express bonds' prices as:

$$P_{t,n} = \exp\left(\mathcal{A}_n + \boldsymbol{\mathcal{B}}_n^\top \mathbf{x}_t\right) \tag{5}$$

Implying that

$$\begin{aligned} P_{t,n} &= \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-r_t\right) P_{t+1,n-1}\right] \\ \exp\left(\mathcal{A}_n + \boldsymbol{\mathcal{B}}_n^\top \mathbf{x}_t\right) &= \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-r_t\right)\exp\left(\mathcal{A}_{n-1} + \boldsymbol{\mathcal{B}}_{n-1}^\top \mathbf{x}_{t+1}\right)\right] \end{aligned} \tag{6}$$

We get the expression for the coefficients $\mathcal{A}_n$ and $\mathcal{B}_n$ by applying the principle of induction:

- Eq (6) for 1
- Eq (6) true for n-1 implies it is true for n

Under assumptions (3)-(4), we conjecture the solution of recursion (2) to be of the exponential affine form:

$$\exp\left(\mathcal{A}_n + \boldsymbol{\mathcal{B}}_n^\top \mathbf{X}_t\right) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-r_t\right)\exp\left(\mathcal{A}_{n-1} + \boldsymbol{\mathcal{B}}_{n-1}^\top \mathbf{X}_{t+1}\right)\right]$$

$$\begin{aligned}
\Leftrightarrow \mathcal{A}_n + \boldsymbol{\mathcal{B}}_n^\top \mathbf{X}_t &= \log\exp\{-\rho_{0\mathrm{X}} - \boldsymbol{\rho}_{1\mathrm{X}}^\top \mathbf{X}_t + \mathcal{A}_{n-1} + \boldsymbol{\mathcal{B}}_{n-1}^\top\ \boldsymbol{\mu}_{\mathbf{X}} + \boldsymbol{\mathcal{B}}_{n-1}^\top \boldsymbol{\Phi}_{\mathbf{X}}\ \mathbf{X}_t\} \\
&+ \underbrace{\log\int_{\mathbb{R}^{N_{\mathrm{X}}}} \exp\{\boldsymbol{\mathcal{B}}_{n-1}^\top \boldsymbol{\Sigma}_{\mathrm{X}}^{1/2}\boldsymbol{\varepsilon}_{t+1}\} f^{\mathbb{Q}}(\boldsymbol{\varepsilon}_{t+1}) d\boldsymbol{\varepsilon}_{t+1}}_{\boldsymbol{\mathcal{B}}_{n-1}^\top \boldsymbol{\Sigma}_{\mathrm{X}} \boldsymbol{\mathcal{B}}_{n-1}} \\
&= -\rho_{0\mathrm{X}} - \boldsymbol{\rho}_{1\mathrm{X}}^\top \mathbf{X}_t + \mathcal{A}_{n-1} + \boldsymbol{\mathcal{B}}_{n-1}^\top\ \boldsymbol{\mu}_{\mathbf{X}} + \boldsymbol{\mathcal{B}}_{n-1}^\top \boldsymbol{\Phi}_{\mathbf{X}}\ \mathbf{X}_t \\
&+ \frac{1}{2}\boldsymbol{\mathcal{B}}_{n-1}^\top \boldsymbol{\Sigma}_{\mathrm{X}} \boldsymbol{\mathcal{B}}_{n-1}
\end{aligned}$$

Details

# Risk-Neutral Pricing and No-Arbitrage: Gaussian case (4)

We can match coefficients on the lhs and rhs of the previous equation to obtain the recursions for $\mathcal{A}_n$ and $\boldsymbol{\mathcal{B}}_n$

$$\mathcal{A}_n = -\rho_{0\mathrm{x}} + \mathcal{A}_{n-1} + \boldsymbol{\mathcal{B}}_{n-1}^\top \ \boldsymbol{\mu}_{\mathbf{x}} + \frac{1}{2}\boldsymbol{\mathcal{B}}_{n-1}^\top \boldsymbol{\Sigma}_{\mathrm{x}} \boldsymbol{\mathcal{B}}_{n-1} \tag{7}$$

$$\mathcal{B}_n^\top = -\boldsymbol{\rho}_{1\mathrm{x}}^\top + \boldsymbol{\mathcal{B}}_{n-1}^\top \boldsymbol{\Phi}_{\mathbf{x}} \tag{8}$$

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The continuously-compounded yield on a zero-coupon bond yield with maturity $n$, $y_{t,n}$, is an affine function of the risk factors:

$$\begin{aligned} P_{t,n} &= \exp(-n y_{t,n}) \\ \Leftrightarrow y_{t,n} &= -\frac{1}{n}\log P_{t,n} \\ y_{t,n} &= -\frac{\mathcal{A}_n}{n} - \frac{\boldsymbol{\mathcal{B}}_n^\top}{n}\mathbf{x}_t \\ &= A_n + \boldsymbol{B}_n^\top \ \mathbf{x}_t \end{aligned}$$

Where $\mathcal{A}_n$ and $\boldsymbol{\mathcal{B}}_n$ are obtained via recursions (7) and (8).

# Historical Dynamics: Gaussian case (1)

- Standard to assume that the risk factors also follow a VAR(1) under $\mathbb{P}$:

$$\mathbf{x}_t = \boldsymbol{\mu}_{\mathbf{x}}^{\mathbb{P}} + \boldsymbol{\Phi}_{\mathbf{x}}^{\mathbb{P}}\ \mathbf{x}_{t-1} + \boldsymbol{\Sigma}_{\mathrm{x}}^{1/2} \boldsymbol{\varepsilon}_t^{\mathbb{P}}$$

where $\boldsymbol{\varepsilon}_t^{\mathbb{P}} \overset{\mathbb{P}}{\sim} \mathcal{N}(\mathbf{0}_{N_{\mathrm{x}}}, \boldsymbol{I}_{\mathrm{N_x}})$.

- The Radon-Nykodym derivative, $\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)_{t,t+1}$, has to satisfy the following condition: Intuition

$$f^{\mathbb{P}}(\boldsymbol{\varepsilon}_{t+1}) = f^{\mathbb{Q}}(\boldsymbol{\varepsilon}_{t+1}) \left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)_{t,t+1}$$

$$\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)_{t,t+1} = \exp\left(-\frac{1}{2}\boldsymbol{\lambda}_{\mathrm{x}t}^{\top}\boldsymbol{\lambda}_{\mathrm{x}t} + \boldsymbol{\lambda}_{\mathrm{x}t}^{\top}\boldsymbol{\varepsilon}_{t+1}\right)$$

where the market price of risk, $\boldsymbol{\lambda}_{\mathrm{x}t}$, is essentially affine in the risk factors:

$$\boldsymbol{\lambda}_{\mathrm{x}t} = \boldsymbol{\Sigma}_{\mathrm{x}}^{-1/2}(\boldsymbol{\lambda}_{0\mathrm{x}} + \boldsymbol{\lambda}_{1\mathrm{x}}\mathbf{x}_t)$$

# Historical Dynamics: Gaussian case (2)

- We obtain the following restriction between our risk-neutral and historical dynamics:

$$\boldsymbol{\varepsilon}_{t+1} = \boldsymbol{\varepsilon}^{\mathbb{P}}_{t+1} + \boldsymbol{\lambda}_{\mathrm{x}t}$$

- Substituting for $\boldsymbol{\varepsilon}_t$ in the risk-neutral VAR equation (4), we obtain the following restrictions on the model dynamics: Show

$$\boldsymbol{\Phi}_{\mathbf{x}} = \boldsymbol{\Phi}^{\mathbb{P}}_{\mathbf{x}} - \boldsymbol{\lambda}_1$$

$$\boldsymbol{\mu}_{\mathbf{x}} = \boldsymbol{\mu}^{\mathbb{P}}_{\mathbf{x}} - \boldsymbol{\lambda}_0$$ Implication

- We can now define the model-implied ex ante holding period return:

$$\begin{aligned}
\mathrm{HPR}_{t,n} &= \mathbb{E}^{\mathbb{P}}_t\left[\log\left(\frac{P_{t+1,n-1}}{P_{t,n}}\right)\right] \\
&= \mathcal{A}_{n-1} - \mathcal{A}_n - \mathcal{B}_n^\top \mathbf{x}_t + \mathcal{B}_{n-1}^\top \mathbb{E}^{\mathbb{P}}_t\left[\mathbf{x}_{t+1}\right] \\
&= r_t - \frac{1}{2}\boldsymbol{\mathcal{B}}_{n-1}^\top \boldsymbol{\Sigma}_{\mathrm{x}} \boldsymbol{\mathcal{B}}_{n-1} + \mathcal{B}_{n-1}^\top\left(\boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1 \mathbf{x}_t\right)
\end{aligned}$$

and the model-implied excess holding period return is defined as:

$$\mathrm{XHPR}_{t,n} = \mathrm{HPR}_{t,n} - r_t$$

# Estimation: Gaussian case

- The measurement equation is given by the link between observed zero-coupon bond yields and model-implied yields:

$$\boldsymbol{y}_t^{\text{OBS}} = \mathbf{A} + \mathbf{B}\mathbf{x}_t + \sigma_y \mathbf{I}_{\text{N}_{\text{MAT}}} \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_{\text{N}_{\text{MAT}}})$$

- The transition equation is given by the VAR-dynamics under $\mathbb{P}$:

$$\mathbf{x}_t = \boldsymbol{\mu}_{\mathbf{x}}^{\mathbb{P}} + \boldsymbol{\Phi}_{\mathbf{x}}^{\mathbb{P}} \ \mathbf{x}_{t-1} + \boldsymbol{\Sigma}_{\text{X}}^{1/2} \boldsymbol{\varepsilon}_t^{\mathbb{P}}$$

- These two equations form the state-space used in the estimation

# Risk-Neutral Pricing and No-Arbitrage: Skew-Gaussian case (1)

We keep the same setting as in the Gaussian DTSM:

$$r_t = \rho_{0\mathrm{x}} + \boldsymbol{\rho}_{1\mathrm{x}}^\top \mathbf{x}_t$$

$$\mathbf{x}_t = \boldsymbol{\mu}_\mathbf{x} + \boldsymbol{\Phi}_\mathbf{x}\ \mathbf{x}_{t-1} + \boldsymbol{\Sigma}_\mathrm{x}^{1/2} \boldsymbol{\varepsilon}_t$$

with the exception that $\boldsymbol{\varepsilon}_t \overset{\mathbb{Q}}{\sim} \mathcal{SN}(\mathbf{0}_{\mathrm{N_x}}; \boldsymbol{I}_{\mathrm{N_x}}, \boldsymbol{\alpha})$. Details

- Following the same approach as in the Gaussian case, the continuously-compounded yield on a zero-coupon bond yield with maturity $n$ is an affine function of the risk factors:

$$\begin{aligned} y_{t,n} &= -\frac{\mathcal{A}_n}{n} - \frac{\boldsymbol{\mathcal{B}}_n^\top}{n} \mathbf{x}_t \\ &= A_n + \boldsymbol{B}_n^\top\ \mathbf{x}_t \end{aligned} \tag{9}$$

Where $\mathcal{A}_n$ and $\boldsymbol{\mathcal{B}}_n$ are obtained via the following recursions:

$$\mathcal{A}_n = -\rho_{0\mathrm{x}} + \mathcal{A}_{n-1} + \boldsymbol{\mathcal{B}}_{n-1}^\top\ \boldsymbol{\mu}_\mathbf{x} + \frac{1}{2}\boldsymbol{\mathcal{B}}_{n-1}^\top \boldsymbol{\Sigma}_\mathrm{x} \boldsymbol{\mathcal{B}}_{n-1} + \log 2\Phi(\boldsymbol{\delta}^\top \boldsymbol{\Sigma}_\mathrm{x}^{1/2} \boldsymbol{\mathcal{B}}_{n-1})$$

$$\mathcal{B}_n^\top = -\boldsymbol{\rho}_{1\mathrm{x}}^\top + \boldsymbol{\mathcal{B}}_{n-1}^\top \boldsymbol{\Phi}_\mathbf{x}$$

where $\boldsymbol{\delta} = \boldsymbol{\alpha} / \left(1 + \boldsymbol{\alpha}^\top \boldsymbol{\alpha}\right)^{\frac{1}{2}}$.

- So far, the shape vector $\boldsymbol{\alpha}$ (or equiv. $\boldsymbol{\delta}$) is not allowed to change over time. Under the following assumptions[2]:

- Investors observe $\boldsymbol{\delta}_t$ at each point in time.
- Investors assume that $\boldsymbol{\delta}_t$ will not change over their investment horizon.

Then, zero-coupon bond yields take the following form:

$$\begin{aligned} y_{t,n} &= -\frac{\mathcal{A}_{t,n}}{n} - \frac{\boldsymbol{\mathcal{B}}_n^\top}{n}\mathbf{x}_t \\ &= A_{t,n} + \boldsymbol{B}_n^\top \mathbf{x}_t \end{aligned} \tag{10}$$

Where $\mathcal{A}_{t,n}$ and $\boldsymbol{\mathcal{B}}_n$ are obtained via the following recursions:

$$\begin{aligned} \mathcal{A}_{t,n} &= -\rho_{0\mathrm{x}} + \mathcal{A}_{t,n-1} + \boldsymbol{\mathcal{B}}_{n-1}^\top \boldsymbol{\mu}_{\mathbf{x}} + \frac{1}{2}\boldsymbol{\mathcal{B}}_{n-1}^\top \boldsymbol{\Sigma}_{\mathrm{x}} \boldsymbol{\mathcal{B}}_{n-1} + \log 2\Phi(\boldsymbol{\delta}_t^\top \boldsymbol{\Sigma}_{\mathrm{x}}^{1/2} \boldsymbol{\mathcal{B}}_{n-1}) \\ \mathcal{B}_n^\top &= -\boldsymbol{\rho}_{1\mathrm{x}}^\top + \boldsymbol{\mathcal{B}}_{n-1}^\top \boldsymbol{\Phi}_{\mathbf{x}} \end{aligned}$$

[2] These assumptions are consistent with the "anticipated utility approach" in macro modelling and has been applied to Shadow-Rate DTSM with TV lower bound by Dewachter, Iania and Wijnandts (2016).

The risk factors also follow a VAR(1) under $\mathbb{P}$:

$$\mathbf{x}_t = \boldsymbol{\mu}^{\mathbb{P}}_{\mathbf{x}} + \boldsymbol{\Phi}^{\mathbb{P}}_{\mathbf{x}} \; \mathbf{x}_{t-1} + \boldsymbol{\Sigma}_{\mathrm{x}}^{1/2} \boldsymbol{\varepsilon}^{\mathbb{P}}_t \tag{11}$$

$$\tag{12}$$

where $\boldsymbol{\varepsilon}^{\mathbb{P}}_t \overset{\mathbb{P}}{\sim} \mathcal{SN}(\mathbf{0}_{N_{\mathrm{x}}}; \boldsymbol{I}_{\mathrm{N_x}}, \boldsymbol{\alpha}_t)$.
To obtain this result, we the Gaussian Radon-Nykodym derivative in the following way:

$$\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)_{t,t+1} = \exp\left(-\frac{1}{2}\boldsymbol{\lambda}_{\mathrm{x}t}^{\top}\boldsymbol{\lambda}_{\mathrm{x}t} + \boldsymbol{\lambda}_{\mathrm{x}t}^{\top}\boldsymbol{\varepsilon}_{t+1}\right) \frac{\Phi\left(\boldsymbol{\alpha}_t^{\top}(\boldsymbol{\varepsilon}_{t+1} - \boldsymbol{\lambda}_{\mathrm{x}t})\right)}{\Phi\left(\boldsymbol{\alpha}_t^{\top}\boldsymbol{\varepsilon}_{t+1}\right)} \tag{13}$$

- The model-implied ex ante holding period return now takes the form:

$$\begin{aligned}
\mathrm{HPR}_{t,n} &= \mathbb{E}^{\mathbb{P}}_t\left[\log\left(\frac{P_{t+1,n-1}}{P_{t,n}}\right)\right] \\
&= \mathcal{A}_{t,n-1} - \mathcal{A}_{t,n} - \mathcal{B}_n^{\top}\mathbf{x}_t + \mathcal{B}_{n-1}^{\top}\mathbb{E}^{\mathbb{P}}_t\left[\mathbf{x}_{t+1}\right] \\
&= r_t - \frac{1}{2}\boldsymbol{\mathcal{B}}_{n-1}^{\top}\boldsymbol{\Sigma}_{\mathrm{x}}\boldsymbol{\mathcal{B}}_{n-1} + \mathcal{B}_{n-1}^{\top}\left(\boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1\mathbf{x}_t + \boldsymbol{\Sigma}_{\mathrm{x}}^{1/2} c_\pi \boldsymbol{\delta}_t\right) \\
&- \log 2\Phi(\boldsymbol{\delta}_t^{\top}\boldsymbol{\Sigma}_{\mathrm{x}}^{1/2}\boldsymbol{\mathcal{B}}_{n-1})
\end{aligned}$$

# State-space form I

- The measurement equation is given by the link between observed zero-coupon bond yields and model-implied yields:

$$\begin{aligned} \mathbf{Y}_t \equiv \boldsymbol{y}_t^{\text{OBS}} &= \mathbf{A}_t(\boldsymbol{\delta}_t) + \mathbf{B}\mathbf{x}_t + \sigma_y \mathbf{I}_{\text{N}_{\text{MAT}}} \boldsymbol{\eta}_t \\ &= \mathbf{A}_t + \mathbf{B}\mathbf{x}_t + \mathbf{D}\boldsymbol{\eta}_t \end{aligned} \tag{14}$$

- We will proxy for $\boldsymbol{\delta_t}$ with the robust measures of conditional asymmetry, $\text{RA}_t$.

# State-space form II

- For the state dynamics, first we can use a weak VAR representation for $\mathbf{x}_t$ under the historical measure:

$$\begin{aligned}
\mathbf{x}_t &= \mathbb{E}_{t-1}[\mathbf{x}_t] + \mathbb{V}\mathrm{ar}_{t-1}[\mathbf{x}_t]^{1/2}\boldsymbol{e}_t \\
&= \boldsymbol{\mu}_{\mathrm{x}}^{\mathbb{P}} + \begin{bmatrix} \boldsymbol{\Phi}_{\mathrm{x}}^{\mathbb{P}} & c_\pi \boldsymbol{\Sigma}_{\mathrm{x}}^{1/2} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{t-1} \\ \boldsymbol{\delta}_{t-1} \end{bmatrix} \\
&+ \sqrt{\boldsymbol{\Sigma}_{\mathrm{x}}^{1/2} \left[ \boldsymbol{I}_{\mathrm{N_x}} - \ c_\pi^2 \boldsymbol{\delta}_{t-1} \boldsymbol{\delta}_{t-1}^\top \right] \boldsymbol{\Sigma}_{\mathrm{x}}^{1/2}}\ \boldsymbol{e}_t \\
&= \boldsymbol{\mu}_{\mathrm{x}}^{\mathbb{P}} + \boldsymbol{\Phi}_{\mathrm{x}}^{\mathbb{P}}\ \mathbf{x}_{t-1} + \boldsymbol{\Phi}_\delta\ \boldsymbol{\delta}_{t-1} + \boldsymbol{\Sigma}_{\mathrm{x}}^{1/2}(\boldsymbol{\delta}_{t-1})\boldsymbol{e}_t
\end{aligned} \tag{15}$$

where $\boldsymbol{e}_t$ is a martingale difference sequence.

- We can thus write our Skew-Gaussian DTSM in state-space form with measurement equation given by equation (14) and transition equation given by (15).

# Time-varying skewness: Quasi Maximum Likelihood Estimation I

Under the following assumptions:

- The error vectors for the measurement and transition equations, $\boldsymbol{\eta}_t$ and $\boldsymbol{e}_t$, are jointly normally distributed and uncorrelated
- The initial state vector is normally distributed: $\mathbf{x}_0 \sim \mathcal{N}(\bar{\boldsymbol{\mu}}_{\mathrm{x}}, \bar{\boldsymbol{\Sigma}}_{\mathrm{x}})$

we may write the likelihood (ignoring a constant) as:

$$-\ln \mathrm{L}(\Theta; \underline{\mathbf{Y}}_{\mathrm{T}}) = \frac{1}{2} \sum_{t=1}^{T} \left( \log |\mathbf{s}_{t|t-1}(\Theta)| + \boldsymbol{\lambda}_t^{\top}(\Theta) \mathbf{s}_{t|t-1}^{-1}(\Theta) \boldsymbol{\lambda}_t(\Theta) \right)$$

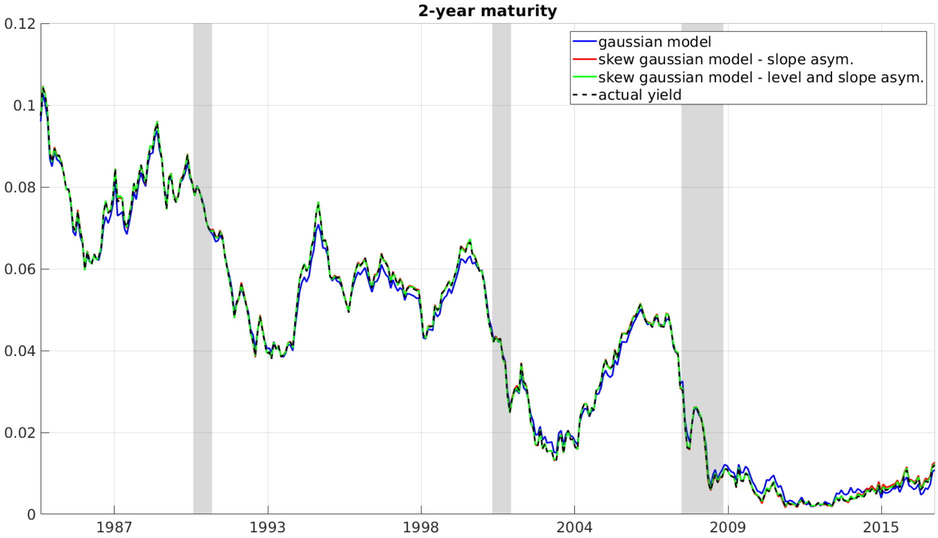
# Outline

# Comparison of model fit: RMSE

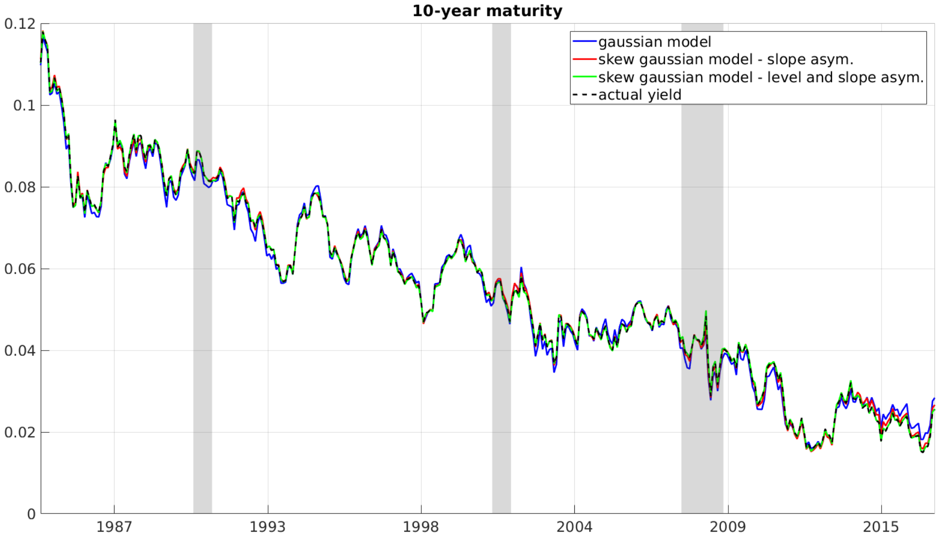
| Model | Maturity in months | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 3 | 6 | 12 | 24 | 36 | 60 | 84 | 120 |
| Specif. (1) | 18.71 | 11.44 | 16.13 | 15.03 | 12.6 | 9.01 | 7.43 | 14.52 |
| Specif. (2) | 6.74 | 8.07 | 9.73 | 2.89 | 4.59 | 5.19 | 3.14 | 5.61 |
| Specif. (3) | 4.1 | 8.55 | 5.41 | 2.52 | 1.64 | 2.14 | 2.37 | 2.31 |

This table reports the yield Root Mean Squared Fitting Errors at different maturities (in bps). The estimation period runs from January 1985 until December 2016 (monthly frequency). We compare the results for a 2-factor Gaussian DTSM (Specif. (1)), a 2-factor Skew-Gaussian DTSM where only the slope factor exhibits conditional asymmetry (Specif. (2)) and a 2-factor Skew-Gaussian DTSM where both factors exhibit conditional asymmetry (Specif. (3))

# Comparison of model fit: 2-year maturity



# Comparison of model fit: 10-year maturity



# Conclusions and ongoing research

Conditional asymmetry of interest rates:

- Is time-varying
- Switches sign over the business cycle
- Is affected by ZLB period (short maturities)
- Is relevant for bond risk premia

Skew-Normal DTSM:

- Allows to introduce conditional asymmetry
- Retains tractable pricing

Ongoing work:

- Implications for model dynamics and pricing of stochastic asymmetry
- Analysis of interactions between financial and macro conditional asymmetries

# Outline

# The Normal Distribution: Moment and Cumulant Generating Functions

Let $\boldsymbol{Z}$, a d-dimensional random vector, distributed as $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.
The Moment Generating Function (MGF or Laplace transform) of $\boldsymbol{Z}$, $\varphi_{\mathbf{Z}}(\boldsymbol{u})$, is given by:

$$\begin{aligned}\varphi_{\mathbf{Z}}(\boldsymbol{u}) &= \int_{\mathbb{R}^d} \exp\left(\boldsymbol{u}^\top \boldsymbol{z}\right) f(\boldsymbol{z}) d\boldsymbol{z} \\ &= \exp\left(\boldsymbol{u}^\top \boldsymbol{\mu} + \frac{1}{2}\boldsymbol{u}^\top \boldsymbol{\Sigma}\, \boldsymbol{u}\right), \quad \boldsymbol{u} \in \mathbb{R}^d \end{aligned} \tag{16}$$

where we notice that the MGF has an exponential-affine form.
The Cumulant Generating Function (CGF or the log-Laplace transform) of $\boldsymbol{Z}$, $\psi_{\mathbf{Z}}(\boldsymbol{u})$, is given by:

$$\psi_{\mathbf{Z}}(\boldsymbol{u}) = \log \varphi_{\mathbf{Z}}(\boldsymbol{u}) = \boldsymbol{u}^\top \boldsymbol{\mu} + \frac{1}{2}\boldsymbol{u}^\top \boldsymbol{\Sigma}\, \boldsymbol{u}, \quad \boldsymbol{u} \in \mathbb{R}^d \tag{17}$$

Go back

# The Skew-Normal distribution: Moment and Cumulant Generating Functions

- Introduce multivariate Skew-Normal distribution:

$$\phi(\mathbf{z};\bar{\boldsymbol{\Omega}},\boldsymbol{\alpha}) = 2\phi(\mathbf{z};\bar{\boldsymbol{\Omega}})\Phi(\boldsymbol{\alpha}^\top \mathbf{z}), \quad \mathbf{z} \in \mathbb{R}^d$$

- We say that $\boldsymbol{Z} \sim \mathcal{SN}(\mathbf{0},\bar{\boldsymbol{\Omega}},\boldsymbol{\alpha})$
- control explicitly for asymmetry through $\boldsymbol{\alpha}$ vector ($\alpha_i \in \mathbb{R}$)
- $\Phi(\boldsymbol{\alpha}^\top \mathbf{z})$ serves as a "symmetry-modulating" mechanism

The log-Laplace transform (or cumulant generating function) of $\boldsymbol{Y} = \boldsymbol{\xi} + \boldsymbol{\omega}\boldsymbol{Z} \sim \mathcal{SN}(\boldsymbol{\xi},\boldsymbol{\Omega},\boldsymbol{\alpha})$ is:

$$\psi_{\mathbf{Y}}(\boldsymbol{u}) = \log \varphi_{\mathbf{Y}}(\boldsymbol{u}) = \boldsymbol{u}^\top \boldsymbol{\xi} + \frac{1}{2}\boldsymbol{u}^\top \boldsymbol{\Omega}\boldsymbol{u} + \log 2\Phi\left(\boldsymbol{\delta}^\top \boldsymbol{\omega}\boldsymbol{u}\right), \quad \boldsymbol{u} \in \mathbb{R}^d \quad (18)$$

Where we defined:

$$\boldsymbol{\Omega} = \boldsymbol{\omega}\bar{\boldsymbol{\Omega}}\boldsymbol{\omega}^\top$$

$$\boldsymbol{\delta} = \frac{\bar{\boldsymbol{\Omega}}\boldsymbol{\alpha}}{\left(1 + \boldsymbol{\alpha}^\top \bar{\boldsymbol{\Omega}}\boldsymbol{\alpha}\right)^{\frac{1}{2}}}$$

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# The Skew-Normal distribution: Negative $\alpha$ implies negative asymmetry

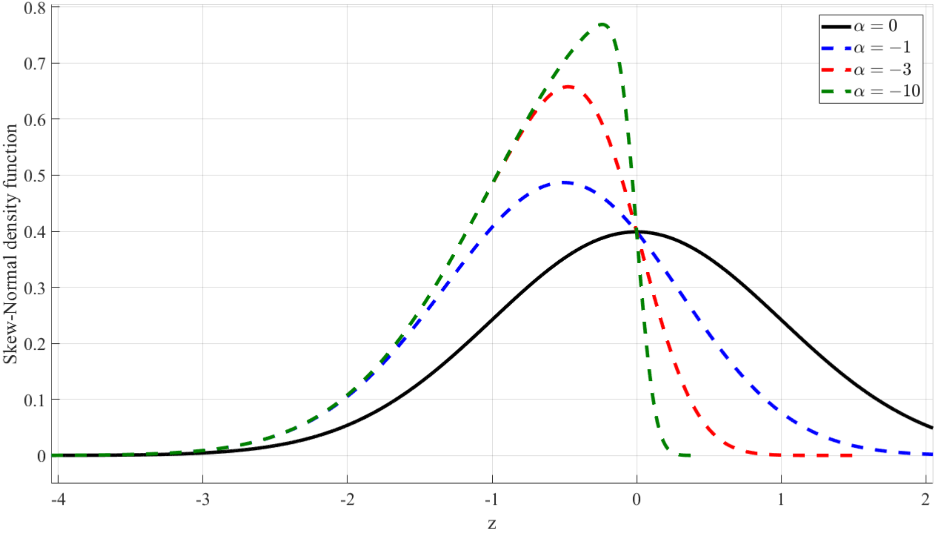


Figure 3: Skew-Normal density functions when $\alpha = 0, -1, -3, -10$.

# The Skew-Normal distribution: Positive $\alpha$ implies positive asymmetry

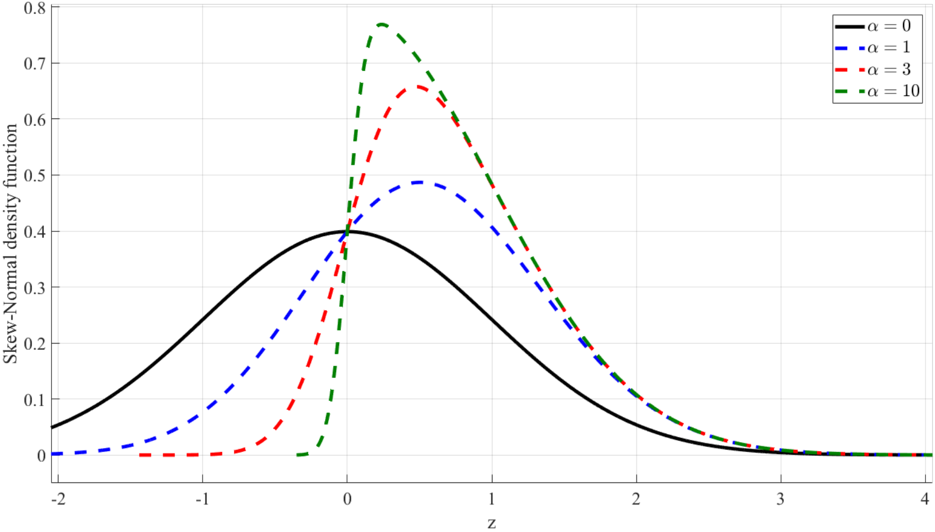


Figure 4: Skew-Normal density functions when $\alpha = 0,\ 1,\ 3,\ 10$.

# Radon-Nykodym Derivative and Change of Measure: Intuition

Idea of changing probabilities is counter-intuitive, illustrate with example of loading a die

- Suppose you make a bet where you roll a dice and you get an amount of money (Euro) equal to the face of the dice
- Expected value of the bet is **3.5** Euro, Variance is 2.9
- By loading the dice, it is possible to change the expected value of the bet while keeping the variance the same (Change of measure). For example the expected value can become **2.5**.
- In term structure models the change of measure is made in order to take into account of risk and in order to price bonds in a "risk adjusted world"

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